ON THE GENERALIZED CIRCLE PROBLEM FOR A RANDOM LATTICE IN LARGE DIMENSION

ANDREAS STRÖMBERGSSON AND ANDERS SÖDERGREN

Abstract. In this note we study the error term $R_{n,L}(x)$ in the generalized circle problem for a ball of volume x and a random lattice L of large dimension n. Our main result is the following functional central limit theorem: Fix an arbitrary function $f:\mathbb{Z}^+\to\mathbb{R}^+$ satisfying $\lim_{n\to\infty} f(n)=\infty$ and $f(n)=O_\varepsilon(e^{\varepsilon n})$ for every $\varepsilon>0$. Then, the random function

$$t\mapsto \frac{1}{\sqrt{2f(n)}}R_{n,L}\left(tf(n)\right)$$

on the interval $[0,1]$ converges in distribution to one-dimensional Brownian motion as $n\to\infty$. The proof goes via convergence of moments, and for the computations we develop a new version of Rogers' mean value formula from [19]. For the individual kth moment of the variable $(2f(n))^{-1/2}R_{n,L}(f(n))$ we prove convergence to the corresponding Gaussian moment more generally for functions f satisfying $f(n)=O(e^{cn})$ for any fixed $c\in(0,c_k)$, where c_k is a constant depending on k whose optimal value we determine.

1. Introduction

Gauss' circle problem is a classical problem in number theory asking for the number of integer lattice points inside a Euclidean circle of radius t centered at the origin. Gauss observed that this quantity equals the area $A(t)=\pi t^2$ enclosed by the circle up to an error term of size at most $O(t)$. Hardy conjectured [7] that the error term can be improved to $O_\varepsilon(t^{1/2+\varepsilon})$; a bound which is known to be essentially optimal. Despite efforts of many mathematicians, Hardy's conjecture remains open and the best known bound is $O_\varepsilon(t^{131/208+\varepsilon})$ due to Huxley [12].

In this paper we will be interested in the circle problem generalized to dimension n and a general n-dimensional lattice L of covolume 1. We denote the space of all such lattices by X_n and recall that X_n can be identified with the homogeneous space $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$ via the correspondence $\mathbb{Z}^n g\leftrightarrow \mathrm{SL}(n,\mathbb{Z})g$. As a consequence of this identification, X_n inherits a right $\mathrm{SL}(n,\mathbb{R})$-invariant probability measure μ_n originating from a Haar measure on $\mathrm{SL}(n,\mathbb{R})$.

Given $n\geq 2$, a lattice $L\in X_n$ and a real number $x\geq 0$, we let $N_{n,L}(x)$ denote the number of non-zero lattice points of L in the closed ball of volume x centered at

Date: November 19, 2016.

The first author is supported by a grant from the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine, and also by the Swedish Research Council Grant 621-2011-3629. The second author was partially supported by a postdoctoral fellowship from the Swedish Research Council, by the National Science Foundation under agreement No. DMS-1128155, as well as by a grant from the Danish Council for Independent Research and FP7 Marie Curie Actions-COFUND (grant id: DFF-1325-00058).

the origin in $\mathbb{R}^n$, i.e. we let

$$N_{n,L}(x) := \#\left\{\boldsymbol{m} \in L \setminus \{\boldsymbol{0}\} : |\boldsymbol{m}| \leq \left(\frac{x}{V_n}\right)^{1/n}\right\}, \tag{1.1}$$

where V_n denotes the volume of the unit ball in $\mathbb{R}^n$. We also define, for $x \geq 0$, the function

$$R_{n,L}(x) := N_{n,L}(x) - x,$$

and formulate, for a given $L \in X_n$, the generalized circle problem as the problem of giving the best possible upper bound on $R_{n,L}(x)$ as $x \to \infty$.

