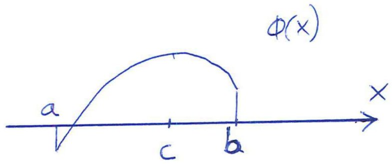
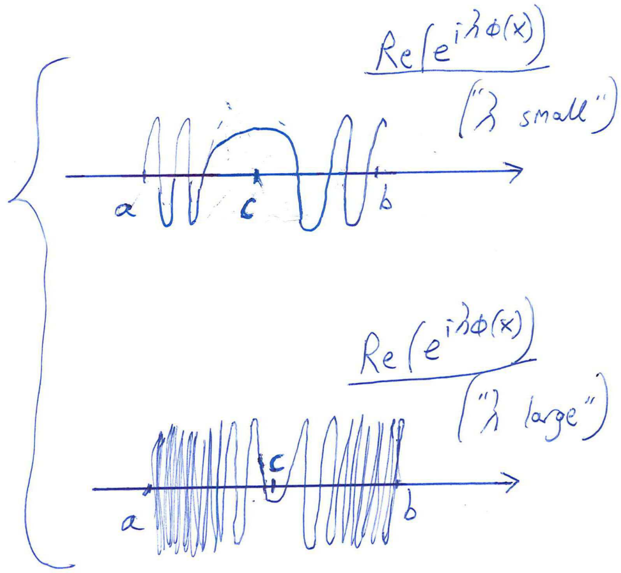
#10. Stationary Phase (see Stein, Ch. 8.1)

Behavior of $\int_a^b e^{i\lambda\phi(x)} \cdot \psi(x)\,dx$ as $\lambda \to \infty$?

$\phi, \psi \in C^\infty([a,b])$
ϕ real-valued

Ex:





Except big cancelations except at $x = a, b$ and c.

$\{x : \phi'(x) = 0\}$

"Localization"

Prop 1: Assume $\phi'(x) \neq 0$ for all $x \in [a,b]$, and that $supp(\psi)$ is a compact subset of (a,b).

Then $\int_a^b e^{i\lambda\phi(x)} \cdot \psi(x)\,dx = O(\lambda^{-N})$ as $\lambda \to +\infty$, for any fixed $N \geq 0$.

proof: Integration by parts!

Let D be the differential operator $(Df)(x) := \frac{1}{i\lambda\phi'(x)} \cdot f'(x)$.

Note $D(e^{i\lambda\phi(x)}) = e^{i\lambda\phi(x)}$!

Hence $\int_a^b e^{i\lambda\phi(x)} \cdot \psi(x)\,dx = \int_a^b D(e^{i\lambda\phi(x)}) \cdot \psi(x)\,dx$

$$= \int_a^b \left(\frac{d}{dx}\left(e^{i\lambda\phi(x)}\right)\right) \cdot \frac{1}{i\lambda\phi'(x)} \cdot \psi(x)\,dx$$

$$= \int_a^b e^{i\lambda\phi(x)} \cdot \underbrace{\left(-\frac{d}{dx}\left(\frac{1}{i\lambda\phi'(x)}\psi(x)\right)\right)}_{\text{Call } ({}^tD\psi)(x)} dx$$

Repeat N times $\Rightarrow$ $\int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx = \int_a^b e^{i\lambda\phi(x)} \cdot \underbrace{\left(({}^tD)^N\psi\right)(x)}_{\lambda^{-N}\cdot[\text{function indep. of } \lambda]}\,dx$

$= O(\lambda^{-N})$

"Scaling"

Prop 2: Assume $|\phi^{(k)}(x)| \geq 1$ for all $x \in [a,b]$, for some fixed $k \in \mathbb{Z}^+$. If $k=1$, assume also that $\phi'(x)$ is monotonic.

Then $$\left|\int_a^b e^{i\lambda\phi(x)}\,dx\right| \ll_k \lambda^{-1/k} \qquad (\forall \lambda > 0)$$

The implied constant is independent of ϕ and λ and a, b!

proof (outline): Proof by induction over k!

