#7. More on Fourier Analysis

Fourier Analysis on the torus

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \left[\mathbb{R}^n/\sim \quad \text{where } [x \sim y \overset{\text{def}}{\Longleftrightarrow} x-y \in \mathbb{Z}^n]\right] = \text{"}[0,1)^n\text{"} \text{ or } \text{"}[-\tfrac{1}{2},\tfrac{1}{2})^n\text{"}$$

Measure space: $(\mathbb{T}^n, \mathcal{B}_{\mathbb{T}^n}, m)$, with $m(\mathbb{T}^n)=1$.

To give $f: \mathbb{T}^n \to \mathbb{C}$ $\Longleftrightarrow$ Giving a $\mathbb{Z}$-periodic $f: \mathbb{R}^n \to \mathbb{C}$.

Function spaces: $C(\mathbb{T}^n)$, $L^p(\mathbb{T}^n)$, $C^k(\mathbb{T}^n)$, ...

Convolution: $f * g(x) = \int_{\mathbb{T}^n} f(x-y)\, g(y)\, dy$ for $f, g \in L^1(\mathbb{T}^n)$ (e.g.)

TYPO 8.19, 8.20

Dual group: $\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n$, isomorphism $k \mapsto E_k$ $(\mathbb{Z}^n \to \widehat{\mathbb{T}^n})$, $E_k(x) = e^{2\pi i k x}$

Fourier transform: For $f \in L^1(\mathbb{T}^n)$; $\hat{f} \in B\cancel{C}(\mathbb{Z}^n)$, $\hat{f}(k) = \int_{\mathbb{T}^n} f(x)\, e^{-2\pi i k x}\, dx$.

Fourier inversion: For $f \in L^1(\mathbb{T}^n)$, if $\sum_{k \in \mathbb{Z}^n} |\hat{f}(k)| < \infty$ then $f = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) E_k$ a.e.

True for $L^2(X)$ of any measure space X!

L^2- theory

Note $L^2(\mathbb{T}^n) \subset L^1(\mathbb{T}^n)$, and $L^2(\mathbb{T}^n)$ is a Hilbert space,

with $\langle f, g \rangle = \int_{\mathbb{T}^n} f \bar{g} \, dm$.

$$\boxed{\text{Thm 8.20: } \{E_k : k \in \mathbb{Z}^n\} \text{ is an ON-basis of } L^2(\mathbb{T}^n).}$$

Key: The finite $\mathbb{C}$-linear combinations of $\{E_k\}_{k \in \mathbb{Z}}$ are dense in $C(\mathbb{T}^n)$, by Stone-Weierstrass.

Hence: $f \mapsto \hat{f}$ is a Hilbert space isomorphism $L^2(\mathbb{T}^n) \xrightarrow{\sim} \ell^2(\mathbb{Z}^n)$;

$\|f\|_2^2 = \sum_{k \in \mathbb{Z}^n} |\hat{f}(k)|^2$, $\forall f \in L^2(\mathbb{T}^n)$

L^2-theory for $\mathbb{R}^n$

Recall that $L^2(\mathbb{R}^n)$ is a Hilbert space; $\langle f,g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx$.

The Plancherel Theorem (8.29): $\mathcal{F}$ maps $L^1 \cap L^2$ into L^2, and $\mathcal{F}|_{L_1 \cap L_2}$ extends uniquely to a unitary isomorphism $L^2 \xrightarrow{\sim} L^2$.

Thus: $\int_{\mathbb{R}^n} f\bar{g} = \int_{\mathbb{R}^n} \hat{f}\,\overline{\hat{g}}$ for all $f,g \in L^2(\mathbb{R}^n)$.

proof, outline: KEY: Prove $\circledast$ $\boxed{\int_{\mathbb{R}^n} f\bar{g} = \int_{\mathbb{R}^n} \hat{f}\,\overline{\hat{g}}}$ for nice f,g (eg C_c^∞).

We know: $\int_{\mathbb{R}^n} f\hat{h} = \int_{\mathbb{R}^n} \hat{f} h$, $\forall f,h \in L^1(\mathbb{R}^n)$.

