#4. Measure and integration theory

Def: A complex measure on a m'ble space $(X, \mathcal{M})$ is a map $\nu : \mathcal{M} \to \mathbb{C}$ such that $\nu(\emptyset) = 0$ and for any pairwise disjoint sets $E_1, E_2, \ldots \in \mathcal{M}$ one has $\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j)$, with the r. h. absolutely convergent.

Set $M(\mathcal{M})$ = the set of complex measures on $(X, \mathcal{M})$.

$M(\mathcal{M})$ is a $\mathbb{C}$-vector space

Defs: Let $\nu \in M(\mathcal{M})$. We set $\nu_r = \operatorname{Re} \nu$, $\nu_i = \operatorname{Im} \nu$.

A set $E \in \mathcal{M}$ is called

a) null for ν if $\nu(F) = 0$ for all $F \in \mathcal{M}$ with $F \subset E$.

b) positive for ν if $\nu(F) \geq 0$ — " —

c) negative for ν if $\nu(F) \leq 0$ — " —.

For $\mu, \nu \in M(\mathcal{M})$: $\mu \perp \nu \overset{def}{\iff} \exists E, F \in \mathcal{M}$ s.t. $X = E \cup F$, E null for μ, F null for ν.

Thm 3.4 Jordan decomposition:
Given any real $\nu \in M(\mathcal{M})$, there exist unique positive, finite measures ν^+, ν^- on $(X, \mathcal{M})$ s.t. $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Thus for any $\nu \in M(\mathcal{M})$: $\nu = \nu_r + i \cdot \nu_i = \nu_r^+ - \nu_r^- + i(\nu_i^+ - \nu_i^-)$

proof, outline: The key is to find some $P, N \in \mathcal{M}$ with $P \cap N = \emptyset$, $P \cup N = X$, P positive for ν, N negative for ν.

– Such $\langle P, N \rangle$ is called a Hahn Decomposition for ν.

Once such P, N is found, we can simply set
$$\begin{cases} \nu^+(E) := \nu(E \cap P) \\ \nu^-(E) := -\nu(E \cap N) \end{cases}$$

Construction: Set $m := \sup\{\nu(E) : E \in \mathcal{M},\ E \text{ positive for } \nu\}$.

Take $P_1, P_2, \ldots \in \mathcal{M}$ positive for ν such that $m = \lim_{k \to \infty} \nu(P_k)$.

Set $P = \bigcup_{k=1}^{\infty} P_k$ and $N = X \setminus P$. ...

Note: $\forall \nu \in M(\mathcal{M})$: $\exists C > 0$: $\forall E \in \mathcal{M}$: $|\nu(E)| \leq C$.

– ν "finite", in a strong sense!

"Def": Given $\nu \in M(\mathcal{M})$, we write $|\nu|$ = total variation (measure) of ν, for the smallest positive measure on $(X, \mathcal{M})$ s.t. $|\nu(E)| \leq |\nu|(E)$ $\forall E \in \mathcal{M}$.

Construction: $|\nu|(E) = \left\{ \sup \sum_{j=1}^{\infty} |\nu(E_j)| : \quad E_1, E_2, \ldots \in \mathcal{M}, \text{ disjoint}, \bigcup_{j=1}^{\infty} E_j = E \right\}$

If ν real then $|\nu| = \nu^+ + \nu^-$.

DEF: If μ is a positive measure on $(X, \mathcal{M})$ and $f \in L^1(\mu)$ then $\nu = f \cdot \mu \in M(\mathcal{M})$ is defined by $\nu(E) = \int_E f \, d\mu$, $\forall E \in \mathcal{M}$.