Start: Assume $k=1$. (Thus: $|\phi'(x)| \geq 1,\ \forall x \in (a,b)$, and $\phi'(x)$ is monotonic.)

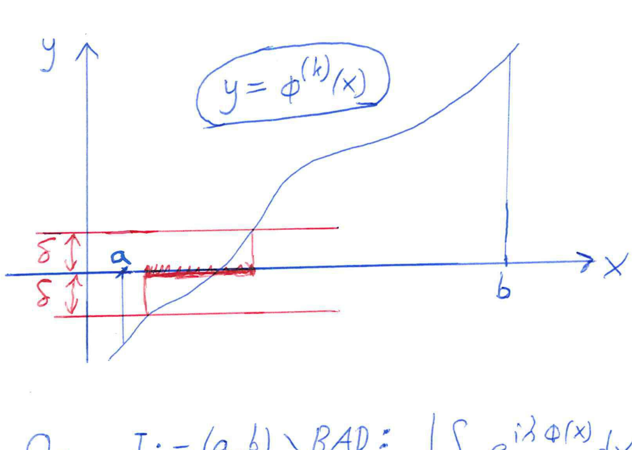
As before:
$$\int_a^b e^{i\lambda\phi(x)}\,dx = \int_a^b \left(\frac{d}{dx} e^{i\lambda\phi(x)}\right)\cdot \frac{1}{i\lambda\phi'(x)}\,dx$$
$$= \underbrace{\left[e^{i\lambda\phi(x)}\cdot\frac{1}{i\lambda\phi'(x)}\right]_{x=a}^{x=b}}_{\text{abs. value} \leq \frac{2}{\lambda}} - \underbrace{\frac{1}{i\lambda}\int_a^b e^{i\lambda\phi(x)}\cdot\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)dx}_{\circledast}$$

$$|\circledast| \leq \frac{1}{\lambda}\int_a^b \left|\frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)\right|dx = \frac{1}{\lambda}\left|\int_a^b \frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)dx\right| = \frac{1}{\lambda}\left|\left[\frac{1}{\phi'(x)}\right]_{x=a}^{x=b}\right| \leq \frac{1}{\lambda}.$$

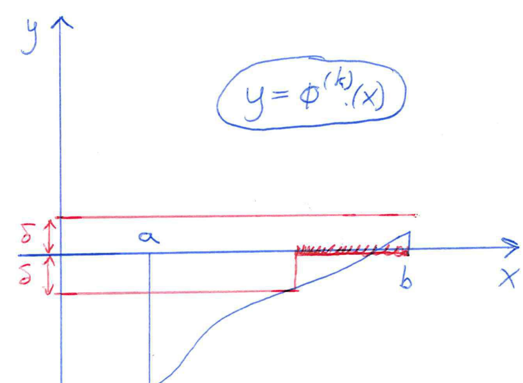
Done!

Induction step: Assume the statement holds "for k".

Also assume $\phi^{(k+1)}(x) \geq 1,\ \forall x \in (a,b)$. For any $\delta > 0$:



OR



On $J := (a,b)\setminus BAD$: $\left|\int_J e^{i\lambda\phi(x)}dx\right| = \left|\int_J e^{i\lambda\delta\cdot\delta^{-1}\phi(x)}dx\right| \ll (\lambda\delta)^{-\frac{1}{k}}$.

