

#9. The J-Bessel function (cf. Lecture Notes, Sec. 8)

DEF: The J-Bessel function is defined by the following formula,
for $\nu \in \mathbb{C}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$: $J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m}$

The function $f(z) = J_\nu(z)$ solves the ODE

$$f''(z) + \frac{1}{z} f'(z) + \left(1 - \frac{\nu^2}{z^2}\right) f(z) = 0.$$

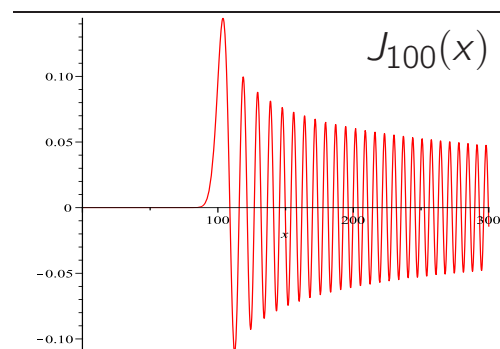
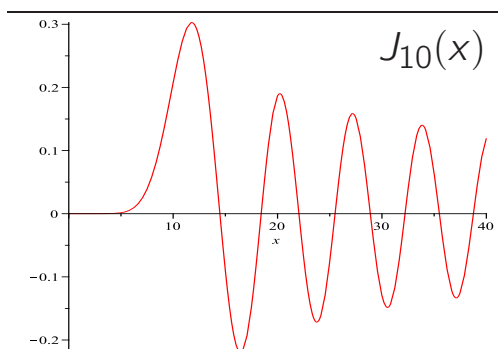
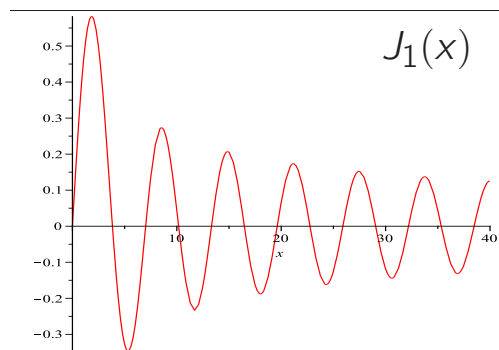
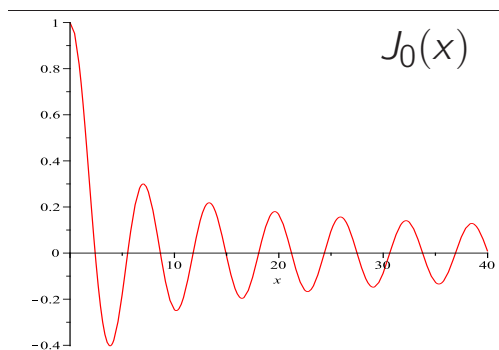
(Use Frobenius method; $f(z) = z^\nu \sum_{n=0}^{\infty} a_n z^n$, $a_0 \neq 0$.)

If $\nu \notin \mathbb{Z}$: $\{J_\nu(z), J_{-\nu}(z)\}$ is a fundamental system of solutions.

If $\nu \in \mathbb{Z}$: $J_{-\nu}(z) \equiv (-1)^\nu J_\nu(z)$

Recurrence relations: $J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z),$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2 J'_\nu(z).$$



Alternative formula:

For $\operatorname{Re}(v) > -\frac{1}{2}$:

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 e^{izt} (1-t^2)^{\nu-\frac{1}{2}} dt,$$

Proof outline: Plug in $e^{izt} = \sum_{n=0}^{\infty} \frac{(izt)^n}{n!}$ in the right hand side.

Lebesgue Dominated Conv. Thm.

$$\Rightarrow \int_{-1}^1 e^{izt} (1-t^2)^{\nu-\frac{1}{2}} dt = \sum_{n=0}^{\infty} \int_{-1}^1 \frac{(izt)^n}{n!} (1-t^2)^{\nu-\frac{1}{2}} dt$$

$$\begin{aligned} \text{For } n=2m: \int_{-1}^1 t^{2m} (1-t^2)^{\nu-\frac{1}{2}} dt &= 2 \int_0^1 t^{2m} (1-t^2)^{\nu-\frac{1}{2}} dt = 2 \int_0^1 u^m (1-u)^{\nu-\frac{1}{2}} \frac{du}{2\sqrt{u}} \\ &= \int_0^1 u^{m-\frac{1}{2}} (1-u)^{\nu-\frac{1}{2}} du = \frac{\Gamma(m+\frac{1}{2}) \Gamma(\nu+\frac{1}{2})}{\Gamma(m+\nu+1)} \end{aligned}$$

Hence get the result.

