

DEF: For $f, g: \mathbb{R}^n \rightarrow \mathbb{C}$, (m'ble), $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y) dy$

Prop 8.6: Assuming all integrals in question exist:

a) $f * g = g * f$

b) $(f * g) * h = f * (g * h)$

c) For $z \in \mathbb{R}^n$: $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$

d) $\text{supp}(f * g) \subset \overline{\text{supp}(f) + \text{supp}(g)}$

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Young's inequality (8.7): If $f \in L^1$ and $g \in L^p$ ($1 \leq p \leq \infty$) then $(f * g)(x)$ exists for a.e. x , and $\|f * g\|_p \leq \|f\|_1 \cdot \|g\|_p$

proof:

For $p = \infty$: $|(f * g)(x)| = \left| \int f(x-y)g(y) dy \right| \leq \int |f(x-y)g(y)| dy \leq \|g\|_\infty \int |f(x-y)| dy = \|g\|_\infty \|f\|_1$

For $p = 1$: Use $\int \int |f(x-y)g(y)| dy dx = \int |f(z)| dz \int |g(y)| dy = \|f\|_1 \|g\|_1$

General p : $(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y) dy$, i.e. $f * g = \int_{\mathbb{R}^n} f(y) \cdot (\tau_y g)(\cdot) dy$

Minkowski's inequality: $\|F_1 + F_2\|_p \leq \|F_1\|_p + \|F_2\|_p$, $\forall F_1, F_2 \in L^p(\mathbb{R}^n)$

$\Rightarrow \|c_1 F_1 + c_2 F_2 + \dots + c_n F_n\|_p \leq |c_1| \|F_1\|_p + \dots + |c_n| \|F_n\|_p$, $\forall F_1, \dots, F_n \in L^p(\mathbb{R}^n)$
 $c_1, \dots, c_n \in \mathbb{C}$

$\Rightarrow \left\| \int_Y F_y d\mu(y) \right\|_p \leq \int_Y \|F_y\|_p d\mu(y)$ if $(Y, \mathcal{N}, \mu) \ni y \mapsto F_y \in L^p(\mathbb{R}^n)$
"nice"

Apply with $F_y = \tau_y g$ and $\mu = f \cdot m$

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Prop 8.8: If $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, $g \in L^q$ then $f * g \in BC(\mathbb{R}^n)$,
 is uniformly continuous, and $\|f * g\|_u \leq \|f\|_p \|g\|_q$.
 If $1 < p, q < \infty$ then $f * g \in C_0(\mathbb{R}^n)$.

proof, outline: Existence and $\|f * g\|_u \leq \|f\|_p \|g\|_q$: From Hölder's inequality.

$h = f * g$ uniformly continuous?

$$\Leftrightarrow \lim_{\substack{y \rightarrow 0 \\ (y \in \mathbb{R}^n)}} \|\tau_y h - h\|_u = 0$$

But: $\|\tau_y h - h\|_u = \|(\tau_y f - f) * g\|_u \leq \|\tau_y f - f\|_p \cdot \|g\|_q \xrightarrow[y \rightarrow 0]{+} 0$

translation continuous in L^p
 - Prop 8.5 (For $p < \infty$)

If $1 < p, q < \infty$: $f * g \in C_0$?

Take $\{f_n\}$ and $\{g_n\}$ in C_c with $f_n \rightarrow f$ and $g_n \rightarrow g$.

Then $f_n * g_n \in C_c$ ($\forall n$) and $f_n * g_n \rightarrow f * g$ in $(BC, \|\cdot\|_u)$.

Hence $f * g \in \overline{C_c} = C_0$.

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Multi-index notation (Ch. 8.1)

On $\mathbb{R}^n$: $x = (x_1, \dots, x_n)$. Write $\partial_j = \frac{\partial}{\partial x_j}$ assume these commute

A multi-index is an n -tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$.

DEF: $|\alpha| = \sum_1^n \alpha_j$, $\alpha! = \prod_1^n \alpha_j!$, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$

and $x^\alpha = \prod_1^n x_j^{\alpha_j}$

Ex $\partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} (\partial^\beta f) (\partial^\gamma g)$

DEF: $C^k(\mathbb{R}^n) = \left\{ f \in C(\mathbb{R}^n) : \partial_{j_1} \dots \partial_{j_m} f \text{ exists and is continuous} \right.$
 $\left. \text{for all } j_1, \dots, j_m \in \{1, \dots, n\}, 0 \leq m \leq k \right\}$

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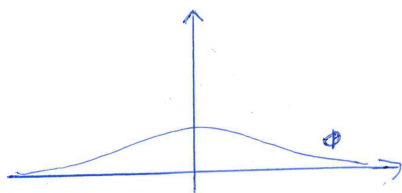
Prop. 8.10: If $f \in L^1$, $g \in C^k$ and $\partial^\alpha g \in BC$ for $|\alpha| \leq k$,
then $f * g \in C^k$ and $\partial^\alpha (f * g) = f * (\partial^\alpha g)$ for $|\alpha| \leq k$.

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Approximate identity & smoothing

DEF: For $\phi: \mathbb{R}^n \rightarrow \mathbb{C}$ and $t > 0$, we set $\phi_t(x) = t^{-n} \phi(t^{-1}x)$

Ex



Note: If $\phi \in L^1$ then $\phi_t \in L^1$
and $\int_{\mathbb{R}^n} \phi_t = \int_{\mathbb{R}^n} \phi$, $\forall t > 0$.

Thm 8.14: Assume $\phi \in L^1(\mathbb{R}^n)$. Set $a = \int_{\mathbb{R}^n} \phi dx$.

Then for any $f \in L^p(\mathbb{R}^n)$ ($1 \leq p < \infty$), we have $f * \phi_t \xrightarrow{(t \rightarrow 0)} af$
in $L^p(\mathbb{R}^n)$.

proof: $(f * \phi_t - af)(x) = \int_{\mathbb{R}^n} (f(x-y) - f(x)) \phi_t(y) dy = \int_{\mathbb{R}^n} (\tau_y f - f)(x) \cdot \phi_t(y) dy$

Hence by Minkowski's ineq. for integrals:

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$$\|f * \phi_t - af\|_p \leq \int_{\mathbb{R}^n} \|\tau_y f - f\|_p \cdot |\phi_t(y)| dy$$

Now conclude by using the Dominated Convergence Theorem!

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Useful fact: $\exists \phi \in C_c^\infty(\mathbb{R}^n)$ s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi = 1$

Ex:
$$\psi(x) = \begin{cases} \exp((|x|^2 - 1)^{-1}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

— then $\psi \geq 0$, $\text{supp}(\psi) \subseteq \overline{B_1^n}$,

and $\psi \in C^\infty$ since $t \mapsto \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$ is smooth!
(Exc 3, Ch. 8.1)

Now take $\phi = \left(\int_{\mathbb{R}^n} \psi \right)^{-1} \cdot \psi$.

Consequences:

Prop 8.17: C_c^∞ is dense in L^p ($1 \leq p < \infty$) and in C_0 .

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The C^∞ Urysohn Lemma (8.18):

If K compact $\subset U$ open $\subset \mathbb{R}^n$, then $\exists f \in C_c^\infty(\mathbb{R})$ s.t.

$\chi_K \leq f \leq \chi_U$ and $\text{supp}(f) \subset U$.

proof: Let $\delta = \frac{1}{3} \text{dist}(K, U^c)$ (> 0).

$$V = K + B_\delta$$

ϕ - as above.

Take $t > 0$ so small that $\text{supp}(\phi_t) \subset B_\delta$.

Set $f = \chi_V * \phi_t$.