On BAD: $\left|\int_{BAD} e^{i\lambda\phi(x)}dx\right| \leq \text{length}(BAD) \leq 2\delta$

Total: $\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \ll (\lambda\delta)^{-\frac{1}{k}} + \delta$. Choose $\delta = \lambda^{-\frac{1}{1+k}}$. $\Rightarrow$ $\boxed{\left|\int_a^b\right| \ll \lambda^{-\frac{1}{k+1}}}$

"Asymptotics"

Prop 3: Assume $k \geq 2$ and

$$\phi(x_0) = \phi'(x_0) = \dots = \phi^{(k-1)}(x_0) = 0; \qquad \phi^{(k)}(x_0) \neq 0.$$

Then if $\operatorname{supp}(\psi)$ is contained in a sufficiently small neighborhood of x_0 (which is contained in (a,b)), then $\exists\, a_0, a_1, a_2, \dots \in \mathbb{C}$ such that

$$\int_a^b e^{i\lambda\phi(x)} \cdot \psi(x)\, dx \quad \text{"}\sim \lambda^{-\frac{1}{k}} \sum_{j=0}^{\infty} a_j \lambda^{-j/k}\text{"}.$$

Proof (outline): Assume $k=2$. Assume $\phi(x) \equiv x^2$ (thus $x_0 = 0$).

Note: $\int_a^b e^{i\lambda\phi(x)} \cdot \psi(x)\, dx = \int_{-\infty}^{\infty} e^{i\lambda x^2} \cdot \psi(x)\, dx \qquad \left(\psi \in C_c^\infty(\mathbb{R})\right)$

Taylor expand $\psi(x)$ at $x=0$. $\rightsquigarrow$ Handle "$\int_{-\infty}^{\infty} e^{i\lambda x^2} \cdot x^\ell\, dx$" $(\ell \geq 0)$.

Instead, write $\int_{-\infty}^{\infty} e^{i\lambda x^2} \cdot \psi(x)\, dx = \int_{-\infty}^{\infty} e^{i\lambda x^2} e^{-x^2} \cdot \left(e^{x^2} \psi(x)\right) dx$,

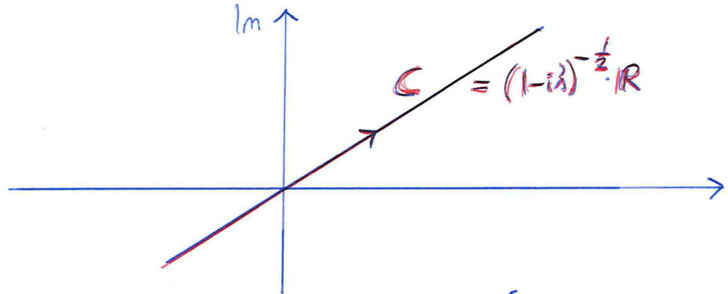
and Taylor expand $e^{x^2}\psi(x)$!

Say $e^{x^2}\psi(x) = \sum_{\ell=0}^{N} b_\ell x^\ell + x^{N+1} R_N(x) \qquad \left(b_0, \dots, b_N \in \mathbb{C},\ R_N \in C^\infty(\mathbb{R})\right)$

Contribution from x^ℓ

$$\int_{-\infty}^{\infty} e^{i\lambda x^2} e^{-x^2} \cdot x^\ell \, dx = \int_{-\infty}^{\infty} e^{-(1-i\lambda)x^2} \cdot x^\ell \, dx =$$

$$x = (1-i\lambda)^{-\frac{1}{2}} z \qquad = (1-i\lambda)^{-\frac{1}{2}(\ell+1)} \cdot \int_C e^{-z^2} z^\ell \, dz$$



Change contour!

$$= (1-i\lambda)^{-\frac{1}{2}(\ell+1)} \int_{-\infty}^{\infty} e^{-z^2} z^\ell \, dz = \begin{cases} 0 & \text{if } \ell \text{ odd} \\ \Gamma\left(\frac{\ell+1}{2}\right) (-i\lambda)^{-\frac{1}{2}(\ell+1)} \left(1+i\lambda^{-1}\right)^{-\frac{1}{2}(\ell+1)} & \text{if } \ell \text{ even} \end{cases}$$

for ℓ even

$$= \Gamma\left(\frac{\ell+1}{2}\right) \cdot (-i\lambda)^{-\frac{1}{2}(\ell+1)} \cdot \sum_{m=0}^{\infty} \binom{-\frac{1}{2}(\ell+1)}{m} i^m \lambda^{-m} \qquad (\text{when } \lambda > 1)$$