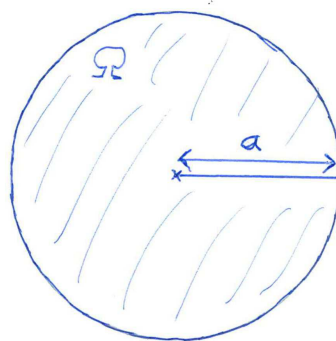
Application: Dirichlet eigenfunctions in a disk

Let $\Omega = \{x \in \mathbb{R}^2 : |x| < a\}$

We are seeking solutions $\lambda \geq 0$

and $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ to

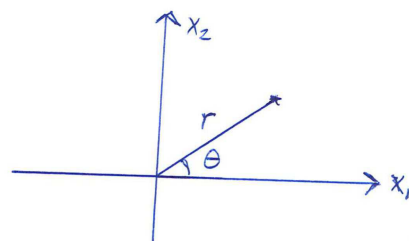
$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$



Recall: $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$

In polar coordinates,
 $(x_1, x_2) = r \cdot (\cos \theta, \sin \theta)$

Ansatz: $u(r, \theta) = R(r) \cdot \Phi(\theta)$



Then need:
$$\begin{cases} (R''(r) + \frac{1}{r} R'(r) + \lambda R(r)) \cdot \phi(\theta) = -\frac{1}{r^2} R(r) \cdot \phi''(\theta) & 0 < r < a \\ R(0+) \text{ exists} \\ R(a-) = 0 \\ \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi) \end{cases}$$

If non-vanishing solution, then $\exists \mu \in \mathbb{R}$ s.t.
$$\begin{cases} R''(r) + \frac{1}{r} R'(r) + \lambda R(r) = \frac{\mu}{r^2} R(r) & \text{on } (0, a) \\ \phi''(\theta) = -\mu \phi(\theta) & \text{on } \mathbb{R}/2\pi\mathbb{Z} \end{cases}$$

The conditions on ϕ imply:
$$\begin{aligned} &\underline{\mu = n^2} \text{ for some } \underline{n \in \mathbb{Z}_{\geq 0}} \\ &\text{and } \underline{\phi(\theta) = A \cos(n\theta) + B \sin(n\theta)} \\ &\text{for some } A, B \in \mathbb{R}. \end{aligned}$$

Setting $\Psi(s) := R(\frac{s}{\sqrt{\lambda}})$, the ODE for R becomes:
$$\underline{\Psi''(s) + \frac{1}{s} \Psi'(s) + (1 - \frac{n^2}{s^2}) \Psi(s) = 0}$$

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With $[\Psi(0) \text{ exists}]$, this implies $\underline{\Psi(s) = [\text{const}] \cdot J_n(s)}$

Finally we also require $\underline{\Psi(\sqrt{\lambda} a) = 0}$.

$\leadsto$ The 'general' solution to the Dirichlet eigenvalue problem in Ω is given by

$$\begin{aligned} &\underline{\lambda = \left(\frac{j_{n,m}}{a}\right)^2}, \quad \underline{u_{n,m}(r, \theta) = J_n\left(\frac{j_{n,m}}{a} r\right) \cdot (A_n \cos n\theta + B_n \sin n\theta)} \\ &\quad (\langle n, m \rangle \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}) \\ &\text{where } \underline{0 < j_{n,1} < j_{n,2} < \dots} \text{ are the zeros of } J_n(x) \text{ for } x > 0. \end{aligned}$$

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Application: The Fourier transform of radial functions

Let σ be the standard surface measure on S_1^{n-1} ($\subset \mathbb{R}^n$).

