6. The Fourier Transform (Ch. 8.3)

DEF: A *character* on a topological (abelian) group G is a continuous homomorphism $G \to \langle \{z \in \mathbb{C} : |z| = 1\}, \cdot \rangle$.

We set $\widehat{G} = $ [the dual of G] = the group of all characters on G.

Theorem ($\approx$ 8.19): We have $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$; an isomorphism is given by $\xi \mapsto \Phi_\xi$, $\mathbb{R}^n \to \widehat{\mathbb{R}^n}$, where $\Phi_\xi(x) = e^{2\pi i \xi \cdot x}$

DEF: For $f \in L^1(\mathbb{R}^n)$, set

$$(\mathcal{F} f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi x}\, dx \qquad (\xi \in \mathbb{R}^n)$$

Note: $\mathcal{F} : L^1(\mathbb{R}^n) \to BC(\mathbb{R}^n)$, $\quad \|\hat{f}\|_u \le \|f\|_1$

Theorem 8.22. For all $f, g \in L^1(\mathbb{R}^n)$:

a) $(\tau_y f)^\wedge(\xi) = e^{-2\pi i \xi y} \cdot \hat{f}(\xi)$ and $\tau_\eta(\hat{f}) = \hat{h}$, $h(x) = e^{2\pi i \eta x} f(x)$.

b) $\forall T \in GL_n(\mathbb{R})$: $(f \circ T)^\wedge = |\det T|^{-1} \cdot \hat{f} \circ (T^*)^{-1}$

Special cases: If T is a rotation then $(f \circ T)^\wedge = \hat{f} \circ T$

If $T(x) \equiv t^{-1} x$ (some $t > 0$) then $(f \circ T)^\wedge(\xi) = t^n \cdot \hat{f}(t\xi)$.

c) $(f * g)^\wedge = \hat{f}\,\hat{g}$

d) If $x^\alpha f \in L^1$ for all α with $|\alpha| \le k$, then $\hat{f} \in C^k$ and $\partial^\alpha \hat{f} = [(-2\pi i x)^\alpha f]^\wedge$.

e) If $f \in C^k$, $\partial^\alpha f \in L^1$ for $|\alpha| \le k$ and $\partial^\alpha f \in C^0$ for $|\alpha| \le k-1$

then $(\partial^\alpha f)^\wedge(\xi) = (2\pi i \xi)^\alpha \cdot \hat{f}(\xi)$.

f) $\mathcal{F}(L^1(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$ (Riemann–Lebesgue)

DEF (inverse Fourier transform)

For $f \in L^1(\mathbb{R}^n)$, $\check{f}(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} f(\xi)\, e^{2\pi i \xi x}\, d\xi$

We also write $\tilde{f}(x) = f(-x)$. Then $\check{f} = \hat{\tilde{f}} = \tilde{\hat{f}}$.

Fourier Inversion Theorem (8.26):

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$ then $(\hat{f})^\vee = (\check{f})^\wedge \in C_0(\mathbb{R}^n)$ and $(\hat{f})^\vee = f$ a.e.

Lemma: If $f, g \in L^1(\mathbb{R}^n)$ then $f * \check{g} = (\hat{f} \cdot g)^\vee$.

proof: $f * \check{g}(x) = \int f(y)\, \check{g}(x-y)\, dy = \iint f(y)\, g(z)\, e^{2\pi i (x-y) z}\, dz\, dy$.

$(\hat{f} \cdot g)^\vee(x) = \int \hat{f}(z)\, g(z)\, e^{2\pi i x z}\, dz = \iint f(y)\, e^{-2\pi i y z}\, g(z)\, e^{2\pi i x z}\, dy\, dz$.

Fubini $\Rightarrow$ Equal!

Explicit approximate unit?

