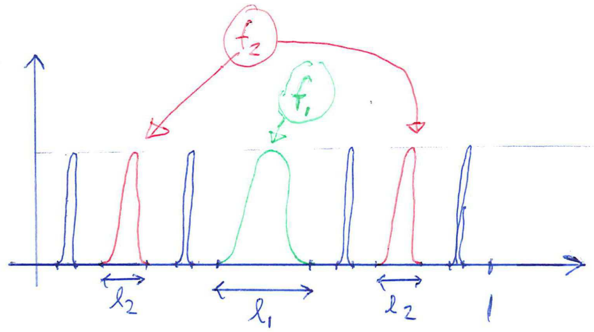
#2. Measure theory

Motivation: We'd like to have $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$ in general!

Problem: $\sum_{1}^{\infty} f_n$ is often not Riemann integrable, even if each f_n is!

Ex 1: $f_n(x) = \begin{cases} 1 & \text{if } x = q_n \\ 0 & \text{otherwise,} \end{cases}$ where $q_1, q_2, q_3, \ldots$ is an enumeration of $\mathbb{Q}$. Then $\sum_{n=1}^{\infty} f_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise,} \end{cases}$ not Riemann integrable

Ex 2:



(with $l_1 + 2l_2 + 2^2 l_3 + 2^3 l_4 + \ldots < 1$)

New ("better") integral?

- First: Just define measure, i.e., define $\int_{\mathbb{R}} f(x)\,dx$ when f is a charactistic function.

Ideally, want a "measure" $m: \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ such that

1. $m(E_1 \cup E_2 \cup \ldots) = m(E_1) + m(E_2) + \ldots$ for any pairwise disjoint sets $E_1, E_2, \ldots \subset \mathbb{R}$
2. $m(t+E) = m(E)$, $\forall E \subset \mathbb{R}$, $t \in \mathbb{R}$
3. $m([0,1)) = 1$.

Impossible!!

Ex: $m(E) = ?$ when E is a "fundamental domain" for $\mathbb{R}/\mathbb{Q}$ (w/t +).
(Folland, Sec. 1.1)

Weaken the goal, a bit...
May just as well consider for general set X.

Def: Let X be a set, $\neq \emptyset$. A **σ-algebra on X** is a non-empty family $\mathcal{A} \subset \mathcal{P}(X)$ s.t.

(1) If $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$

(2) If $E_1, E_2, \ldots \in \mathcal{A}$ then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$

Note:
- Then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.
- If $E_1, E_2, E_3, \ldots \in \mathcal{A}$ then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{A}$. $\left(\text{Since } \bigcap_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} E_n^c\right)^c.\right)$

Def: If $\mathcal{E} \subset \mathcal{P}(X)$ then $\mathcal{M}(\mathcal{E}) := \left[\text{the unique smallest } \sigma\text{-algebra on } X \text{ containing } \mathcal{E}\right]$

{the "σ-algebra generated by $\mathcal{E}$"}

Def: If X is a topological space, then the **Borel σ-algebra** on X is $\mathcal{B}_X := \mathcal{M}(\{U : U \text{ open} \subset X\})$

Def: Let $\mathcal{M}$ be a σ-algebra on a set X.

A **measure** on $\mathcal{M}$ (or "on $(X, \mathcal{M})$") is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $E_1, E_2, \ldots$ are pairwise disjoint sets in $\mathcal{M}$ then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$.

Then $(X, \mathcal{M}, \mu)$ is called a **measure space**.

Def:
- μ is **finite** if $\mu(X) < \infty$.
- μ is **σ-finite** if $\exists E_1, E_2, \ldots \in \mathcal{M}$ s.t. $\mu(E_j) < \infty$ $(\forall j)$ and $X = \bigcup_{j=1}^{\infty} E_j$.
- μ is a **Borel measure** if X is a topological space and $\mathcal{M} = \mathcal{B}_X$.

Examples of measures: For any set X, set $\mathcal{M} = \mathcal{P}(X)$ and

(1) $\mu(E) = \begin{cases} \# E & \text{if } E \text{ finite} \\ \infty & \text{if } E \text{ infinite} \end{cases}$ — this is the **counting measure** on X.

(2) Given $x_0 \in X$, set $\mu(E) = I(x_0 \in E)$ — this is the **point mass** at x_0 (or **Dirac measure**)

Basic properties

{Folland's Thm 1.8}

Let $(X, \mathcal{M}, \mu)$ be a measure space.

a) If $E, F \in \mathcal{M}$ and $E \subset F$ then $\mu(E) \leq \mu(F)$

b) If $\{E_j\}_1^\infty \subset \mathcal{M}$ then $\mu\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty \mu(E_j)$

c) If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \ldots$ then $\mu\left(\bigcup_1^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j)$.

d) If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \supset E_2 \supset \ldots$ and $\mu(E_1) < \infty$, then $\mu\left(\bigcap_1^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j)$.