Contribution from $x^{N+1} R_N(x)$

$$\int_{-\infty}^{\infty} e^{i\lambda x^2} x^{N+1} \eta(x) \, dx, \qquad \text{where } \eta(x) = e^{-x^2} R_N(x)$$

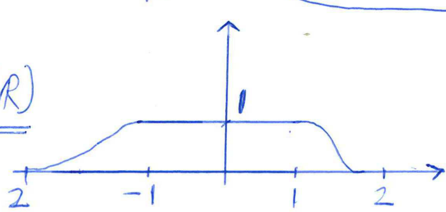
Note $\eta \in \mathcal{S}$, i.e. η and all its derivatives decay superpolynomially.

As in Prop 1 & 2; use $e^{i\lambda x^2} = \left(\frac{d}{dx} e^{i\lambda x^2}\right) \cdot \frac{1}{2i\lambda x}$ and integration by parts!

Problem at $x = 0$!

Split the integral into $\int_{-\varepsilon}^{\varepsilon}$ and "the rest" – but do it smoothly!

Fix $\alpha \in C_c^\infty(\mathbb{R})$



For $\varepsilon > 0$, write (with $l := N+1$)

$$\int_{-\infty}^{\infty} e^{i\lambda x^2} x^l \eta(x)\,dx = \underbrace{\int_{-\infty}^{\infty} e^{i\lambda x} x^l \eta(x)\,\alpha\!\left(\frac{x}{\varepsilon}\right)dx}_{\ll\, \varepsilon^{l+1}} + \underbrace{\int_{-\infty}^{\infty} e^{i\lambda x} x^l \eta(x)\left(1-\alpha\!\left(\frac{x}{\varepsilon}\right)\right)dx}_{\circledast}$$

Integrating by parts M times (using $\eta \in \mathcal{S}$)

$$\Rightarrow \circledast = \int_{-\infty}^{\infty} e^{i\lambda x^2} ({}^tD^M)\left\{x^l \eta(x)\left(1-\alpha\!\left(\frac{x}{\varepsilon}\right)\right)\right\}dx \ll \sum_{\substack{j+k\leq M \\ (j,k\geq 0)}} \lambda^{-M} \int_{|x|\geq\varepsilon} |x|^{l-M-k}\varepsilon^{-j}\,dx$$

$$({}^tD f)(x) := -\frac{d}{dx}\left(\frac{f(x)}{2i\lambda x}\right)$$

Take $M \geq l+2$

$$\ll \sum_{\substack{j+k\leq M \\ (j,k\geq 0)}} \lambda^{-M} \varepsilon^{l-M-k+1-j} \ll \lambda^{-M}\varepsilon^{l-2M+1}.$$

Choose $\varepsilon = \lambda^{-\frac{1}{2}}$ $\Rightarrow$ $\boxed{\text{TOTAL} \ll \lambda^{-\frac{l+1}{2}}}$ $\square$

Next: $k = 2$ but general ϕ

Then write $\phi(x) = c(x-x_0)^2(1+\varepsilon(x))$ ← $\varepsilon \in C^\infty([a,b])$ and $|\varepsilon(x)| \ll |x-x_0|$

Substitute $y = (x-x_0)\cdot\sqrt{1+\varepsilon(x)}$; then $x \mapsto y$ is a C^∞ diffeo

$[\text{nbhd of } x_0] \xrightarrow{\sim} [\text{nbhd of } 0]$.

Assume Ψ has compact support contained in this neighborhood!

$\Rightarrow$ Get $\int e^{i\lambda c y^2}\cdot\tilde{\Psi}(y)\,dy$! Etc...

$\square\square\square$