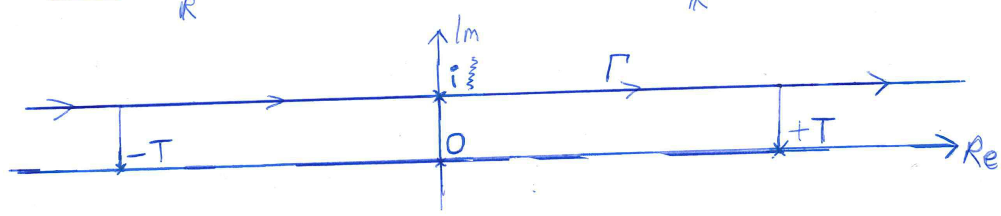
Prop 8.24: If $\phi(x) = e^{-\pi |x|^2}$ then $\hat{\phi}(\xi) = e^{-\pi |\xi|^2}$.

proof: $\hat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-\pi |x|^2} e^{-2\pi i \xi x}\, dx = \prod_{j=1}^{n} \int_{\mathbb{R}} e^{-\pi x_j^2 - 2\pi i \xi_j x_j}\, dx_j$

$\boxed{\text{If Prop 8.24 holds for } n=1} \longrightarrow \; = \prod_{j=1}^{n} e^{-\pi \xi_j^2} = e^{-\pi |\xi|^2}$.

Remains verifying for $n=1$:

$\hat{\phi}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2 - 2\pi i \xi x}\, dx = e^{-\pi \xi^2} \cdot \int_{\mathbb{R}} \exp\left(-\pi (x + i\xi)^2\right) dx = e^{-\pi \xi^2} \int_{\Gamma} e^{-\pi z^2}\, dz$



change contour! $= e^{-\pi \xi^2} \int_{\mathbb{R}} e^{-\pi z^2}\, dz = e^{-\pi \xi^2}$ ← Prop 2.53 or "$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$"

Explicit approximate unit?

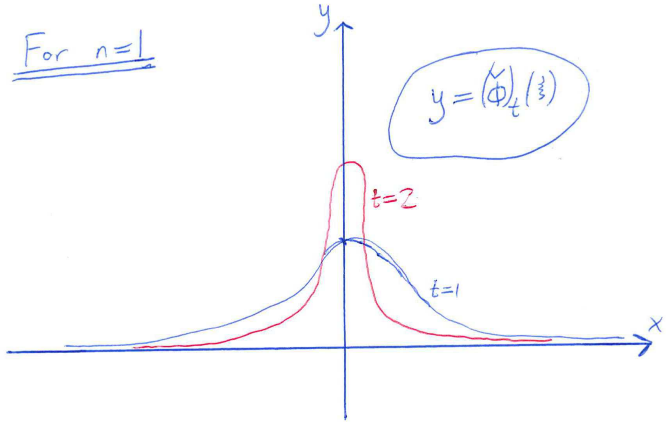
Let $\underline{\underline{\phi(x) = e^{-\pi|x|^2}}}$. Then $\underline{\underline{\check{\Phi}(\xi) = e^{-\pi|\xi|^2}}}$.

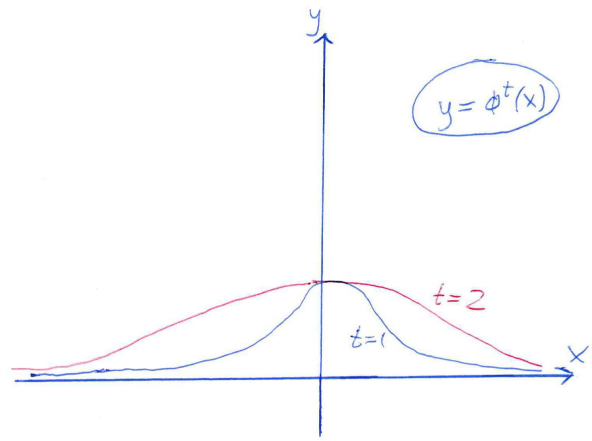
For $t > 0$: $\underline{\underline{(\check{\Phi})_t(\xi) = t^{-n}\, \check{\Phi}(t^{-1}\xi)}}$ (appr. unit for $*$ as $t \to 0$)