For $g \in L^1$ with $\hat{g} \in L^1$, set $h = \overline{\check{g}}$; then $\hat{h} = \bar{g}$ by Fourier Inversion; also $h = \overline{\hat{g}}$. Hence: $\boxed{\circledast \text{ holds, } \forall f \in L^1,\ g \in L^1 \text{ with } \hat{g} \in L^1}$

For $f \in L^2$ ((not in L^1)): Take any sequence $f_1, f_2, \ldots \in L^1 \cap L^2$ with $f_n \to f$ in L^2; then $\hat{f} = \underbrace{\lim_{n\to\infty}}_{\text{limit in } L^2} \hat{f}_n$.

Consequence:

The Hausdorff-Young Inequality (8.30): Suppose $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Then $\forall f \in L^p(\mathbb{R}^n)$: $\hat{f} \in L^q(\mathbb{R}^n)$ and $\|\hat{f}\|_q \le \|f\|_p$

proof: True for $p=1$ and for $p=2$.
Hence true for all $1 \le p \le 2$ by Riesz–Thorin.

Fourier analysis of measures (Ch. 8.6)

DEF: For $\mu, \nu \in M(\mathbb{R}^n)$, define $\mu * \nu \in M(\mathbb{R}^n)$ by

$$(\mu * \nu)(E) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_E(x+y)\, d\mu(x)\, d\nu(y) \qquad (E \in \mathcal{B}_{\mathbb{R}^n})$$

and define $\hat{\mu} \in BC(\mathbb{R}^n)$ by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x}\, d\mu(x) \qquad (\xi \in \mathbb{R}^n)$$

Fact: $\hat{\mu}$ is uniformly continuous, and $\|\hat{\mu}\|_u \leq \|\mu\|$.

Ex: $\delta_a * \delta_b = \delta_{a+b}$ $\quad (a, b \in \mathbb{R}^n)$

$\delta_a * \mu = \tau_a(\mu)$, $\quad (\mu \in M(\mathbb{R}^n),\ a \in \mathbb{R}^n)$

Properties {Prop 8.48 + p. 272...}

a) Convolution on $M(\mathbb{R}^n)$ is commutative and associative.

b) For $h : \mathbb{R}^n \to \mathbb{C}$ bounded and Borel measurable: $\int_{\mathbb{R}^n} h \, d(\mu * \nu) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(x+y) \, d\mu(x) \, d\nu(y)$

c) $\|\mu * \nu\| \leq \|\mu\| \cdot \|\nu\|$

d) If $\mu = f \cdot m$ and $\nu = g \cdot m$ where $f, g \in L^1(\mathbb{R}^n)$ then $\mu * \nu = (f * g) \cdot m$.

d') If $\mu = f \cdot m$ then $\hat{\mu} = \hat{f}$.

e) For $\mu, \nu \in M(\mathbb{R}^n)$: $\widehat{\mu * \nu} = \hat{\mu} \cdot \hat{\nu}$.

Ex: $(\tau_y f)^\wedge(\xi) = (\delta_y * f)^\wedge(\xi) = \hat{\delta}_y(\xi) \, \hat{f}(\xi) = e^{-2\pi i \xi \cdot y} \cdot \hat{f}(\xi)$ (This is Thm 8.22 (a))

The Poisson Summation Formula (8.32)

Assume that $f \in C(\mathbb{R}^n)$ satisfies $|f(x)| \ll (1+|x|)^{-n-\varepsilon}$ and $|\hat{f}(\xi)| \ll (1+|\xi|)^{-n-\varepsilon}$ for some fixed $\varepsilon > 0$. Then $\underline{\underline{\sum_{k \in \mathbb{Z}^n} f(k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)}}$.

proof, outline: Define $g: \mathbb{R}^n \to \mathbb{C}$ by $g(x) = \sum_{k \in \mathbb{Z}^n} f(k+x)$.

"Periodization" of f.

Then g is $\mathbb{Z}^n$-periodic, i.e. we can view g as $\mathbb{T}^n \to \mathbb{C}$.

Set $Q = [-\frac{1}{2}, \frac{1}{2})^n$.