Folland writes: $d\nu = f \cdot d\mu$

Facts, if $\nu = f \cdot \mu$:

- $\nu_r = Re(f) \cdot \mu$, $\nu_i = Im(f) \cdot \mu$
- If ν real: $\nu^+ = f^+ \cdot \mu$, $\nu^- = f^- \cdot \mu$
  Recall $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$
- $|\nu| = |f| \cdot \mu$

DEF: For $\nu \in M(\mathcal{M})$, set $L^1(\nu) := L^1(|\nu|)$ and for $f \in L^1(\nu)$, define $\int_X f \, d\nu$ using $\nu = \nu_r^+ - \nu_r^- + i(\nu_i^+ - \nu_i^-) \ldots$

Fact, if $\nu = f \cdot \mu$:

$g \in L^1(\nu) \Leftrightarrow \int_X |gf| \, d\mu < \infty$, and then $\int_X g \, d\nu = \int_X gf \, d\mu$.

DEF: For μ a positive measure on $(X, \mathcal{M})$ and $\nu \in M(\mathcal{M})$, we say

$\nu \ll \mu$ iff $\forall E \in \mathcal{M}: \mu(E) = 0 \Rightarrow \nu(E) = 0$.

ν absolutely continuous w.r.t. μ

Radon-Nikodym Theorem (3.13): If μ is a σ-finite positive measure on $(X, \mathcal{M})$, $\nu \in M(\mathcal{M})$, and $\nu \ll \mu$, then there exists a unique $f \in L^1(\mu)$ s.t. $\nu = f \cdot \mu$.

More generally, for any $\nu \in M(\mathcal{M})$, there exist unique $\lambda \in M(\mathcal{M})$ and $f \in L^1(\mu)$ s.t. $\nu = \lambda + f \cdot \mu$ and $\lambda \perp \mu$.

proof ~~outline~~ start: Reduce to μ, ν both positive & finite.

Let $\mathcal{F} = \left\{ f \in L^+(\mu) : \left[\int_E f \, d\mu \leq \nu(E), \ \forall E \in \mathcal{M} \right] \right\}$

Set $a = \sup \left\{ \int_X f \, d\mu : f \in \mathcal{F} \right\}$; note $a \leq \nu(X) < \infty$.

Take $f_1, f_2, f_3, \ldots \in \mathcal{F}$ s.t. $\int_X f_k \, d\mu \longrightarrow a$. May assume $f_1 \leq f_2 \leq \ldots$

Now our $f := \lim f_n$!! .....

Application 1: Conditional expectation & probability

Let $(X, \mathcal{M}, \mu)$ be a finite measure space (viz., $\mu(X) < \infty$),

let $f \in L^1(\mu)$ and $\lambda = f \cdot \mu$. Let $\mathcal{N}$ be a sub-σ-algebra of $\mathcal{M}$.

Now $\lambda \ll \mu$; hence $\lambda|_{\mathcal{N}} \ll \mu|_{\mathcal{N}}$; hence $\exists! \, g \in L^1(\mu|_{\mathcal{N}}) : \lambda|_{\mathcal{N}} = g \cdot \mu|_{\mathcal{N}}$.

Now: $\forall E \in \mathcal{N}: \int_E f \, d\mu = \int_E g \, d\mu|_{\mathcal{N}} = \int_E g \, d\mu$

If $(X, \mathcal{M}, \mu)$ is a probability space, i.e. $\mu(X) = 1$, then $f \in L^1(\mu)$ is a "random variable", and we write

$g = \mathbb{E}(f \mid \mathcal{N})$

Also, for $A \in \mathcal{M}$: $\mu(A \mid \mathcal{N}) := \mathbb{E}(\chi_A \mid \mathcal{N})$

Application 2: $(L^p)^*$ (Folland, Ch 6.1)

DEF: Let $(X, \mathcal{M}, \mu)$ be a measure space and let $0 < p < \infty$.
For $f: X \to \mathbb{C}$ measurable, set $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$

and $L^p = \{f: X \to \mathbb{C} : f \text{ measurable and } \|f\|_p < \infty\}$

or $L^p(\mu)$ or $L^p(X)$

Theorem: For $1 \leq p < \infty$, L^p with $\|\cdot\|_p$ is a Banach space.