- We have $\underline{\underline{(\check{\Phi})_t}} = (\Phi^t)^\vee$, where $\underline{\underline{\phi^t(x) = \phi(tx)}}$ (true for *any* $\phi \in L^1$)

Note $\boxed{\int_{\mathbb{R}^n} \check{\Phi}(\xi)\, d\xi = 1 \quad \text{and} \quad \phi(0) = 1}$

For $n=1$





One alternative: The Poisson kernel

$$\boxed{\text{Lemma: If } \phi(\xi) = e^{-2\pi|\xi|} \text{ then } \check{\Phi}(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\left(1+|x|^2\right)^{-\frac{n+1}{2}}.}$$

Again: $\boxed{\phi, \check{\phi} \in BC \cap L^1, \quad \int_{\mathbb{R}^n} \check{\Phi}(\xi)\, d\xi = 1, \quad \phi(0) = 1.}$

proof of the Fourier Inversion Theorem

Fix some $\phi \in L^1 \cap BC$ with $\check{\phi} \in L^1 \cap BC$, $\int_{\mathbb{R}^n} \check{\phi} = 1$, $\phi(0) = 1$.

Set $\phi^t(x) = \phi(tx)$; then $(\phi^t)^\vee(\xi) = (\check{\phi})_t(\xi) = t^{-n}\check{\phi}(t^{-1}\xi)$.

Apply $\boxed{f * \check{g} = (\hat{f} \cdot g)^\vee}$ with $g = \phi^t$:

$$\Rightarrow \quad \underbrace{(f * (\phi^t)^\vee)(x)}_{\to f \text{ in } L^1} = \underbrace{(\hat{f} \cdot \phi^t)^\vee(x)}_{\to (\hat{f})^\vee(x) \text{ for each } x \in \mathbb{R}^n}, \quad \forall x \in \mathbb{R}^n, \ t > 0.$$

Let $t \to 0$ — by the Dominated Convergence Theorem.

The limits must be equal (e.g. by Cor. 2.32), i.e. $f(x) = (\hat{f})^\vee(x)$ for a.e. x.

□

Fourier Inversion Theorem (8.26):
If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$ then $(\hat{f})^\vee = (\check{f})^\wedge \in C_0(\mathbb{R}^n)$ and $(\hat{f})^\vee = f$ a.e.

$\Rightarrow$ $\mathcal{F}$ (as a map $L^1 \to BC$) is injective.

The proof of Thm 8.26 shows how to recover f from $\hat{f}$ even if $\hat{f} \notin L^1$:

For any $f \in L^1$, we have $(\hat{f} \cdot \phi^t)^\vee \xrightarrow{t \to 0} f$ in L^1.

See also Thm 8.35.

Application to the heat equation

For $u(x,t)$ $(x \in \mathbb{R}^n,\ t \geq 0)$: $\begin{cases} (\partial_t - \Delta_x) u = 0 \\ u(x,0) = f(x) \end{cases}$

$$\Delta_x u = \sum_{j=1}^{n} \partial_{x_j}^2 u$$

$$\because \quad \widehat{\Delta_x u}(\xi) = \sum_{j=1}^{n} (2\pi i \xi_j)^2 \hat{u}(\xi) = -4\pi |\xi|^2 \cdot \hat{u}(\xi)$$

Thus $\begin{cases} (\partial_t + 4\pi^2|\xi|^2)\hat{u} = 0 \\ \hat{u}(\xi, 0) = \hat{f}(\xi) \end{cases}$

$$\Rightarrow \hat{u}(\xi, t) = \hat{f}(\xi)\, e^{-4\pi^2|\xi|^2 t}$$

Set $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-|x|^2/4t}$; then $\widehat{G_t}(\xi) = e^{-4\pi^2|\xi|^2 t}$,

and so $\hat{u}(\xi, t) = \hat{f}(\xi) \cdot \widehat{G_t}(\xi) = \widehat{f * G_t}(\xi)$.

Hence: $\boxed{u(x,t) = (f * G_t)(x)}$