Some examples of dynamical systems

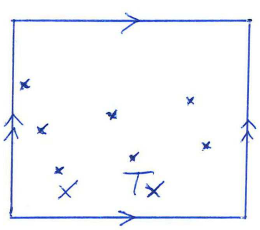
DEF: A $\left\{\begin{matrix}\text{smooth}\\ \text{topological}\end{matrix}\right\}$ dynamical system (of simplest type!) is a pair (X,T), where X is a $\left\{\begin{matrix}C^\infty \text{ manifold}\\ \text{metric space}\end{matrix}\right\}$ and T is a $\left\{\begin{matrix}C^\infty\\ \text{continuous}\end{matrix}\right\}$ map $X \to X$.

One studies iterations of T, e.g. the orbit of a point $x \in X$:

$x, Tx, T^2x, T^3x, \ldots$

Examples

- Translation on a torus: $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$,
  $T: X \to X$; $Tx = x + \alpha$ ($\alpha \in \mathbb{R}^n$ fixed)

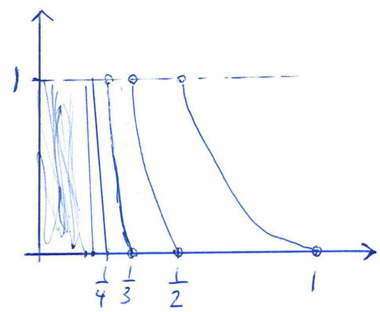


- One-sided full shift: $X = \{1,2,\ldots,n\}^{\mathbb{N}} = \{1,\ldots,n\} \times \{1,\ldots,n\} \times \ldots$
  $T((a_1,a_2,\ldots)) = (a_2,a_3,\ldots)$ ("left shift")

- Two-sided full shift: $X = \{1,2,\ldots,n\}^{\mathbb{Z}}$, T left-shift.

• The Gauss map: $Y = [0,1] \setminus \mathbb{Q}$, $\quad T : Y \to Y$; $\quad T(x) = \{x^{-1}\} = x^{-1} - \lfloor x^{-1} \rfloor$

(cf. E&W, Ch 3.2)



"Coding" of T: $\Psi : Y \to \mathbb{N}^{\mathbb{N}}$;

$\Psi(x)$ gives the continued fraction expansion of x, that is:

$$\Psi(x) = (a_1, a_2, \dots) \quad \overset{\text{def.}}{\Longleftrightarrow} \quad x = [a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}}$$

Ψ is a homeomorphism, and $\Psi \circ T \circ \Psi^{-1}$ is the left shift map on $\mathbb{N}^{\mathbb{N}}$. 3

Some details

Consider

$$\begin{tikzcd} \mathbb{N}^{\mathbb{N}} \arrow[r, "\sigma = \text{left-shift}"] \arrow[d, "\Psi^{-1}"] & \mathbb{N}^{\mathbb{N}} \arrow[d, "\Psi^{-1}"] \\ Y \arrow[r, "?"'] & Y \end{tikzcd}$$

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \quad \longmapsto \quad \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}$$

$\Downarrow$

$$x^{-1} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}} \quad \Rightarrow \quad \{x^{-1}\} = x^{-1} - \lfloor x^{-1} \rfloor = (*)$$

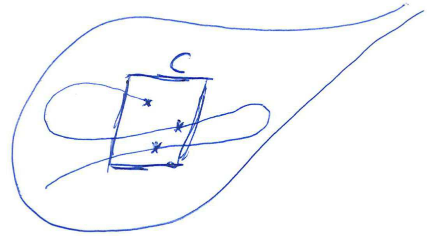
!

ASIDE: Coding of the geodesic flow 'on' the modular surface

E & W Ch. 9

$\mathcal{M} = PSL(2,\mathbb{Z}) \backslash \mathbb{H}$; consider geodesic flow on $T_1\mathcal{M}$.

For a well-chosen cross-section C



we have:

$$\left(C, \begin{smallmatrix}\text{return map of}\\ \text{geodesic flow}\end{smallmatrix}\right) \cong \underbrace{(\overline{Y}, \overline{T})}_{} \cong (\mathbb{N}^{\mathbb{Z}}, \sigma)$$

$$\boxed{\begin{aligned} \overline{Y} &= \left\{(y,z) \in [0,1)^2 : \quad 0 \le z \le \tfrac{1}{1+y}\right\} \\ \overline{T}(y,z) &= \left(\{y^{-1}\}, \; y(1-yz)\right) \end{aligned}}$$

Invariant measures

Important information about a dynamical system (X, T) comes from knowing one/some/all of its invariant (probability) measures, i.e., measures μ on X satisfying $T_* \mu = \mu$ ($\Leftrightarrow \mu(T^{-1}A) = \mu(A)$ for all measurable $A \subset X$)

"Purified" situation:

DEF: If $(X, \mathcal{M}, \mu)$ is a measure space, a measurable map $T: X \to X$ is called a measure-preserving transformation if $T_* \mu = \mu$.
(Also, μ is then said to be a $(T-)$invariant measure.)