$$\text{Then } \underline{\hat{\sigma}}\left(\frac{z}{|z|}\right) = \int_{S_1^{n-1}} e^{-2\pi i \frac{z}{|z|} \cdot w} d\sigma(w) = \int_{S_1^{n-1}} e^{-2\pi i |z| \cdot w_1} d\sigma(w),$$

↑
rotational symmetry

Let $P: \mathbb{R}^n \rightarrow \mathbb{R}$; $P(w_1, \dots, w_n) = w_1$. Then we get:

$$\begin{aligned} &= \int_{-1}^1 e^{-2\pi i |z| \cdot w_1} d(P_*\sigma)(w_1) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \cdot \int_{-1}^1 e^{-2\pi i |z| \cdot w_1} (1-w_1^2)^{\frac{n-3}{2}} dw_1 \\ &= \underline{\underline{2\pi \cdot |z|^{1-\frac{n}{2}} \cdot J_{\frac{n}{2}-1}(2\pi |z|)}}} \end{aligned}$$

Thus: $\hat{\sigma}\left(\frac{z}{|z|}\right) = 2\pi \cdot |z|^{1-\frac{n}{2}} \cdot J_{\frac{n}{2}-1}(2\pi |z|)$

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Hence for any radial function $F \in L^1(\mathbb{R}^n)$, say $F\left(\frac{z}{|z|}\right) = f(|z|)$:

$$\begin{aligned} \underline{\underline{\hat{F}\left(\frac{z}{|z|}\right)}} &= \int_{\mathbb{R}^n} F(x) e^{-2\pi i \frac{z}{|z|} \cdot x} dx = \int_0^\infty \int_{S_1^{n-1}} f(r) e^{-2\pi i \frac{z}{|z|} \cdot rw} r^{n-1} d\sigma(w) dr \\ &= \int_0^\infty f(r) \left(\int_{S_1^{n-1}} e^{-2\pi i \frac{z}{|z|} \cdot rw} d\sigma(w) \right) \cdot r^{n-1} dr \\ &= \int_0^\infty f(r) \hat{\sigma}\left(\frac{z}{|z|}\right) \cdot r^{n-1} dr \\ &= \underline{\underline{2\pi |z|^{1-\frac{n}{2}} \cdot \int_0^\infty f(r) \cdot r^{\frac{n}{2}} \cdot J_{\frac{n}{2}-1}(2\pi r |z|) dr =: \tilde{f}(|z|)}}} \end{aligned}$$

Inversion - SAME formula: $f(r) = 2\pi r^{1-\frac{n}{2}} \int_0^\infty \tilde{f}(y) y^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi y r) dy$

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Asymptotic formula for $J_\nu(x)$ as $x \rightarrow \infty$?

First order asymptotic:

$$J_\nu(x) = \underbrace{\sqrt{\frac{2}{\pi x}} \left(\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1}) \right)}_{:= F_\nu^{(1)}(x)} \quad \text{as } x \rightarrow \infty \quad (\nu \text{ fixed}).$$

More precise (“second order”) asymptotic formula:

$$J_\nu(x) = \underbrace{\sqrt{\frac{2}{\pi x}} \left(\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \cdot \frac{\frac{1}{2}\nu^2 - \frac{1}{8}}{x} + O(x^{-2}) \right)}_{:= F_\nu^{(2)}(x)}$$

as $x \rightarrow \infty$ (ν fixed).

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Asymptotic expansion: For any fixed $N \in \mathbb{Z}^+$ and $\nu \in \mathbb{C}$,