In a series of papers Bentkus and Götze [1, 2] and Götze [6] proved strong explicit bounds on $R_{n,L}(x)$ for an arbitrary given lattice $L \in X_n$. In particular, Götze proved in [6] that $|R_{n,L}(x)| = O(x^{1-2/n})$ holds for every $L \in X_n$ when $n \geq 5$. This result is best possible for all rational lattices $L \in X_n$, while for irrational lattices Götze proved the stronger bound $R_{n,L}(x) = o(x^{1-2/n})$ as $x \to \infty$.[1] However, it turns out that for most lattices (in the measure sense) one can do much better. In fact, Schmidt [25] proved that for any $n \geq 2$ and μ_n-almost every $L \in X_n$ we have $R_{n,L}(x) = O_\varepsilon(x^{1/2}(\log x)^{5/2+\varepsilon})$. This upper bound should be compared to Landau's result $R_{n,L}(x) = \Omega(x^{1/2-1/(2n)})$ (cf. [16]). Hence, for large n, Schmidt's bound is close to optimal. In this vein it should also be noted that, for $n \geq 3$, [2]

$$\operatorname{Var}(R_{n,L}(x)) = \mathbb{E}\left(R_{n,L}(x)^2\right) := \int_{X_n} R_{n,L}(x)^2 \, d\mu_n(L) \asymp x \tag{1.2}$$

(cf., e.g., [25, p. 518] or [30, Lemma 3.1]).

In a closely related direction, the second author has recently studied the distribution of lengths of lattice vectors in a μ_n-random lattice of large dimension n. Given a lattice $L \in X_n$, we order its non-zero vectors by increasing lengths as $\pm\boldsymbol{v}_1, \pm\boldsymbol{v}_2, \pm\boldsymbol{v}_3, \ldots$ and define, for each $j \geq 1$,

$$\mathcal{V}_j(L) := V_n |\boldsymbol{v}_j|^n.$$

We stress that the first few vectors in this list, that is, the shortest non-zero vectors in L, encode important geometric information attached to L. Indeed, these short vectors play a crucial role in, for example, the lattice sphere packing problem where the quantity $2^{-n} \sup_{L \in X_n} \mathcal{V}_1(L)$ determines the maximal density of a lattice sphere packing in $\mathbb{R}^n$. In [27], by calculating the limits as $n \to \infty$ of mixed moments of the form

$$\mathbb{E}\left(\prod_{j=1}^{k} N_{n,L}(x_j)\right) \tag{1.3}$$

for any fixed $k \geq 1$ and $0 < x_1 \leq x_2 \leq \ldots \leq x_k$, the following theorem is established:

Theorem 1.1 (Södergren)**.** *The sequence $\{\mathcal{V}_j(\cdot)\}_{j=1}^\infty$ converges in distribution, as $n \to \infty$, to the sequence $\{T_j\}_{j=1}^\infty$, where $0 < T_1 < T_2 < T_3 < \cdots$ denote the points of a Poisson process $\mathcal{P} = \{\mathcal{N}(x), x \geq 0\}$ on $\mathbb{R}^+$ with constant intensity $\frac{1}{2}$.*

[1] Here we call a lattice L irrational if the Gram matrix for every $\mathbb{Z}$-basis of L is not proportional to a matrix with integer entries only.

[2] Throughout the paper, $\mathbb{E}$ will denote the expected value with respect to the measure μ_n on X_n.

The convergence in Theorem 1.1 is equivalent to the convergence of all finite dimensional distributions, i.e. to the fact that the truncated sequence $\{\mathcal{V}_j(\cdot)\}_{j=1}^N$ converges in distribution to the corresponding truncated sequence $\{T_j\}_{j=1}^N$, for every fixed $N \in \mathbb{Z}^+$. This raises the question whether it is possible to allow for more flexibility in Theorem 1.1 in the sense of allowing $N = N(n)$ to grow as a function of the dimension n? It seems reasonable to expect that for moderately growing N the Possion characteristic of the limit sequence should remain intact, but that the Poissonian behavior will eventually disappear as N is allowed to grow faster. A first result in this direction, indicating a Poissonian behavior for $N \leq cn$ where $c > 0$ is a small absolute constant, is proved in a recent paper by Kim [15] using a sieving argument (cf. also [14] where the range $N \leq (n/2)^{1/2-\varepsilon}$ was obtained). The following result extends this range, giving an indication of Poissonian behavior for any N growing *sub-exponentially* with respect to n.