Now for each $k \in \mathbb{Z}^n$,

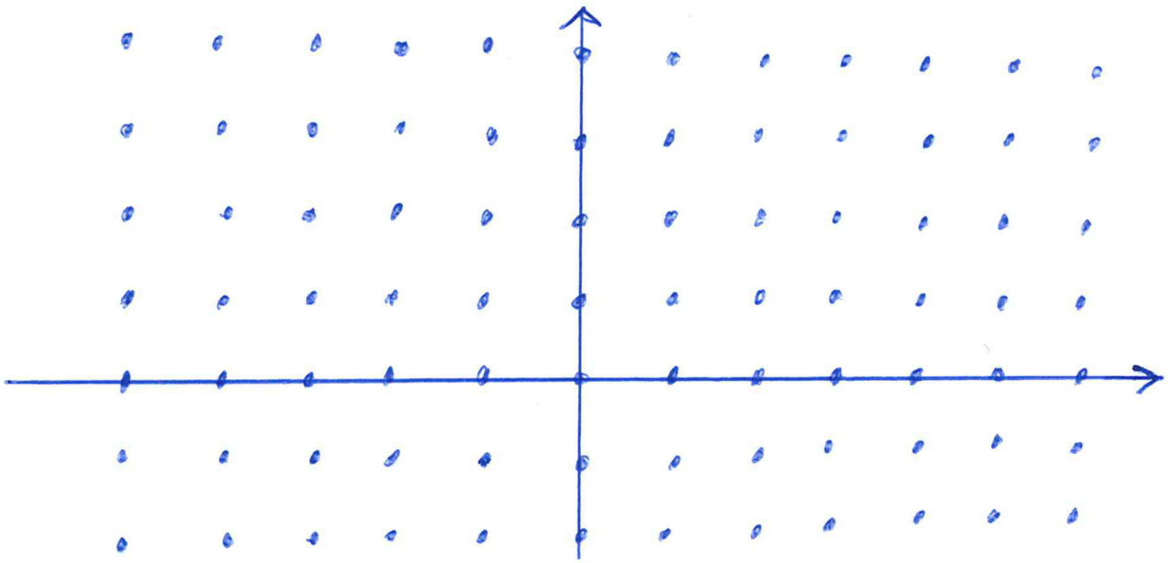
$$\hat{g}(k) = \int_{\mathbb{T}^n} g(x)\, e^{-2\pi i k \cdot x}\, dx = \int_{\mathbb{T}^n} \sum_{k \in \mathbb{Z}^n} f(x+k)\, e^{-2\pi i k \cdot x}\, dx = \sum_{k \in \mathbb{Z}^n} \int_Q f(x+k) e^{-2\pi i k \cdot x}\, dx$$

$$= \sum_{k \in \mathbb{Z}^n} \int_{Q+k} f(x)\, e^{-2\pi i k \cdot x}\, dx = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i k \cdot x}\, dx = \hat{f}(k)$$

Also $g(x) = \sum_{k \in \mathbb{Z}^n} \hat{g}(k) e^{2\pi i k x} = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)\, e^{2\pi i k \cdot x}$. Here take $x = 0$!

Ex: Counting integer points in large convex sets (Lecture notes, Sec. 6)

Given a nice set E, get precise asymptotics for $\#(\mathbb{Z}^n \cap RE)$ as $R \to \infty$.



Note $\#(\mathbb{Z}^n \cap RE) = \sum_{k \in \mathbb{Z}^n} \chi_{RE}(k)$

Poisson summation formula $\rightsquigarrow \sum_{k \in \mathbb{Z}^n} \widehat{\chi_{RE}}(k)$?

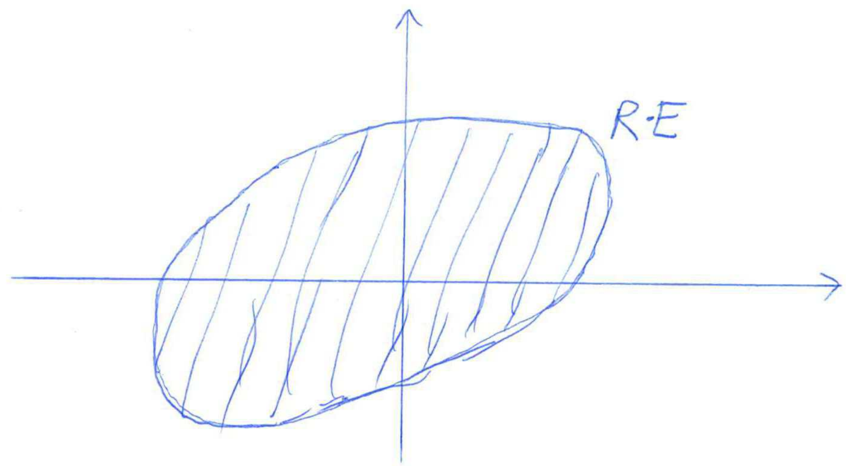
NO!

