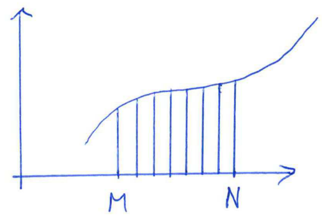
#1. Sums and integrals

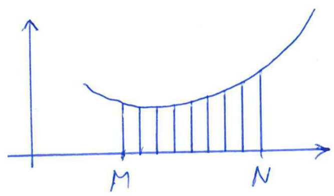
Ex: $M < N$ integers, $f : [M-1, N+1] \to \mathbb{R}$

If f increasing:



$$\int_{M-1}^{N} f(x)\,dx \leqslant \sum_{n=M}^{N} f(n) \leqslant \int_{M}^{N+1} f(x)\,dx$$

If f convex:

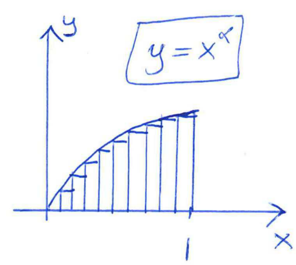


$$\sum_{M}^{N} f(n) \leqslant \int_{M-\frac{1}{2}}^{N+\frac{1}{2}} f(x)\,dx$$

Other (related) connection: Riemann sum

Ex: Given $\alpha > -1$, study the asymptotic behavior of $\sum_{n=1}^{N} n^{\alpha}$ as $N \to \infty$!

solution #1: $$\sum_{1}^{N} n^{\alpha} = N^{\alpha+1} \cdot \underbrace{\sum_{1}^{N} \left(\frac{n}{N}\right)^{\alpha} \cdot \frac{1}{N}}_{\text{Riemann sum for } \int_0^1 x^{\alpha}\,dx}$$



Hence $$\sum_{1}^{N} \left(\frac{n}{N}\right)^{\alpha} \cdot \frac{1}{N} \to \int_0^1 x^{\alpha}\,dx = \frac{1}{\alpha+1} \quad \text{as } N \to \infty$$

and so $$\boxed{\sum_{n=1}^{N} n^{\alpha} \sim \frac{N^{\alpha+1}}{\alpha+1} \quad \text{as } N \to +\infty}$$

Def: "$f(x) \sim g(x)$ as $x \to a$" means $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$.

Solution #2: Say $\alpha \geq 0$; then $f(x) = x^{\alpha}$ is $\nearrow$, thus

$$\int_0^N x^{\alpha}\,dx \leq \sum_{n=1}^{N} n^{\alpha} \leq \int_1^{N+1} x^{\alpha}\,dx$$

$$\Rightarrow \frac{N^{\alpha+1}}{\alpha+1} \leq \sum_{n=1}^{N} n^{\alpha} \leq \frac{(N+1)^{\alpha+1}-1}{\alpha+1}$$

$$\Rightarrow \boxed{\sum_{n=1}^{N} n^{\alpha} = \frac{N^{\alpha+1}}{\alpha+1} + O(N^{\alpha}), \quad \forall N \in \mathbb{Z}^{+}}$$

Def: For $a \geq 0$, "$O(a)$" denotes a number $b \in \mathbb{C}$ which ~~as~~ satisfies $|b| \leq Ca$, where C is a "constant".

Ex: Given $0 < w_1 \leq w_2 \leq \cdots$ satisfying

$$A(T) := \#\{n \in \mathbb{N} : w_n \leq T\} \sim cT^2 \quad \text{as } T \to \infty$$

for some fixed $c > 0$,

(1) For which $\alpha \in \mathbb{R}$ does $\sum_{n=1}^{\infty} w_n^{-\alpha}$ converge?

(2) Then, study the asymptotics of $\sum_{w_n > T} w_n^{-\alpha}$ as $T \to \infty$!