Theorem 1.2. *Let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ be any function satisfying $\lim_{n\to\infty} f(n) = \infty$ and $f(n) = O_\varepsilon(e^{\varepsilon n})$ for every $\varepsilon > 0$. Let $\mathcal{N}(x)$ be a Poisson distributed random variable with expectation $x/2$. Then*

$$\text{Prob}_{\mu_n}(N_{n,L}(x) \leq 2N) - \text{Prob}(\mathcal{N}(x) \leq N) \to 0 \quad \textit{as } n \to \infty, \tag{1.4}$$

uniformly with respect to all $N, x \geq 0$ satisfying $\min(x, N) \leq f(n)$.

We will deduce Theorem 1.2 from Theorem 1.1 combined with the following result, a central limit theorem for the normalized error term in the generalized circle problem for a random lattice L.

Theorem 1.3. *Let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ be any function satisfying $\lim_{n\to\infty} f(n) = \infty$ and $f(n) = O_\varepsilon(e^{\varepsilon n})$ for every $\varepsilon > 0$. Let $Z_n^{(\mathrm{B})}$ be the random variable*

$$Z_n^{(\mathrm{B})} := \frac{1}{\sqrt{2f(n)}} R_{n,L}(f(n)), \tag{1.5}$$

with L picked at random in (X_n, μ_n). Then

$$Z_n^{(\mathrm{B})} \xrightarrow{\mathrm{d}} N(0,1) \qquad \textit{as } n \to \infty.$$

The "B" in $Z_n^{(\mathrm{B})}$ stands for "ball". In fact, the same convergence holds even if we consider completely general subsets of $\mathbb{R}^n$ symmetric about the origin.

Theorem 1.3'. *Let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ be as in Theorem 1.3, and for each n let S_n be a Borel measurable subset of $\mathbb{R}^n$ satisfying $\mathrm{vol}(S_n) = f(n)$ and $S_n = -S_n$. Set*

$$Z_n := \frac{\#(L \cap S_n \setminus \{\mathbf{0}\}) - f(n)}{\sqrt{2f(n)}}, \tag{1.6}$$

with L picked at random in (X_n, μ_n). Then

$$Z_n \xrightarrow{\mathrm{d}} N(0,1) \qquad \textit{as } n \to \infty.$$

Remark 1.4. Theorem 1.3' remains true if we consider $L \cap S_n$ instead of $L \cap S_n \setminus \{\mathbf{0}\}$ in (1.6), since $f(n) \to \infty$. However, the fact that we remove $\mathbf{0}$ in (1.1) is essential for Theorem 1.2 to hold, namely in the case when x stays bounded as $n \to \infty$.

In Theorem 4.2 below we generalize Theorem 1.3' to the case of r pairwise disjoint subsets of $\mathbb{R}^n$, for any fixed $r \in \mathbb{Z}^+$, showing that the joint distribution of the normalized counting variables approaches r independent normal distributions. In the special case of balls centered at the origin, we also have the following *functional* central limit theorem, generalizing Theorem 1.3:

Theorem 1.5. *Let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ be any function satisfying $\lim_{n\to\infty} f(n) = \infty$ and $f(n) = O_\varepsilon(e^{\varepsilon n})$ for every $\varepsilon > 0$. Consider, for $n \in \mathbb{Z}^+$ and L picked at random in (X_n, μ_n), the random function*

$$t \mapsto \widetilde{Z}_n^{(\mathrm{B})}(t) := \frac{1}{\sqrt{2f(n)}} R_{n,L}\left(tf(n)\right)$$

on the interval $[0,1]$. Let P_n denote the corresponding probability measure on the space $\mathcal{D}[0,1]$ of cadlag functions on $[0,1]$. Then $\widetilde{Z}_n^{(\mathrm{B})}(t)$ converges in distribution to one-dimensional Brownian motion, or equivalently, P_n converges weakly to Wiener measure, as $n \to \infty$.

Remark 1.6. In a different direction, for $n = 2$ and fixed $L \in X_2$, a result by Bleher [4] (cf. also Heath-Brown [8] for the case $L = \mathbb{Z}^2$) implies the existence of a limit distribution of $t^{-1/4} R_{2,L}(t)$ for t random in $(0,T)$, as $T \to \infty$. This limit distribution is non-Gaussian; however the corresponding limit for the number of lattice points in thin annuli is Gaussian in certain situations; cf. [11] and [32]. We are not aware of any similar results in dimension $n \geq 3$; cf. however Peter [17].

