

#11. Stationary phase; examples (Lecture notes, Sec. 9)

① Asymptotics of $J_m(r)$ as $r \rightarrow +\infty$ for fixed m

Recall
$$J_m(r) = \frac{\left(\frac{r}{2}\right)^m}{\Gamma(m+\frac{1}{2}) \Gamma(\frac{1}{2})} \cdot \int_{-1}^1 e^{irx} (1-x^2)^{m-\frac{1}{2}} dx.$$

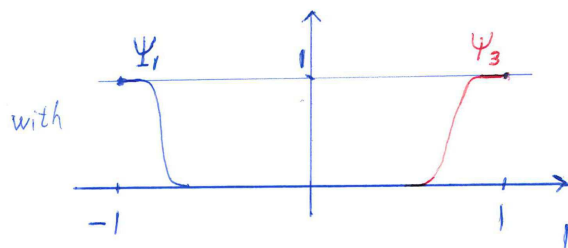
$\forall r \in \mathbb{R}^+$
 $m \in \mathbb{C}$ with
 $\operatorname{Re}(m) > -\frac{1}{2}$

Hence wish to understand
$$\int_{-1}^1 e^{irx} (1-x^2)^{m-\frac{1}{2}} dx \text{ as } r \rightarrow +\infty.$$

This is of the form we studied in #10,

with $\phi(x) = x$, $\psi(x) = (1-x^2)^{m-\frac{1}{2}}$, $\lambda = r$. No critical points!

Split as
$$\sum_{j=1}^3 \int_{-1}^1 e^{irx} (1-x^2)^{m-\frac{1}{2}} \psi_j(x) dx$$



Contribution from ψ_2 : Since $x \mapsto (1-x^2)^{m-\frac{1}{2}} \psi_2(x)$ is C^∞ and has compact support contained in $(-1, 1)$

Prop 1 (in last lecture) gives

$$\int_{-1}^1 e^{irx} (1-x^2)^{m-\frac{1}{2}} \psi_2(x) dx = O(r^{-N}) \text{ as } r \rightarrow +\infty,$$

for any fixed $N \geq 0$.

Contribution from Ψ_1 :

$$\int_{-1}^1 e^{irx} (1-x^2)^{m-\frac{1}{2}} \Psi_1(x) dx = \boxed{x = x_{\text{new}} - 1}$$

$$= e^{-ir} \int_0^2 e^{irx} x^{m-\frac{1}{2}} (2-x)^{m-\frac{1}{2}} \Psi_1(x-1) dx$$

$:= \Psi(x)$

May view $\Psi \in C_c^\infty([0, \infty))$

By Lemma 9.1 in lecture notes, this is

$$= e^{-ir} \sum_{j=0}^{N-1} c_j r^{-j-m-\frac{1}{2}} + O(r^{-N-\frac{1}{2}-\text{Re}(m)}) \quad \text{as } r \rightarrow +\infty.$$

with $c_j = i^{j+m+\frac{1}{2}} \cdot \frac{\Gamma(j+m+\frac{1}{2})}{j!} \cdot \Psi^{(j)}(0)$

$$= \frac{d^j}{dx^j} \left((2-x)^{m-\frac{1}{2}} \right) \Big|_{x=0}$$

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Contribution from Ψ_3 : Symmetric! Choose $\Psi_3(x) = \Psi_1(-x)$;

then $\int_{-1}^1 e^{irx} (1-x^2)^{m-\frac{1}{2}} \Psi_3(x) dx = \int_{-1}^1 e^{irx} (1-x^2)^{m-\frac{1}{2}} \Psi_1(x) dx$. Etc!

...

Final result (cf lecture #9):

$$J_m(r) = \sqrt{\frac{2}{\pi r}} \left(\cos\left(r - \frac{m\pi}{2} - \frac{\pi}{4}\right) \cdot \sum_{0 \leq k < \frac{N}{2}} (-1)^k A_{m,2k} \cdot r^{-2k} \right.$$

$$\left. - \sin\left(r - \frac{m\pi}{2} - \frac{\pi}{4}\right) \cdot \sum_{0 \leq k < \frac{N-1}{2}} (-1)^k A_{m,2k+1} \cdot r^{-2k-1} + O(r^{-N}) \right)$$

where $A_{m,n} = \frac{\prod_{j=-n}^{n-1} (m+\frac{1}{2}+j)}{2^n \cdot n!}$.

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② Special case of ①, via other formula.