{sum over all n with $w_n > T$.}

Solution #(1): {Positive sum ~~w~~ for order of magnitude, a dyadic decomposition should suffice.}

Assume $\alpha > 0$. (We have obvious divergence if $\alpha \leq 0$.)

$$\sum_{m=1}^{\infty} w_n^{-\alpha} = \sum_{m=0}^{\infty}\left(\sum_{2^m < w_n \leq 2^{m+1}} w_n^{-\alpha}\right) + \sum_{w_n \leq 1} w_n^{-\alpha} \qquad \circledast$$

$$\circledast \geq \sum_{m=0}^{\infty} \underbrace{\#\{2^m < w_n \leq 2^{m+1}\}}_{= A(2^{m+1}) - A(2^m) \sim 3c \cdot 2^{2m}} \cdot 2^{-(m+1)\alpha} \quad \text{and} \quad \circledast \leq \sum_{m=0}^{\infty} \#\{2^m < w_n \leq 2^{m+1}\} \cdot 2^{-m\alpha} + O(1)$$

$\therefore$ $\circledast$ converges iff $\sum_{m=0}^{\infty} 2^{2m - \alpha m} < \infty$, i.e. iff $\boxed{\alpha > 2}$.

Solution #2: "Summation by parts"

$\boxed{\text{Assume } \alpha > 2}$

$$\sum_{w_n > T} w_n^{-\alpha} = \sum_{w_n>T} \int_{w_n}^{\infty} \alpha x^{-\alpha-1}\,dx = \int_T^{\infty} \sum_{T<w_n\leq x} \alpha x^{-\alpha-1}\,dx$$

Change order $\sum \int = \int \sum$. No problems since $\alpha x^{-\alpha-1} > 0$

$$= \int_T^{\infty} \big(A(x)-A(T)\big)\,\alpha x^{-\alpha-1}\,dx = \underbrace{\int_T^{\infty} A(x)\,\alpha x^{-\alpha-1}\,dx}_{\sim \int_T^{\infty} c x^2 \cdot \alpha x^{-\alpha-1}\,dx \,=\, \frac{c\alpha T^{2-\alpha}}{\alpha-2}} - \underbrace{A(T)\,T^{-\alpha}}_{\sim c T^{2-\alpha}}$$

Requires some thought...

$$\boxed{\therefore \sum_{w_n>T} w_n^{-\alpha} \sim \frac{2c}{\alpha-2}\,T^{2-\alpha} \quad \text{as} \quad T \to \infty}$$

Alt. presentation:
$$\sum_{w_n>T} w_n^{-\alpha} = \underbrace{\int_T^{\infty} x^{-\alpha}\,dA(x)}_{\text{Riemann-Stieltes integral}} = \Big[x^{-\alpha}A(x)\Big]_{x=T}^{x=\infty} + \alpha \int_T^{\infty} x^{-\alpha-1} A(x)\,dx = \ldots\ldots$$

Def. (Riemann integral)

Let $A < B$, $g : [A,B] \to \mathbb{C}$.

A sequence $\{x_n\}_{n=0}^{N}$ with $A = x_0 \leq x_1 \leq \cdots \leq x_N = B$ is called a partition of $[A,B]$.

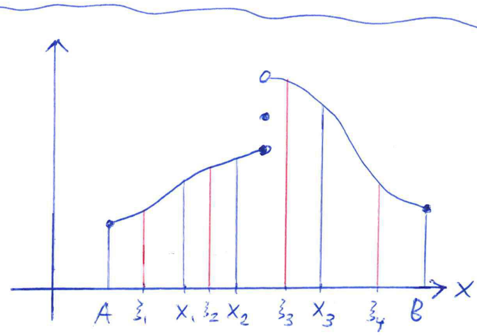
For any partition $\{x_n\}_0^N$ of $[A,B]$ and any points $\xi_n \in [x_{n-1}, x_n]$, $n = 1, \ldots, N$,

set $S(\{x_n\}, \{\xi_n\}) := \sum_{n=1}^{N} g(\xi_n)\cdot(x_n - x_{n-1})$

"Riemann sum"

Call such $\langle \{x_n\}, \{\xi_n\} \rangle$ a tagged partition of $[A,B]$ = t.p.