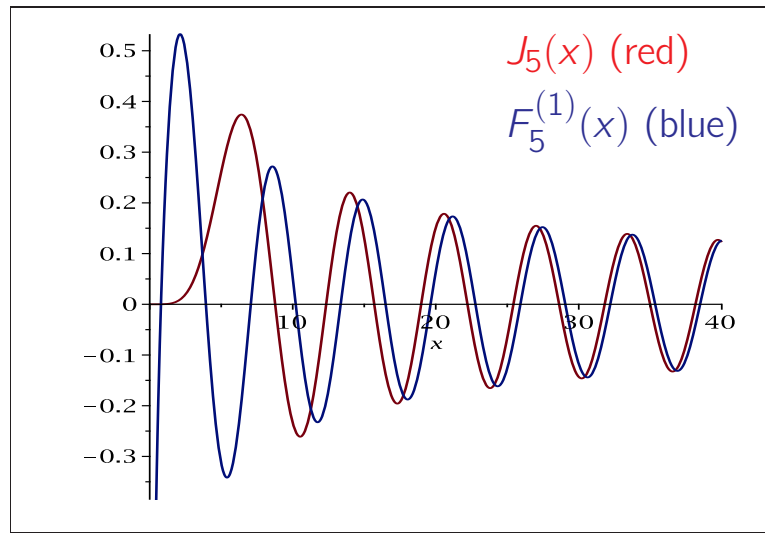
$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left(\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \cdot \sum_{0 \leq k < N/2} (-1)^k A_{\nu, 2k} x^{-2k} \right. \\ \left. - \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \cdot \sum_{0 \leq k < (N-1)/2} (-1)^k A_{\nu, 2k+1} x^{-2k-1} + O(x^{-N}) \right)$$

as $x \rightarrow \infty$ (ν fixed), with

$$A_{\nu, n} := \frac{\prod_{j=-n}^{n-1} (\nu + \frac{1}{2} + j)}{2^n \cdot n!}.$$

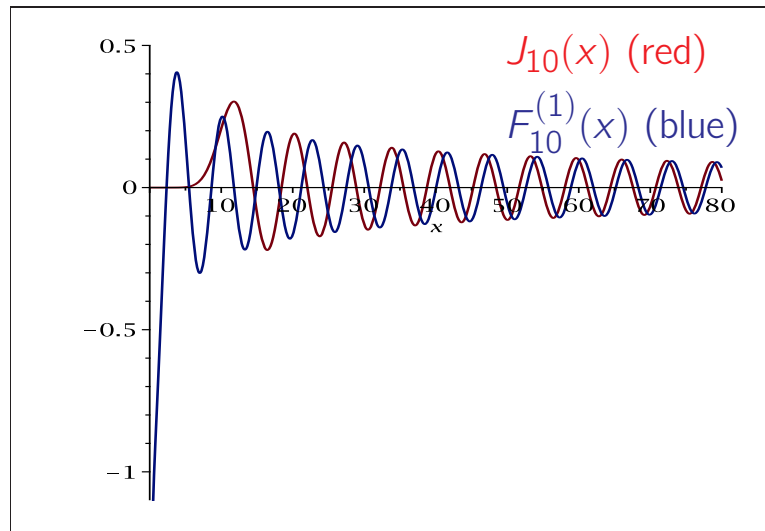
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$$\nu = 5$$

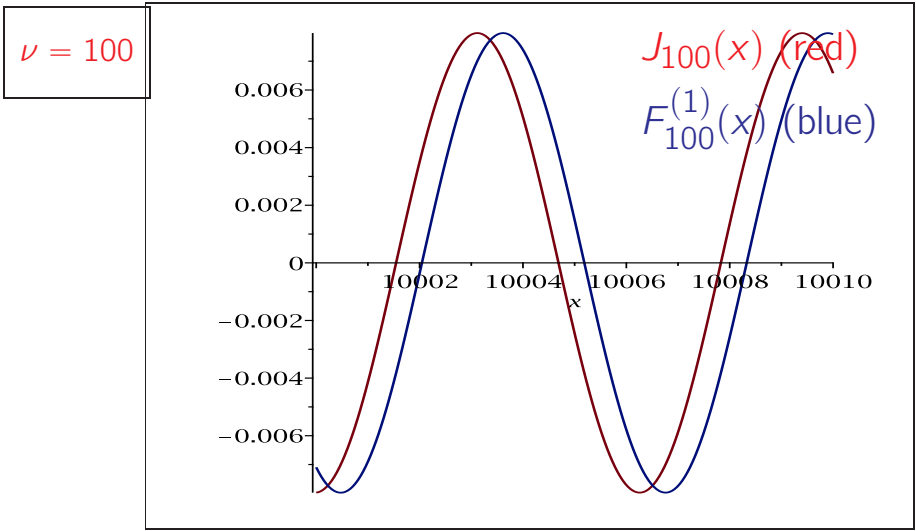
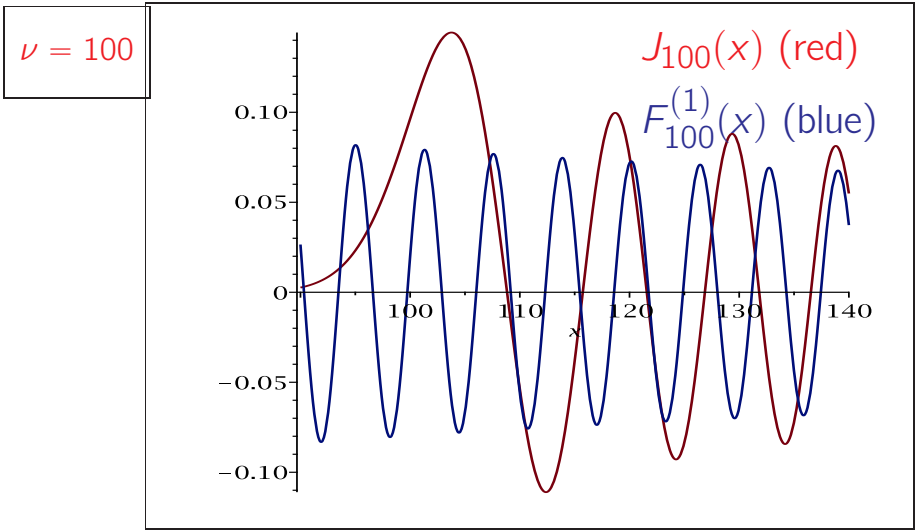


