Analysis for PhD students (2025); Assignment 2

Problem 1. Prove that if $f \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} \big|\xi \, \widehat{f}(\xi)\big| \, d\xi < \infty$, then f coincides almost everywhere with a differentiable function f_0 whose derivative is

$$f_0'(x) = 2\pi i \int_{-\infty}^{\infty} \xi \, \widehat{f}(\xi) \, e^{2\pi i \xi x} \, d\xi \qquad (\forall x \in \mathbb{R}).$$

(15p)

Problem 2. Prove that if μ is a positive Borel measure on $\mathbb{T}$ with $\mu(\mathbb{T}) = 1$, then $|\widehat{\mu}(k)| < 1$ for all $k \neq 0$ unless μ is a linear combination, with nonnegative coefficients, of the point masses at $y, y + \frac{1}{m}, \ldots, y + \frac{m-1}{m}$ for some $m \in \mathbb{N}$ and $y \in \mathbb{T}$.

(15p)

Problem 3. Folland p. 255, Exercise 15, all parts a–c (the Sampling Theorem).

(15p)

Problem 4. Prove that for any $k \geq 1$:

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + |x - a|)^k (1 + |x|)^k} \asymp_k \begin{cases} (1 + |a|)^{-k} & \text{if } k > 1 \\ (1 + |a|)^{-1} \log(2 + |a|) & \text{if } k = 1, \end{cases}$$

uniformly over all $a \in \mathbb{R}$.

(10p)

Problem 5. Let a_0 be a fixed number with $0 < a_0 < 1$. Prove that

$$\int_1^{\infty} e^{-bx} x^{ab} \, dx \sim \frac{e^{-b}}{(1 - a)b}$$

as $b \to \infty$, uniformly over all $a \in [0, a_0]$.

(15p)

Problem 6. Let $n \in \mathbb{Z}^+$. Find (with proof) the largest possible constant $\alpha > 0$ such that

$$\int_0^1 e(x_1 t + x_2 t^2 + \cdots + x_n t^n) \, dt \ll_n (1 + |x|)^{-\alpha}, \qquad \forall x \in \mathbb{R}^n.$$

$$(15\text{p})$$

Problem 7. Using stationary phase, prove that for any fixed $\alpha > 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda(\alpha \sin x - x)} \, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\cos\left(\lambda\left(\sqrt{\alpha^2 - 1} - \arccos\left(\alpha^{-1}\right)\right) - \frac{\pi}{4}\right)}{\sqrt[4]{\alpha^2 - 1}\sqrt{\lambda}}$$

$$+ O\left(\frac{1}{\lambda}\right)$$

as $\lambda \to +\infty$ (through real numbers), where the implied constant may depend on α.

[Comment: Specializing to $\lambda = n \in \mathbb{Z}^+$ this gives an asymptotic formula for $J_n(\alpha n)$, via (8.9) in the lecture notes. It is satisfactory to note that this asymptotic formula is consistent with Prop. 10.1 in the lecture notes.]

$$(15\text{p})$$

To be returned: Wednesday, November 19, before midnight. Please send your solutions by email, or put them in my mailbox.