For g real-valued:



$S(\{x_n\}, \{\xi_n\})$ = Area

We say that the Riemann integral $\int_A^B g(x)\,dx$ exists (and g is Riemann integrable)

if

$$\boxed{\begin{array}{l} \exists\, I \in \mathbb{C}:\ \forall\, \varepsilon > 0:\ \exists\, \delta > 0: \\ \qquad \forall \text{ t.p. } \langle \{x_n\}_0^N, \{\xi_n\}_1^N \rangle \text{ of } [A,B]: \\ \qquad\qquad \operatorname{mesh}(\{x_n\}) \leqslant \delta \ \Rightarrow\ \left| S(\{x_n\},\{\xi_n\}) - I \right| < \varepsilon \end{array}}$$

Def: $= \max_{1 \le n \le N} (x_n - x_{n-1})$

— and then $\int_A^B g(x)\,dx := I$

(Basic facts: g Riemann integrable on $[A,B]$ $\Rightarrow$ g bounded on $[A,B]$.
g continuous on $[A,B]$ $\Rightarrow$ g Riemann integrable on $[A,B]$.)

Def (Riemann-Stieltjes integral)

Let $A < B$, $f, g : [A,B] \to \mathbb{C}$.

For any t.p. $\langle \{x_n\}_0^N, \{\xi_n\}_1^N \rangle$ of $[A,B]$, set $S(\{x_n\},\{\xi_n\}) := \sum_{n=1}^{N} g(\xi_n) \cdot (f(x_n) - f(x_{n-1}))$

only difference!

We say that $\int_A^B g(x)\, df(x) = I \in \mathbb{C}$ if

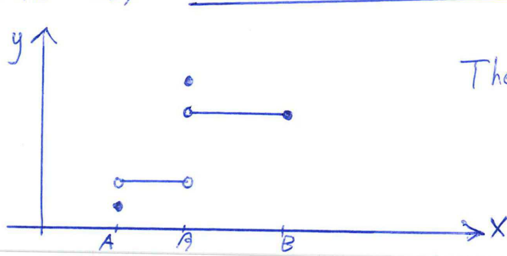
$$\boxed{\begin{array}{l} \forall\, \varepsilon > 0:\ \exists\, \delta > 0:\ \forall \text{ t.p. } \langle \{x_n\}_0^N, \{\xi_n\}_1^N \rangle \text{ of } [A,B]: \\ \qquad \operatorname{mesh}(\{x_n\}) \leqslant \delta \ \Rightarrow\ \left| S(\{x_n\},\{\xi_n\}) - I \right| < \varepsilon \end{array}}$$

Same as before!

See Ex 1.5 in notes

Ex: $g \in C([A,B])$, $f : [A,B] \to \mathbb{C}$ piecewise constant,

say



Then $\int_A^B g\,df = \big(f(A+) - f(A)\big) \cdot g(A)$

$+ \big(f(\beta+) - f(\beta-)\big) \cdot g(\beta)$

$+ \big(f(B) - f(B-)\big) \cdot g(B)$.

Basic properties:

- $\underline{g \in C([A,B]) \text{ and } f \in \overbrace{BV([A,B])}^{\text{will define...}}} \Rightarrow \underline{\int_A^B g\,df \text{ exists}}$ — Thm. 1.10

- For any $f, g : [A,B] \to \mathbb{C}$, if $\int_A^B g\,df$ exists, then $\int_A^B f\,dg$ exists and $\underline{\underline{\int_A^B g\,df = \left[g(x)f(x)\right]_A^B - \int_A^B f\,dg}}$ — Thm 1.12 – easy!