What is $L^p(X)^*$?

Recall (Folland, Ch. 5.1-2): For V a normed $\mathbb{C}$-linear space,

$V^* := \{f: V \to \mathbb{C} : f \text{ linear and bounded}\}$

the dual of V

$\overset{\text{def}}{\Longleftrightarrow} \exists B > 0 : \forall x \in V : |f(x)| \leq B\|x\|$

Theorem 6.15: Assume $1 \leq p < \infty$. If $p = 1$, assume μ is σ-finite.
Let $q = \frac{p}{p-1}$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$).
Then each $g \in L^q$ gives a $\phi_g \in (L^p)^*$ by $\phi_g(f) = \int_X g f \, d\mu$,
and the map $g \mapsto \phi_g$ is an isometric isomorphism $L^q \xrightarrow{\sim} (L^p)^*$

Thus: $\boxed{(L^p)^* = L^q}$

proof outline: $\forall g \in L^q : \forall f \in L^p : \left|\int_X gf \, d\mu\right| \leq \int_X |gf| \, d\mu \leq \|g\|_q \cdot \|f\|_p$.

Hence: $\|\phi_g\| \leq \|g\|_q$. Now easy $\rightsquigarrow$ $\|\phi_g\| = \|g\|_q$

Now let $\phi \in (L^p)^*$. Reduce to μ finite.
Now $E \mapsto \phi(\chi_E)$ is a complex measure on $(X, \mathcal{M})$, clearly $\ll \mu$.

Radon-Nikodym $\Rightarrow$ $\exists!\, g \in L^1(\mu)$ s.t. $\phi(\chi_E) = \int_E g \, d\mu$, $\forall E \in \mathcal{M}$

$\Rightarrow$ For every simple function $f: X \to \mathbb{C}$, $\Phi(f) = \int_X fg\, d\mu$.

and $\left|\int_X fg\, d\mu\right| = |\Phi(f)| \leq \|\Phi\| \cdot \|f\|_p \rightsquigarrow g \in L^q$.

....

Riesz' Representation Theorem (Folland Ch. 7.3)

Def: Let X be a topological space,

$C(X) = \{f: X \to \mathbb{C} : f \text{ continuous}\}$.

For $f \in C(X)$: $\text{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}$ and $\|f\|_u = \sup\{|f(x)| : x \in X\}$.

$C_c(X) = \{f \in C(X) : \text{supp}(f) \text{ is compact}\}$.

$C_0(X) = \{f \in C(X) : [\forall \varepsilon > 0 : \exists\ K \text{ compact} \subset X \text{ s.t. } [|f| < \varepsilon \text{ outside } K]]\}$

Note: $C_c \subset C_0 \subset C$, and $\|\cdot\|$ is a norm on the first two!

Lemma: $C_0(X)$ is complete (thus: a Banach space). If X is LCH then $\overline{C_c(X)} = C_0(X)$.

Outline of proof (see Prop 4.13 and 4.35): Given a Cauchy sequence $\{f_j\}$ in C_0, a (unique) limit function exists, is continuous and "vanish at ∞".

To prove $\overline{C_c} = C_0$, need LCH, and

Urysohn's Lemma (4.32) For X LCH, if K compact $\subset U$ open $\subset X$, then $\exists f \in C_c(X)$: $\chi_K \leq f \leq \chi_U$ and $\text{supp}(f) \subset U$.

Riesz' Representation Theorem (7.17): Let X be an LCH space in which every open set is σ-compact. For any $\mu \in M(\mathcal{B}_X)$ and $f \in C_0(X)$, set $I_\mu(f) = \int_X f \, d\mu$.

Then $\mu \mapsto I_\mu$ is an isometric isomorphism $\underbrace{M(\mathcal{B}_X)}_{\text{Norm: } \|\mu\| = |\mu|(X)} \xrightarrow{\sim} C_0(X)^*$.