– Approximate χ_{RE} by a smoother function!

Standard approach: Consider $\chi_{RE} * \phi_\delta$ with ϕ a C_c^∞ "bump function"...

Technically simpler: $\boxed{\text{Assume } E \text{ convex \& open \& bounded}}$, and then

consider $\chi_{RE} * \chi_{hE}$ for some small $h \neq 0$.



For any $R > h > 0$:

$$\chi_{RE} \geq \frac{1}{\mathrm{vol}(hE)} \chi_{(R-h)E} * \chi_{hE}$$

For any $R, h > 0$:

$$\chi_{RE} \leq \frac{1}{\mathrm{vol}(hE)} \chi_{(R+h)E} * \chi_{-hE}$$

NOW the Poisson summation formula applies!

$$\underline{\underline{\#(\mathbb{Z}^n \cap RE)}} = \sum_{k\in\mathbb{Z}^n} \chi_{RE}(k) \geq \frac{1}{\mathrm{vol}(hE)} \sum_{k\in\mathbb{Z}^n} \left(\chi_{(R-h)E} * \chi_{hE}\right)(k)$$

$$= \frac{1}{\mathrm{vol}(hE)} \sum_{k\in\mathbb{Z}^n} \widehat{\chi_{(R-h)E}}(k) \cdot \widehat{\chi_{hE}}(k) = \underline{\underline{\frac{(R-h)^n\, h^n}{h^n\, \mathrm{vol}(E)} \sum_{k\in\mathbb{Z}^n} \widehat{\chi_E}((R-h)k) \cdot \widehat{\chi_E}(hk)}}$$

$\underline{[k=0]\text{-contribution:}}$ $\frac{(R-h)^n}{\mathrm{vol}(E)} \cdot \mathrm{vol}(E)^2 = \left(R^n + O(R^{n-1}h)\right)\mathrm{vol}(E) = \underline{\underline{\mathrm{vol}(RE) + O(R^{n-1}h)}}$

$\underline{\underline{\text{For } E \text{ "nice"}}}$: $\left|\widehat{\chi_E}(\xi)\right| \ll (1+|\xi|)^{-\frac{n+1}{2}} \qquad \forall\, \xi \in \mathbb{R}^n$

Hence: $\underline{\underline{\#(\mathbb{Z}^n \cap RE) \geq \mathrm{vol}(RE) - O(R^{n-1}h) - O\left(R^n \sum_{\substack{k\in\mathbb{Z}^n \\ (k\neq 0)}} (1+|Rk|)^{-\frac{n+1}{2}} (1+|hk|)^{-\frac{n+1}{2}}\right)}}$

$\underline{\text{Similarly } \leq.}$

Hence:

$$\left|\#(\mathbb{Z}^n \cap RE) - \mathrm{vol}(RE)\right| \ll R^{n-1} h + R^n \underbrace{\sum_{\substack{k \in \mathbb{Z}^n \\ (k \neq 0)}} (1+|Rk|)^{-\frac{n+1}{2}} (1+|hk|)^{-\frac{n+1}{2}}}_{\ll R^{-\frac{n+1}{2}} h^{-\frac{n-1}{2}}}$$

will prove in #8!

Choose $h = R^{-\frac{n-1}{n+1}}$

$$\Longrightarrow \boxed{\#(\mathbb{Z}^n \cap RE) = \mathrm{vol}(RE) + O\left(R^{\frac{(n-1)n}{n+1}}\right)} \quad \forall R \geq 1$$

(For $n = 2$ and E = a disc, this is Sierpinski's $O(R^{2/3})$, (1908) for the Gauss circle problem!)