

#12. Distribution Theory

DEF: Let U be an open subset of $\mathbb{R}^n$.

A distribution on U is a continuous linear functional on $C_c^\infty(U)$.

We set $\mathcal{D}'(U)$ = the space of all distributions on U .

Also: $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^n)$

Topology on $C_c^\infty(U)$?

Remark: Any $f \in L_{loc}^1(U)$ is identified with the distribution
 $\phi \mapsto \int f \phi dx$ ($\phi \in C_c^\infty(U)$). Hence a distribution is "a generalized function"!

Definition of $C_c^\infty(U)$?

DEF: For any Borel set $E \subset \mathbb{R}^n$:

$$\underline{C_c^\infty(E)} = \{f \in C_c^\infty(\mathbb{R}^n) : \text{supp}(f) \text{ is compact and } \subseteq E\}$$

DEF: For K compact $\subset \mathbb{R}^n$, we give $C_c^\infty(K)$ the topology generated by the seminorms $\phi \mapsto \|2^\alpha \phi\|_\alpha$ ($\alpha \in (\mathbb{Z}_{\geq 0})^n$).

This makes $C_c^\infty(K)$ into a Fréchet space.

Thm 5.14: Let $\{p_\alpha\}$ be a family of seminorms on a $\mathbb{C}$ -vector space X .

For $x \in X$, $\alpha \in A$, $\varepsilon > 0$, set $B_{x,\alpha,\varepsilon} = \{y \in X : p_\alpha(y-x) < \varepsilon\}$,

and let τ be the topology generated by $\{B_{x,\alpha,\varepsilon}\}$.

Then (X, τ) is a topological vector space. Also for any net $\langle x_i \rangle_{i \in I}$ in X , we have $x_i \rightarrow x$ iff $[p_\alpha(x_i - x) \rightarrow 0 \text{ for all } \alpha \in A]$.

Prop. 5.15: Assume that $\langle X, \{p_\alpha\}_{\alpha \in A} \rangle$ and $\langle Y, \{q_\beta\}_{\beta \in B} \rangle$ are t.v.s as above, and let $T: X \rightarrow Y$ be a linear map.

Then T is continuous i.f.f.

$$\forall A \in B: \exists \alpha_1, \dots, \alpha_k \in A, \exists C > 0 \text{ s.t. } \forall x \in X: q_A(Tx) \leq C \sum_{j=1}^k p_{\alpha_j}(x).$$

Prop. 5.16: Let $\langle X, \{p_\alpha\}_{\alpha \in A} \rangle$ be a t.v.s. as above.

a) X is Hausdorff i.f.f. $\forall x \neq 0: \exists \alpha \in A: p_\alpha(x) \neq 0$

b) If X is Hausdorff and A countable, then X is metrizable with a translation-invariant metric.

DEF: A t.v.s. as in 5.16(b) which is complete is called a Fréchet space.

3

Topology on $C_c^\infty(U)$ (for U open $\subset \mathbb{R}^n$)

There is a certain natural topology on $C_c^\infty(U)$ (locally convex, but not metrizable) such that

- a sequence $\phi_1, \phi_2, \dots$ in $C_c^\infty(U)$ converges to ϕ i.f.f. there is a compact set $K \subset U$ such that $\phi_1, \phi_2, \dots \in C_c^\infty(K)$ and $\phi_j \rightarrow \phi$ in $C_c^\infty(K)$.

- a linear map $F: C_c^\infty(U) \rightarrow \mathbb{C}$ is continuous i.f.f.

$F|_{C_c^\infty(K)}$ is continuous for each compact set $K \subset U$.

Hence: A linear functional $F: C_c^\infty(U) \rightarrow \mathbb{C}$ is a distribution on U

($\Leftrightarrow$ is continuous) i.f.f.: $\forall$ compact $K \subset U: \exists C > 0, k \in \mathbb{Z}_{\geq 0}$:

$$\forall \phi \in C_c^\infty(K): |F(\phi)| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha \phi\|_U$$

4

Ex: For any $f \in L^1_{loc}(U)$:

$$\forall K \text{ compact } \subset U, \phi \in C_c^\infty(K): \quad \left| \int_U f \phi dx \right| = \left| \int_K f \phi dx \right| \leq \left(\int_K |f| \right) \cdot \|\phi\|_u$$

Hence, indeed, $L^1_{loc}(U) \subset \mathcal{D}'(U)$.