x	$J_5(x)$	$J_5(x) - F_5^{(1)}(x)$	$J_5(x) - F_5^{(2)}(x)$
1	2.497577e-04	-1.70e-01	-9.82e+00
10	-2.340615e-01	-2.87e-01	1.87e-02
10^2	-7.419574e-02	3.02e-03	5.39e-04
10^3	5.025407e-03	3.06e-04	-3.38e-07
10^4	3.638933e-03	-8.78e-06	-2.57e-09
10^5	1.846551e-03	-2.13e-07	-1.30e-11

$$\nu = 10$$



x	$J_{10}(x)$	$J_{10}(x) - F_{10}^{(1)}(x)$	$J_{10}(x) - F_{10}^{(2)}(x)$
1	2.630615e-10	7.80e-01	-7.69e+00
10	2.074861e-01	-3.93e-02	-3.02e-01
10^2	-5.473218e-02	-3.46e-02	3.86e-03
10^3	-2.452062e-02	2.65e-04	3.01e-05
10^4	7.114312e-03	1.81e-05	-8.66e-08
10^5	1.720124e-03	9.21e-07	-2.10e-10



x	$J_{100}(x)$	$J_{100}(x) - F_{100}^{(1)}(x)$	$J_{100}(x) - F_{100}^{(2)}(x)$
1	8.431829e-189	-7.80e-01	8.49e+02
10	6.597316e-89	2.47e-01	2.66e+01
10^2	9.636667e-02	7.63e-02	-3.78e+00
10^3	1.167614e-02	-1.31e-02	1.05e-02
10^4	-7.976516e-03	-8.80e-04	9.44e-04
10^5	-1.809353e-03	-9.01e-05	2.19e-06
10^6	3.346687e-04	3.63e-06	-4.15e-09
10^7	-8.695579e-05	-1.18e-07	1.09e-11

Uniform asymptotics for $J_\nu(x)$, for ν large (Olver, 1954)

$$J_\nu(\nu t) = \nu^{-\frac{1}{3}} \left(\frac{4\zeta}{1-t^2} \right)^{1/4} \left(\text{Ai}(\xi) + O\left(\nu^{-1} \frac{e^{-\frac{2}{3}(\xi^+)^{3/2}}}{(1+|\xi|)^{1/4}} \right) \right) \quad (\nu \geq 1, t > 0).$$

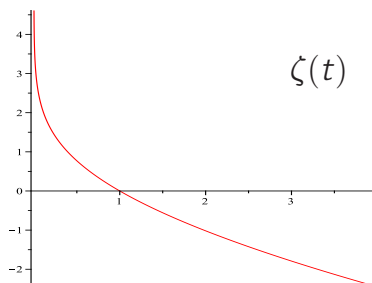
— with an *absolute* implied constant.