It is an interesting question whether the above limit results could be extended to more rapidly growing functions $f(n)$. Our proof of Theorem 1.3' goes by establishing convergence of all moments of Z_n. For any *fixed* moment $\mathbb{E}(Z_n^k)$, the method actually yields the desired limit result even for $f(n)$ of modest exponential growth; however for more rapidly growing $f(n)$ the moment diverges (if $k \geq 3$). In the case of balls, we have determined the precise growth rate where this transition occurs: Set

$$c_2 = +\infty \quad \text{and} \quad c_k = \frac{k-1}{k-2}\log(k-1) - \log k \quad (k \geq 3). \tag{1.7}$$

Note that $\{c_k\}_{k\geq 3}$ is a positive, strictly decreasing sequence; its first values are $c_3 = 0.28768\ldots$, $c_4 = 0.26162\ldots$, $c_5 = 0.23895\ldots$, and $c_k \sim k^{-1}\log k$ as $k \to \infty$.

Theorem 1.7. *Let $k \geq 2$ and $0 < c < c_k$, and let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ be any function satisfying $\lim_{n\to\infty} f(n) = \infty$ and $f(n) = O(e^{cn})$. For each n let S_n be a Borel measurable subset of $\mathbb{R}^n$ satisfying $\mathrm{vol}(S_n) = f(n)$ and $S_n = -S_n$, and define Z_n as in Theorem 1.3'. Then*

$$\lim_{n\to\infty} \mathbb{E}\big(Z_n^k\big) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (k-1)!! & \text{if } k \text{ is even.} \end{cases} \tag{1.8}$$

On the other hand, if $k \geq 3$ and $c > c_k$, and if $f : \mathbb{Z}^+ \to \mathbb{R}^+$ is any function satisfying $f(n) \gg e^{cn}$ as $n \to \infty$, then $\mathbb{E}\big(\big(Z_n^{(\mathrm{B})}\big)^k\big) \to +\infty$ as $n \to \infty$.

The last result shows in particular that the assumption of sub-exponential growth imposed in Theorem 1.3' is best possible for our method of proof via convergence of moments; however the question remains open whether a limit distribution of Z_n exists (Gaussian or not) also for more rapidly growing $f(n)$. Theorem 1.7 shows in this regard that any limit distribution of any subsequence of Z_n is necessarily *close* to the Gaussian $N(0,1)$ distribution, in the weak topology, so long as $f(n) = O(e^{cn})$ with $c > 0$ sufficiently small.

Remark 1.8. In the setting of balls as in Theorem 1.3, taking $f(n) = e^{cn}$ corresponds to counting all lattice vectors $\boldsymbol{m} \in L \setminus \{\boldsymbol{0}\}$ of length $|\boldsymbol{m}| \lesssim e^c \sqrt{\frac{n}{2\pi e}}$. In this connection we note that for any fixed N, with probability tending to one as $n \to \infty$, the first N shortest non-zero vectors $\pm \boldsymbol{v}_1, \ldots, \pm \boldsymbol{v}_N$ of a random lattice $L \in X_n$ all have length $\sqrt{\frac{n}{2\pi e}}(1 + O(\frac{\log n}{n}))$. This follows e.g. from Theorem 1.1, using the asymptotics $V_n \sim (\frac{2\pi e}{n})^{n/2} (\pi n)^{-1/2}$.

Remark 1.9. Kelmer has recently obtained a bound on the mean square of $R_{n,L}(x)$ for fixed $n \geq 2$ and large x; cf. [13, Thm. 2]. This bound supports the conjecture that for almost every $L \in X_n$, $R_{n,L}(x) \ll x^{\frac{1}{2} - \frac{1}{2n} + \varepsilon}$ holds as $x \to \infty$ (cf. also [5], [9]). Kelmer's bound implies that if $f(n)$ grows *sufficiently rapidly* (the growth condition could be made explicit with further work), then $Z_n^{(\mathrm{B})}$ converges in distribution to 0 as $n \to \infty$, showing that the normalization in (1.5) is inappropriate in this regime.