Ex: For any $\mu \in M(\mathcal{B}_U)$:

The functional $\phi \mapsto \int_U \phi d\mu$ is a distribution on U .

$$(\text{Indeed: } \forall \phi \in C_c^\infty(U): \left| \int_U \phi d\mu \right| \leq \|\mu\| \cdot \|\phi\|_u)$$

Hence $M(\mathcal{B}_U) \subset \mathcal{D}'(U)$.

Ex: For any $x_0 \in U$ and $\alpha \in (\mathbb{Z}_{\geq 0})^n$:

The functional $\phi \mapsto \partial^\alpha \phi(x_0)$ is a distribution on U .

5

DEF: If $V \subset U$ are open subsets of $\mathbb{R}^n$, and $F \in \mathcal{D}'(U)$, then we define $F|_V \in \mathcal{D}'(V)$ through $\langle F|_V, \phi \rangle = \langle F, \phi \rangle, \forall \phi \in C_c^\infty(V)$.

Prop. 9.2: Let $\{V_\eta\}$ be a family of open subsets of U , and set $V = \bigcup_\eta V_\eta$. If $F, G \in \mathcal{D}'(U)$ and $F|_{V_\eta} = G|_{V_\eta}, \forall \eta$, then $F|_V = G|_V$.

proof: Assume $\phi \in C_c^\infty(V)$.

Can then find $\eta_1, \eta_2, \dots, \eta_m$ s.t. $\text{supp}(\phi) \subset \bigcup_{j=1}^m V_{\eta_j}$.

Now pick $\phi_1, \phi_2, \dots, \phi_m \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp}(\phi_j) \subset V_{\eta_j}$ and $\sum_{j=1}^m \phi_j = \phi$.

(This uses a partition of unity.)

$$\text{Now } \langle F, \phi \rangle = \sum_{j=1}^m \langle F, \phi_j \rangle = \sum_{j=1}^m \langle G, \phi_j \rangle = \langle G, \phi \rangle.$$

6

DEF: For $F \in \mathcal{D}'(U)$, $\text{supp}(F) := U \setminus \left[\begin{array}{l} \text{The largest open set } V \subset U \\ \text{with } F|_V = 0 \end{array} \right]$

Equivalently, $\text{supp}(F) = \left\{ x \in U : x \text{ has no open neighbourhood } V \subset U \text{ with } F|_V = 0 \right\}$

7

Operations on distributions (start)

Differentiation

Note that if $f \in C^1(U)$ and $\phi \in C_c^\infty(U)$ then

$$\int_U (\partial_j f) \cdot \phi \, dx = \underbrace{[f \cdot \phi]}_{=0} - \int_U f \cdot (\partial_j \phi) \, dx$$

$\leadsto$ DEF: For $F \in \mathcal{D}'(U)$, $\partial_j F \in \mathcal{D}'(U)$ is defined by $\langle \partial_j F, \phi \rangle := -\langle F, \partial_j \phi \rangle$, $\forall \phi \in C_c^\infty(U)$.

Is really $\partial_j F \in \mathcal{D}'(U)$? Yes; prove by showing $\phi \mapsto \partial_j \phi$ is a continuous map $C_c^\infty(U) \rightarrow C_c^\infty(U)$. For this, it suffices to note that $\phi \mapsto \partial_j \phi$ is continuous $C_c^\infty(K) \rightarrow C_c^\infty(K)$, for every compact $K \subset U$.

For general multi-index α : $\langle \partial^\alpha F, \phi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \phi \rangle$

8

Multiplication by smooth functions

DEF: For $F \in \mathcal{D}'(U)$ and $\psi \in C^\infty(U)$:

$\psi F \in \mathcal{D}'(U)$ is defined by $\langle \psi F, \phi \rangle = \langle F, \psi \phi \rangle$, $\forall \phi \in C_c^\infty(U)$.

If $\psi \in C_c^\infty(U)$ then we even have $\psi F \in \mathcal{D}'(\mathbb{R}^n)$!