- If $f \in C^1([A,B])$ and $g : [A,B] \to \mathbb{C}$ Riemann integrable, then $\underline{\underline{\int_A^B g\,df = \int_A^B g(x) f'(x)\,dx}}$ (both exists). — Thm 1.13

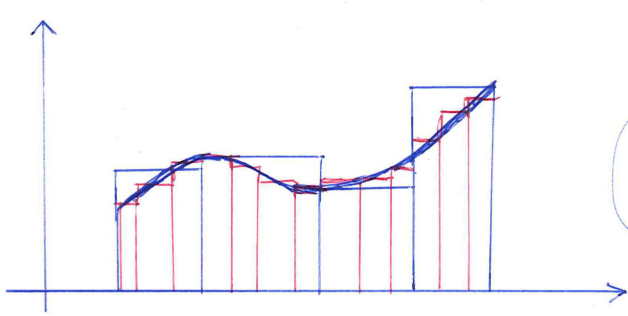
[Note: Now "alt presentation" on p. 5 is justified!
Warning: $\int_A^B \neq \int_A^C + \int_C^B$ in general!]

Proof of existence, for $\int_A^B g(x)\,dx$, assuming $g \in C([A,B])$

① One easily verifies (cf. Lemma 1.11): It suffices to prove the following "Cauchy property":

$$(*)\quad \boxed{\begin{array}{l} \forall \varepsilon > 0 : \exists \delta > 0 : \quad \forall \text{ t.p. } \langle \{x_n\}, \{\xi_n\} \rangle, \langle \{x_n'\}, \{\xi_n'\} \rangle \\ \left[\operatorname{mesh}(\{x_n\}) \leq \delta \text{ and } \operatorname{mesh}(\{x_n'\})\right] \Rightarrow \left| S(\{x\}, \{\xi\}) - S(\{x'\}, \{\xi'\}) \right| < \varepsilon \end{array}}$$

② Any two t.p.'s have a common refinement; hence in $(*)$ we may assume that $\{x_n'\}$ is a refinement of $\{x_n\}$.



– Must compare the areas!

Use g continuous $\Rightarrow$ uniformly continuous on $[A,B]$!

Thus, given $\varepsilon > 0$ we can take $\delta > 0$ s.t.

$$|g(x) - g(x')| < \varepsilon \quad \text{whenever} \quad |x - x'| < \delta$$

Now if $\text{mesh}(\{x_n\}) \leqslant \delta$ and $\{x_n'\}$ is a refinement of $\{x_n\}$:

$$S(\{x_n'\},\{\xi_n'\}) - S(\{x_n\},\{\xi_n\}) = \sum_{n=1}^{N'} g(\xi_n')\cdot(x_n' - x_{n-1}') - \sum_{n=1}^{N} g(\xi_n)\cdot(x_n - x_{n-1})$$

$$= \sum_{n=1}^{N} \sum_{\substack{j \\ [x_{j-1}',x_j'] \subset [x_{n-1},x_n]}} \left(g(\xi_j') - g(\xi_n)\right)\cdot(x_j' - x_{j-1}')$$

$$\text{Abs. value} \leqslant \sum_{n=1}^{N} \sum_{\substack{j \\ [\;] \subset [\;]}} \varepsilon\cdot(x_j' - x_{j-1}') = \varepsilon(B-A)$$

Done! Replace $\varepsilon \rightsquigarrow \varepsilon(B-A)$

$\square$

Same approach works in much more general situations!

Proof of existence of $\int_A^B g(x)\,df(x)$, for $g \in C([A,B])$, $f \in BV([A,B])$

Def: For $f : [A,B] \to \mathbb{C}$, $\quad \text{Var}_{[A,B]}(f) = \sup_{\{x_n\}_0^N} \sum_{n=1}^{N} |f(x_n) - f(x_{n-1})|$

any partition of $[A,B]$

We say $f \in BV([A,B])$ if $\text{Var}_{[A,B]}(f) < \infty$

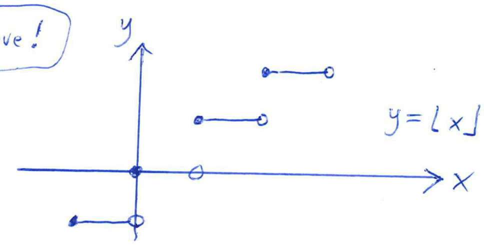
Now (essentially) the SAME existence proof (p. 10–11) works!

