#17. Some more about dynamical systems

Birkhoff's Pointwise Ergodic Theorem (E & W 2.30):
Let $(X, \mathcal{M}, \mu)$ be a probability measure space, and let $T: X \to X$ be a measure-preserving transformation.
Let $f \in L^1(X, \mu)$. Then

$$A_N^f(x) := \frac{1}{N} \sum_{k=0}^{N-1} f(T^k(x))$$

converges μ-a.e. and in L^1 to some $f^* \in L^1(X, \mu)$.
We have $f^* \circ T = f^*$ a.e., and $\int_X f^* \, d\mu = \int_X f \, d\mu$.
If T is ergodic then $f^*(x) = \int_X f \, d\mu$ for μ-a.e. x.

General application

Theorem (cf. E&W, Cor. 4.20; here more general): Let X be a locally compact second countable Hausdorff space; let $T: X \to X$ be Borel measurable, and let μ be an ergodic T-invariant Borel probability measure on X. Then for μ-almost every $x \in X$:

$$\forall f \in BC(X): \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k(x)) = \int_X f \, d\mu$$

DEF (E & W, Def. 4.19): A sequence $x_0, x_1, x_2, \ldots$ in X is said to be equidistributed w.r.t. μ if

$$\forall f \in BC(X): \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(x_k) = \int_X f \, d\mu$$

Example: Let (X, μ, T) be a Bernoulli shift, say $X = \{1, 2, \dots n\}^{\mathbb{N}}$ and $\mu = \mu_1 \times \mu_1 \times \dots$. Recall: μ is ergodic!

Let $f \in BC(X)$ be given by $f((x_n)) = f_1(x_0)$ for some $f_1 : \{1, \dots, n\} \to \mathbb{C}$.

Then the Pointwise Ergodic Theorem says that for μ-a.e. $x = (x_n) \in X$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(x_n) = \sum_{j=1}^{n} \mu_1(\{j\}) \cdot f_1(j) = \mathbb{E}_{\mu_1}(f_1).$$

This is the (strong) Law of Large Numbers!

Example: Translation on torus: $X = \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, μ = Lebesgue

$T : \mathbb{T}^n \to \mathbb{T}^n$, $T(x) = x + \alpha$ ($\alpha \in \mathbb{R}^n$, fixed),

where we assume $\alpha = (\alpha_1, \dots, \alpha_n)$ with $1, \alpha_1, \alpha_2, \dots, \alpha_n$ linearly independent over $\mathbb{Q}$. Then μ is ergodic (E & W, Cor 4.15)

Hence for μ-a.e. $x \in \mathbb{T}^n$, the sequence $x, x+\alpha, x+2\alpha, x+3\alpha, \dots$ is μ-equidistributed in $\mathbb{T}^n$. $\circledast$

But, by "obvious" translation argument, if $\circledast$ holds for some $x \in \mathbb{T}^n$ then it holds for all $x \in \mathbb{T}^n$. Hence $\circledast$ holds for all $x \in \mathbb{T}^n$.

Example: The Gauss map $Y = [0,1] \setminus \mathbb{Q}$. $T: Y \to Y$; $T(x) = \{x^{-1}\}$.

Let μ = Gauss measure on Y, i.e. $d\mu(x) = \frac{1}{\log 2} \cdot \frac{dx}{1+x}$.

Recall: μ is T-invariant and ergodic.

Write $x = [a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$

Choose $f \in BC(Y)$: $\boxed{f(x) = I(a_1 = 2,\ a_2 = 5)}$.

Then $f(T^k(x)) = I(a_{k+1} = 2,\ a_{k+2} = 5)$. $(\forall k \geq 0)$

Also $\int_Y f(x)\, d\mu(x) = \int_{5/11}^{6/13} \frac{1}{\log 2} \cdot \frac{dx}{1+x} = \frac{\log\left(\frac{19}{13} \cdot \frac{11}{16}\right)}{\log 2} = \frac{\log\left(\frac{209}{208}\right)}{\log 2} \approx 0{,}00691\ldots$

(a computation shows $[a_1 = 2, a_2 = 5] \Leftrightarrow \frac{5}{11} < x < \frac{6}{13}$)

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Hence, by the Pointwise Ergodic Theorem, for μ-a.e. $x \in Y$:

$$\lim_{N \to \infty} \frac{1}{N} \#\{k \in \{1, 2, \ldots, N\} : a_k = 2, a_{k+1} = 5\} = 0{,}00691\ldots$$

Similarly for $I(a_1 = j)$ (E & W, Cor. 3.8)

or $I(a_1 = a_2 = \ldots = a_k = 1)$ or $I(a_1 = a_2 = \ldots = a_k = 2)$ (Assignment 3:7)

But, if we take $f(x) = a_1(x)$ then

$$\int_Y f\, d\mu = \sum_{n=1}^{\infty} n \cdot \mu\left(\left(\frac{1}{n+1}, \frac{1}{n}\right)\right) \geq \sum_{n=1}^{\infty} n \cdot \frac{1}{\log 2} \int_{1/n+1}^{1/n} \frac{dx}{2} \gg \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Hence $f \notin L^1(Y, \mu)$. However, truncating f from below and then applying PET gives: $\boxed{\text{For } \mu\text{-a.e. } x \in Y: \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} a_j(x) = +\infty}$