Here:

$$\xi = \nu^{2/3} \zeta \quad \text{and} \quad \xi^+ = \max(\xi, 0);$$

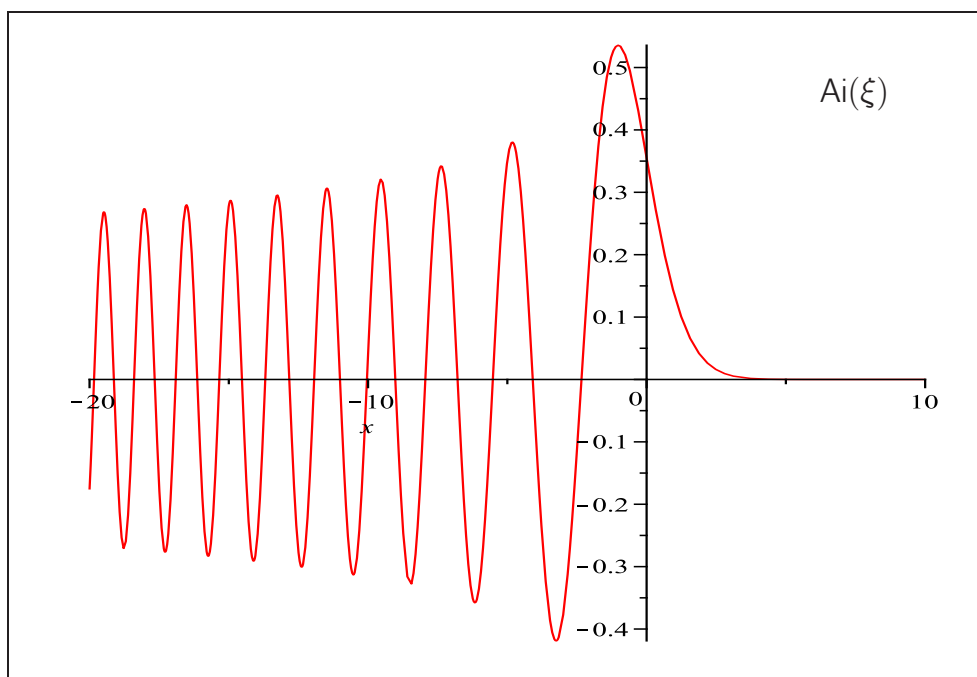
$$\zeta = \zeta(t) = \left(\frac{3}{2} u(t) \right)^{2/3} \text{sgn}(1-t);$$

$$u(t) = \begin{cases} \text{arctanh}(\sqrt{1-t^2}) - \sqrt{1-t^2} & \text{if } 0 < t \leq 1 \\ \sqrt{t^2-1} - \arctan(\sqrt{t^2-1}) & \text{if } t \geq 1. \end{cases}$$



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The Airy function:



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Uniform asymptotics in simplified form:

Fix an arbitrary $C > 0$. Then for all $\nu \geq 1$ and $x > 0$ we have

$$J_\nu(x) = \begin{cases} \frac{e^{\sqrt{\nu^2 - x^2}}}{\sqrt{2\pi} \sqrt[4]{\nu^2 - x^2} \left(\frac{\nu}{x} + \sqrt{\left(\frac{\nu}{x}\right)^2 - 1}\right)^\nu} \left(1 + O\left(\frac{\sqrt{\nu}}{(\nu - x)^{3/2}}\right)\right) & \text{if } x \leq \nu - C\nu^{\frac{1}{3}} \\ O(\nu^{-\frac{1}{3}}) & \text{if } |x - \nu| \leq C\nu^{\frac{1}{3}} \\ \frac{\sqrt{2}}{\sqrt{\pi} \sqrt[4]{x^2 - \nu^2}} \left\{ \cos\left(\sqrt{x^2 - \nu^2} - \nu \arccos\left(\frac{\nu}{x}\right) - \frac{\pi}{4}\right) + O\left(\frac{\sqrt{\nu}}{(x - \nu)^{3/2}} + \frac{1}{\nu}\right) \right\} & \text{if } x \geq \nu + C\nu^{\frac{1}{3}}. \end{cases}$$

The implied constants *depend only on C*.

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