Ex: The Euler-Maclaurin summation formula

For $f:[A,B]\to\mathbb{C}$ "slowly varying", estimate $\displaystyle\sum_{\substack{A<n\le B\\(n\in\mathbb{Z})}} f(n)$! ?

{ We will get better estimates than on p. 1-3 above! }

Let $\lfloor x\rfloor$ = "floor of x" $= \max\left(\mathbb{Z}\cap(-\infty,x]\right)$



$$\sum_{A<n\le B} f(n) = \int_A^B f(x)\,d\lfloor x\rfloor \overset{\text{If } f\in C^1([A,B])}{=} \Big[f(x)\cdot\lfloor x\rfloor\Big]_{x=A}^{x=B} - \int_A^B f'(x)\lfloor x\rfloor\,dx$$

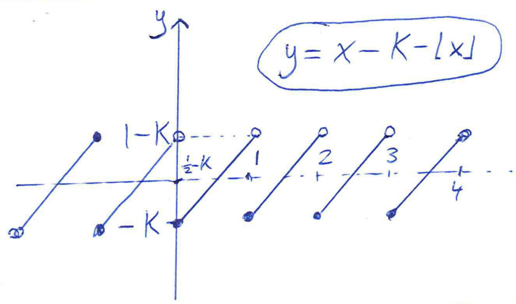
Also, for any $K\in\mathbb{C}$:

$$\int_A^B f(x)\,dx = \Big[f(x)\cdot(x-K)\Big]_{x=A}^{x=B} - \int_A^B f'(x)\,(x-K)\,dx$$

Combine $\Rightarrow$ $$\sum_{A<n\le B} f(n) = \int_A^B f(x)\,dx - \Big[f(x)\cdot(x-K-\lfloor x\rfloor)\Big]_A^B + \int_A^B f'(x)\cdot(x-K-\lfloor x\rfloor)\,dx$$

Remains to understand: $\displaystyle\int_A^B f'(x)\cdot(x-K-\lfloor x\rfloor)\,dx$

Here: $y = x-K-\lfloor x\rfloor$ — oscillates around $y=\frac{1}{2}-K$



{ **Principle**: An integral of [slowly varying] × [oscillating around 0] is small, and this can be proved by integrating by parts. }

Thus we choose $K=\frac{1}{2}$ in ⊛, then integrate by parts! Repeat!!

$$\int_A^B f'(x)\cdot(\underbrace{x-\lfloor x\rfloor}_{\tilde{x}:=x-\lfloor x\rfloor}-\tfrac{1}{2})\,dx \curvearrowright -\int_A^B f''(x)\cdot\underbrace{\left(\frac{\tilde{x}^2}{2}-\frac{\tilde{x}}{2}+\frac{1}{12}\right)}\,dx \curvearrowright \int_A^B f'''(x)\cdot\underbrace{\left(\frac{\tilde{x}^3}{6}-\frac{\tilde{x}^2}{4}+\frac{\tilde{x}}{12}\right)}\,dx \curvearrowright \cdots$$

Bernoulli polynomials!

Def: The Bernoulli polynomials $B_0(x), B_1(x), B_2(x), \ldots$ are defined by

$$\underline{B_0(x) \equiv 1} \quad \text{and} \quad \underline{\left[B_r'(x) = r \cdot B_{r-1}(x) \text{ and } \int_0^1 B_r(x)\,dx = 0\right]} \quad (r = 1, 2, 3, \ldots)$$

The r:th Bernoulli number is $\underline{B_r := B_r(0)}$.

The Euler-Maclaurin summation formula (Thm 1.19)

For $A < B$, $h \in \mathbb{N}$, $f \in C^h([A,B])$ we have

$$\sum_{A<n\leq B} f(n) = \int_A^B f(x)\,dx + \sum_{r=1}^{h} \frac{(-1)^r}{r!}\left[\widetilde{B}_r(x) \cdot f^{(r-1)}(x)\right]_{x=A}^{x=B} + (-1)^{h+1}\int_A^B \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x)\,dx$$

where $\widetilde{B}_r(x) := B_r(x - \lfloor x \rfloor)$

$B_0(x) = 1$

$B_1(x) = x - \frac{1}{2}$

$B_2(x) = x^2 - x + \frac{1}{6}$

$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$

$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$

Ex, again: Asymptotic behavior of $\sum_{n=1}^{N} n^\alpha$ as $N \to \infty$, for $\alpha > -1$ fixed?