Def: μ is complete if

$\forall F \subset X: \quad [\exists E \in \mathcal{M}: \mu(E) = 0 \text{ and } F \subset E] \Rightarrow F \in \mathcal{M}$

Theorem: Every measure μ has a unique completion.

(Folland Thm 1.9; notation $\overline{\mathcal{M}}, \overline{\mu}$.)

Ex (important): There is a unique measure μ on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ which is invariant under all translations and satisfies $\mu([0,1]^n) = 1$.

The completion of this measure is called Lebesgue measure: $(\mathbb{R}^n, \mathcal{L}^n, m)$.

= standard volume measure on $\mathbb{R}^n$.

Integration

Given $(X, \mathcal{M}, \mu)$ and $f: X \to \mathbb{C}$, want to define $\int_X f \, d\mu$.

First case: $f = \chi_E$ for some $E \subset X$. Then want: $\int_X \chi_E \, d\mu := \mu(E)$

$\rightsquigarrow$ must require $E \in \mathcal{M}$!

For general $f: X \to \mathbb{C}$: Must require f $(\mathcal{M}$-$)$measurable

Def: If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces then a function $f: X \to Y$ is said to be $(\mathcal{M}, \mathcal{N})$-measurable if $f^{-1}(E) \in \mathcal{M}$, $\forall E \in \mathcal{N}$.

If Y is a topological space: $f: X \to Y$ measurable (or $\mathcal{M}$-measurable) $\overset{\text{def}}{\iff}$ $(\mathcal{M}, \mathcal{B}_Y)$-measurable.

- this applies in particular when $Y = \mathbb{R}^n$, $\mathbb{C}^n$ or $\overline{\mathbb{R}}$.

(Thus: A function $f: \mathbb{R}^n \to \mathbb{R}^k$ is Borel-measurable if it is $(\mathcal{B}_{\mathbb{R}^n}, \mathcal{B}_{\mathbb{R}^k})$-m'ble, Lebesgue-measurable if it is $(\mathcal{L}^n, \mathcal{B}_{\mathbb{R}^k})$-m'ble.)

Basic properties of measurability

Folland pp. 44-45

- For $E \subset X$, χ_E is measurable iff $E \in \mathcal{M}$.
- The family of $\mathcal{M}$-measurable functions $f: X \to \mathbb{C}$ is closed under $+$, $\cdot$, limit.
- Same for the family of $\mathcal{M}$-measurable functions $f: X \to \overline{\mathbb{R}}$; it is also closed under sup, inf, limsup, liminf.

Def: A simple function on X is a finite $\mathbb{C}$-linear combination of functions in $\{\chi_E : E \in \mathcal{M}\}$

Any simple function $f: X \to \mathbb{C}$ has a standard representation

$$f = \sum_{j=1}^{n} z_j \cdot \chi_{E_j} \quad \text{where } \underbrace{\{z_1, \ldots, z_n\}}_{\text{distinct!}} = \operatorname{range}(f) \text{ and } E_j = f^{-1}(\{z_j\}), \ \forall j.$$

Def: Given a measure space $(X, \mathcal{M}, \mu)$, we set

Folland Ch. 2.2

$$L^+ = \{ f : X \to [0, \infty] \ : \ f \text{ is } \mathcal{M}\text{-measurable} \}$$

or "$L^+(X)$" or "$L^+(\mathcal{M})$"

For any simple function $\phi \in L^+$, set

$$\int_X \phi \, d\mu = \sum_{j=1}^{n} \underbrace{a_j \cdot \mu(E_j)}_{\text{note: } 0 \cdot \infty = 0}$$

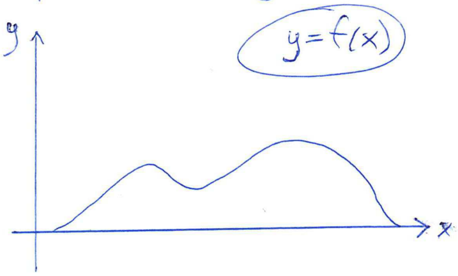
if ϕ has standard repr. $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$.

For any arbitrary $f \in L^+$, set

$$\int_X f \, d\mu = \sup \left\{ \int_X \phi \, d\mu \ : \ \phi \text{ simple, } 0 \leq \phi \leq f \right\}$$

- Def. ok for f simple.
- $f \leq g \Rightarrow \int f \leq \int g$
- $\int cf = c \int f$ for all $c \geq 0$.

Note: For $X = \mathbb{R}$, i.e. $f : \mathbb{R} \to [0, \infty]$, the def. of $\int_X f \, d\mu$ means that "we partition along the y-axis":



Theorem: (alt. def.) For any $f \in L^+$,

$$\int_X f \, d\mu = \underbrace{\int_0^\infty \mu(\{ x \in X : f(x) \geq t \}) \, dt}_{\text{generalized Riemann integral!}}$$

(Cf. Lieb & Loss "Analysis")