We will only consider the case when μ is a probability measure.

Examples

Translation on torus: $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, $T(x) = x+\alpha$ ($\alpha \in \mathbb{R}^n$ fixed)

Then μ = Lebesgue is an invariant probability measure.

Bernoulli shift: $X = \{1,2,\dots,n\}^{\mathbb{N}}$ or $\{1,2,\dots,n\}^{\mathbb{Z}}$, and T = left-shift.

Then for any probability measure μ_1 on $\{1,2,\dots,n\}$, $\mu_1 \times \mu_1 \times \dots$ is a T-invariant probability measure on X.

This measure preserving transformation (X,μ,T) is called a Bernoulli shift.

(Cf. E&W, Ex. 2.8, 2.9.) (Also: Folland Thm 7.28.)

The Gauss map: $Y = [0,1] \setminus \mathbb{Q}$, $T(x) = \{x^{-1}\}$

The Gauss measure on Y is $d\mu(x) = \frac{1}{\log 2} \cdot \frac{1}{1+x} \cdot dx$.

It is a T-invariant probability measure on Y. (E&W Lemma 3.5)

Discussion/proof: Write $d\mu(x) = f(x) \cdot dx$, $f \in L^1((0,1), dx)$.

By Folland Thm. 1.16, suffices to check:

$$\mu(\underbrace{T^{-1}(0,s)}_{= \bigcup_{n=1}^{\infty} \left(\frac{1}{s+n}, \frac{1}{n}\right)}) = \mu((0,s)), \quad \forall s \in (0,1)$$

$$\Leftrightarrow \sum_{n=1}^{\infty} \int_{\frac{1}{s+n}}^{\frac{1}{n}} f(x)\,dx = \int_0^s f(x)\,dx, \quad \forall s \in (0,1)$$

formally...

$$\Longrightarrow \sum_{n=1}^{\infty} f\left(\frac{1}{s+n}\right) \frac{1}{(s+n)^2} = f(s)$$

(transfer operator)

Check that $f(x) = \frac{C}{1+x}$ satisfies the above:

$$L.H. = C\cdot\sum_{n=1}^{\infty} \frac{s+n}{s+n+1}\cdot\frac{1}{(s+n)^2} = C\sum_{n=1}^{\infty}\left(\frac{1}{s+n}-\frac{1}{s+n+1}\right) = \frac{C}{s+1} = R.H.$$

OK!

Now recall

$$\begin{tikzcd} \mathbb{N}^{\mathbb{N}} \arrow[r, "\sigma = \text{left-shift}"] & \mathbb{N}^{\mathbb{N}} \\ Y \arrow[u, "\Psi"] \arrow[r, "T"] & Y \arrow[u, "\Psi"] \end{tikzcd}$$

It follows that $\Psi_*\mu$ is a σ-invariant measure on $\mathbb{N}^{\mathbb{N}}$.

Claim: $\Psi_*\mu$ is not $= \prod_{j=1}^{\infty} \mu_1$ with μ_1 a probability measure on $\mathbb{N}$.

proof: (See E&W Exc. 3.24)

$$(\Psi_*\mu)\{a_1=1\} = \int_{(\frac{1}{2},1)} d\mu(x) = \frac{\log 2 - \log\frac{3}{2}}{\log 2} = \frac{2\log 2-\log 3}{\log 2} = 0,415\ldots$$

Thus $\left((\Psi_*\mu)\{a_1=1\}\right)^2 = 0,172\ldots$

But $(\Psi_*\mu)\{a_1=a_2=1\} = \int_{(\frac{1}{2},\frac{2}{3})} d\mu(x) = \frac{\log\left(\frac{5}{3}/\frac{3}{2}\right)}{\log 2} = \frac{\log\left(\frac{10}{9}\right)}{\log 2} = 0,152\ldots$

Not equal! □

DEF (E & W, 2.13): A T-invariant probability measure μ on $(X, \mathcal{M})$ is called **ergodic** if $[\forall A \in \mathcal{M}:\ T^{-1}A = A \Rightarrow [\mu(A) = 0 \text{ or } 1]]$.

Note: ergodic $\approx$ "indecomposable" in an appropriate sense!

Ergodic decomposition: Every T-invariant measure can be expressed as a superposition of ergodic T-invariant measures. (E & W Thm 4.8, Thm 6.2)

Ex: Translation on torus: $X = \mathbb{T}^n$, $T(x) = x + \alpha$ ($\alpha \in \mathbb{R}^n$ fixed)

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ linearly independent over $\mathbb{Q}$, then the T-invariant measure $\mu =$ Lebesgue is ergodic (E & W, Cor. 4.15.)

Ex: Any Bernoulli shift is ergodic. (E & W, Prop. 2.15)

Ex: The Gauss measure $\left(d\mu(x) = \frac{1}{\log 2} \cdot \frac{dx}{1+x}\right)$ for the Gauss map $T(x) = \{x^{-1}\}$ on $Y = [0,1] \setminus \mathbb{Q}$, is ergodic. (E & W, Thm 3.7)