(Can now do arbitrary $\alpha \in \mathbb{C}$!)

Apply Euler-Maclaurin with ~~[illegible]~~ $A = $ "$1-$" and $B = N$, $f(x) = x^\alpha$, $h \in \mathbb{N}$:

$$\sum_{n=1}^{N} n^\alpha = \int_1^N x^\alpha\,dx + \sum_{r=1}^{h} \frac{(-1)^r}{r!} B_r \left(f^{(r-1)}(N) - f^{(r-1)}(1)\right) + 1 + (-1)^{h+1}\int_1^N \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x)\,dx$$

For $r \geq 2$: $\widetilde{B}_r(N) = \widetilde{B}_r(1-) = B_r$.
For $r = 1$: $\widetilde{B}_1(1-) = \frac{1}{2}$ while $B_1 = -\frac{1}{2}$

Assuming $\alpha \neq -1$

$f^{(h)}(x) = \alpha(\alpha-1)\cdots(\alpha-h+1)\,x^{\alpha-h}$

$$= \frac{N^{\alpha+1} - 1}{\alpha+1} + \sum_{r=1}^{h} (-1)^r B_r \cdot \frac{1}{\alpha+1}\binom{\alpha+1}{r} \cdot \left(N^{\alpha-r+1} - 1\right) + 1 + (-1)^{h+1}\binom{\alpha}{h} \cdot \int_1^N \widetilde{B}_h(x)\, x^{\alpha-h}\,dx$$

$$\text{Error term} = \int_1^N O\left(x^{\operatorname{Re}\alpha - h}\right)dx = \begin{cases} O\left(N^{\operatorname{Re}\alpha - h+1}\right) & \operatorname{Re}\alpha > h-1 \\ O(\log N) & \operatorname{Re}\alpha = h-1 \\ O(1) & \operatorname{Re}\alpha < h-1 \end{cases}$$

Ex For $\alpha = \frac{3}{2}$, $h=3$: $\sum_{n=1}^{N} n^{\frac{3}{2}} = \frac{2}{5} N^{\frac{5}{2}} + \frac{1}{2} N^{\frac{3}{2}} + \frac{1}{8} N^{\frac{1}{2}} + \underbrace{O(1)}_{\text{Seems to } \to -0.0254\ldots}, \quad \forall N \geq 1$

Asymptotic expansion!

Can improve "$O(1)$"?

- Write $\int_1^N = \int_1^\infty - \int_N^\infty$, then continue integrating by parts!

Get
$$\boxed{\sum_{n=1}^{N} n^\alpha = \frac{1}{\alpha+1} \sum_{r=0}^{k} (-1)^r B_r \binom{\alpha+1}{r} N^{\alpha-r+1} + C(\alpha) + \underbrace{(-1)^k \binom{\alpha}{k} \cdot \int_N^\infty \widetilde{B}_k(x) \cdot x^{\alpha-k} dx}_{\text{Error} = O(N^{\operatorname{Re}\alpha - k+1})}}$$

for any $\alpha \in \mathbb{C}$, $\alpha \neq -1$, $k, h \in \mathbb{N}$, $k \geq h > \operatorname{Re} \alpha + 1$

Here
$$C(\alpha) = 1 - \frac{1}{\alpha+1} \sum_{r=0}^{h} (-1)^r B_r \binom{\alpha+1}{r} + (-1)^{h+1} \binom{\alpha}{h} \int_1^\infty \widetilde{B}_h(x) \cdot x^{\alpha-h} dx$$

Actually $C(\alpha) = $ "$\sum_{n=1}^{\infty} n^\alpha$" in a sense!