(Basic) examples:

a) Let $X = \mathbb{R}^n$ and fix $x_0 \in \mathbb{R}^n$. Define $I \in C_0(X)^*$ by $I(f) := f(x_0)$.

Then $I = I_\mu$ for $\mu =$ Dirac at x_0.

b) Let $X = \mathbb{R}^2$ and define $I \in C_0(X)^*$ by $I(f) = \int_0^1 f(x,0)\,dx$.

Then $I = I_\mu$ where $\mu \in M(\mathcal{B}_X)$ is "Lebesgue measure on the line segment $[0,1] \times \{0\}$":

$$\mu(E) = m^1(\{x \in [0,1] : (x,0) \in E\})$$

Weak * convergence of measures

DEF: Let X be LCH. A sequence $\mu_1, \mu_2, \ldots$ in $M(\mathcal{B}_X) = C_0(X)^*$ is said to converge vaguely (or weak *) to $\mu \in M(\mathcal{B}_X)$ if

$$\int_X f \, d\mu_j \to \int_X f \, d\mu, \quad \forall f \in C_0(X).$$

Ex: For $N \in \mathbb{N}$, let $I_N \in C_0(\mathbb{R}^n)$ be given by $I_N(f) = \frac{1}{N^n} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} f\left(\frac{j_1}{N}, \ldots, \frac{j_n}{N}\right)$.

Then $\{I_N\}_{N=1}^\infty$ converges vaguely to m^n restricted to $[0,1]^n$!

Ex: If $I_N \in C_0(\mathbb{R})$ is given by $I_N(f) = f(N)$, then $I_N \xrightarrow{\text{vaguely}} 0$.

Regularity (of measures)

DEF: Let X be a topological space. A positive measure μ on $(X, \mathcal{B}_X)$ is said to be <u>outer regular</u> on $E \in \mathcal{B}_X$ if $\mu(E) = \inf\{\mu(U) : U \text{ open} \supset E\}$ and <u>inner regular</u> on $E \in \mathcal{B}_X$ if $\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}$.
If μ is both, on every $E \in \mathcal{B}_X$, then μ is said to be <u>regular</u>.

DEF: $\nu \in M(\mathcal{B}_X)$ is said to be <u>regular</u> if $|\nu|$ is regular.

Thm. 7.8: Let X be an <u>LCH space</u> in which <u>every open set is σ-compact</u>. Then every positive measure on $(X, \mathcal{B}_X)$ which is <u>finite on all compact sets</u> is <u>regular</u>. Hence also <u>every $\nu \in M(\mathcal{B}_X)$ is regular</u>.

$\therefore$ For such X: $M(\mathcal{B}_X) =$ the space of <u>Radon measures</u> on X; "$M(X)$" ← Folland Ch. 7

Application of regularity

Prop 7.9: For $(X, \mathcal{B}_X, \mu)$ as in Thm. 7.8, <u>$C_c(X)$ is dense in $L^p(X)$</u>, for each <u>$1 \le p < \infty$</u>.

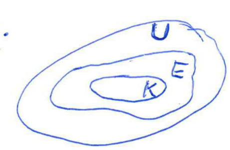
proof outline:

<u>Simple</u> functions are dense in L^p.
Hence it suffices to prove that any <u>characteristic function</u> belongs to $\overline{C_c(X)}$ in L^p.

Given $E \in \mathcal{B}_X$ with $\mu(E) < \infty$; <u>approximate χ_E</u>?

<u>Given $\varepsilon > 0$</u>:
<u>By regularity</u>, $\exists$ open U, compact K s.t. <u>$K \subset E \subset U$</u> and <u>$\mu(U \setminus K) < \varepsilon$</u>.

Urysohn $\Rightarrow$ <u>$\exists f \in C_c(X)$ s.t. $\chi_K \le f \le \chi_U$</u>.



Then <u>$\|\chi_E - f\|_p \le \mu(U \setminus K)^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}}$</u>.