Using now $J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{irs \sin x - imx} dx.$

$m \in \mathbb{Z}$

(8.9) in course notes

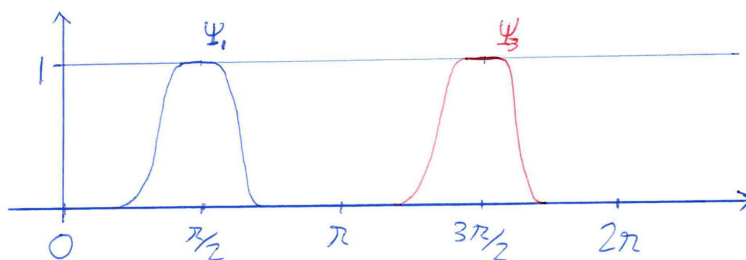
This is of the form studied in #10,

with $\phi(x) = \sin x$; critical points at $\frac{\pi}{2}, \frac{3\pi}{2},$

and $\psi(x) = e^{imx}$ and $\lambda = r \rightarrow +\infty.$

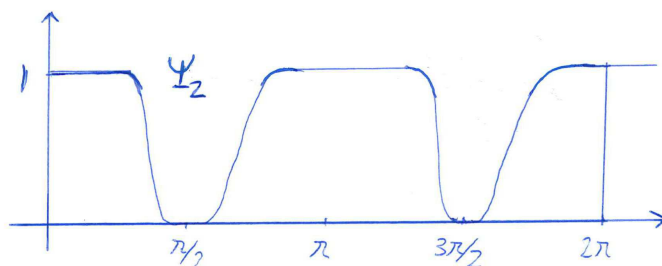
Split as $\sum_{j=1}^3 \frac{1}{2\pi} \int_0^{2\pi} e^{irs \sin x} \cdot e^{-imx} \cdot \Psi_j(x) dx$

with



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Contribution from Ψ_2



May view as $\frac{1}{2\pi} \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{irs \sin x} \cdot e^{-imx} \cdot \Psi_2(x) dx$ $\otimes$

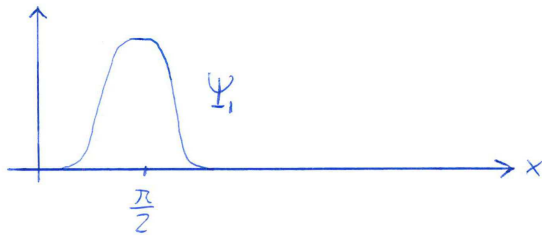
circle!

Hence Prop. 1 (in last lecture) applies, with no endpoint contributions,

$\therefore \otimes = O(r^{-N})$ as $r \rightarrow +\infty$, for any fixed $N \geq 0$.

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Contribution from Ψ_1



$\phi(x) = \sin x$, with critical point at $x_0 = \frac{\pi}{2}$.

Stein's Prop. 3 applies directly to give

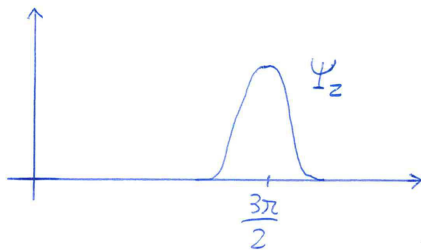
$$\frac{1}{2\pi} \int_0^{2\pi} e^{ir \sin x} e^{-imx} \underbrace{\Psi_1(x)}_{:= \psi(x)} dx = \frac{e^{ir}}{2\pi} \int_{-\infty}^{\infty} e^{ir(\sin x - 1)} \psi(x) dx$$

$$\sim \frac{e^{ir}}{2\pi} \cdot r^{-\frac{1}{2}} \cdot \sum_{j=0}^{\infty} a_j \cdot r^{-j/2}$$

Stein's § 1.3.4 $\Rightarrow \underline{a_0 = \left(\frac{2\pi}{-i \phi''(x_0)} \right)^{\frac{1}{2}} \psi(x_0) = \sqrt{-2\pi i} \cdot (-i)^m}$

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Contribution from Ψ_2



Take $\Psi_2(x) \equiv \Psi_1(x - \pi)$ and use symmetry!

$$\Rightarrow \frac{1}{2\pi} \int_0^{2\pi} e^{ir \sin x} e^{-imx} \Psi_2(x) dx \sim \frac{e^{-ir}}{2\pi} r^{-\frac{1}{2}} \sum_{j=0}^{\infty} a'_j \cdot r^{-j/2}$$

with $a'_0 = \sqrt{2\pi i} \cdot i^m$

Conclusion: $J_m(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi m}{2} - \frac{\pi}{4}\right) + O(r^{-1})$ as $r \rightarrow +\infty$

(and also get asymptotic expansion!)