Our original motivation for studying the limit distribution of $Z_n^{(\mathrm{B})}$ comes from questions concerning the Epstein zeta function of a random lattice $L \in X_n$ as $n \to \infty$; cf. [23, 29, 30]. Recall that for $\operatorname{Re} s > \frac{n}{2}$ and $L \in X_n$ the Epstein zeta function is defined by the absolutely convergent series

$$E_n(L, s) := \sum_{\boldsymbol{m} \in L \setminus \{\boldsymbol{0}\}} |\boldsymbol{m}|^{-2s}.$$

The function $E_n(L, s)$ can be meromorphically continued to $\mathbb{C}$ and satisfies a functional equation of "Riemann type" relating $E_n(L, s)$ and $E_n(L^*, \frac{n}{2} - s)$. (Here L^* denotes the dual lattice of L.) An outstanding question from [30] is whether $E_n(L, s)$ for s on or near the central point $s = \frac{n}{4}$, possesses, after appropriate normalization, a limit distribution as $n \to \infty$? This question turns out to be closely related to the behavior of the random function $\widetilde{Z}_n^{(\mathrm{B})}(t)$, and we expect that Theorem 1.5 in this paper in combination with the methods of [30] will make it possible to give an answer in the case of $s = cn$ with $c > \frac{1}{4}$ tending to $\frac{1}{4}$ sufficiently slowly as a function of n. However in order to handle $c = \frac{1}{4}$ or c arbitrarily near $\frac{1}{4}$, it appears that we need a precise understanding of the limit of $\widetilde{Z}_n^{(\mathrm{B})}(t)$ when the volume $f(n)$ is allowed to grow as rapidly as $e^{\frac{1}{2}(1 - \log 2)n}$, and furthermore we need to understand this distribution jointly with the corresponding distribution for the dual lattice of L. We hope to return to these matters in future work.

The organization of the paper is as follows. As mentioned, Theorem 1.3' is proved by computing the moments of Z_n; similarly Theorem 1.5 is proved by computing the mixed moments of the finite dimensional distributions of $\widetilde{Z}_n^{(\mathrm{B})}(t)$. The standard tool for calculating moments of this form is Rogers' mean value formula [19]; however, the assumption $\lim_{n \to \infty} f(n) = \infty$ causes divergence problems. To get around these, we develop, in Section 2, a new version of Rogers' formula suitable for calculating moments of functions that can be represented in the form

$$\sum_{\boldsymbol{m} \in L \setminus \{\boldsymbol{0}\}} \rho(V_n |\boldsymbol{m}|^n) - \int_0^\infty \rho(x)\, dx$$

for suitable test functions ρ; in particular the formula can be applied to calculate moments of $\widetilde{Z}_n^{(\mathrm{B})}(t)$. The proof of this formula is combinatorial in nature. Using the formula, in Section 3 we prove Theorems 1.3' and 1.2, and in Section 4 we prove Theorem 1.5. Finally in Section 5 we prove Theorem 1.7, by a careful analysis of the sizes of the various non-leading order terms appearing in the moment computation used to prove Theorem 1.3'.

Acknowledgments. We are grateful to Svante Janson for helpful discussions.

2. A new version of Rogers' mean value formula

To begin, we describe Rogers' original formula. Let $1 \leq k \leq n-1$ and let $\rho : (\mathbb{R}^n)^k \to \mathbb{R}_{\geq 0}$ be a non-negative Borel measurable function. In [19] Rogers proved the following remarkable identity:

$$\int_{X_n} \sum_{\boldsymbol{m}_1,\ldots,\boldsymbol{m}_k \in L\setminus\{\boldsymbol{0}\}} \rho(\boldsymbol{m}_1,\ldots,\boldsymbol{m}_k)\, d\mu_n(L) \tag{2.1}$$
$$= \sum_{q=1}^{\infty} \sum_{D} \Big(\frac{e_1}{q}\cdots\frac{e_m}{q}\Big)^n \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} \rho\Big(\sum_{i=1}^m \frac{d_{i1}}{q}\boldsymbol{x}_i,\ldots,\sum_{i=1}^m \frac{d_{ik}}{q}\boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1\ldots d\boldsymbol{x}_m.$$

Here the inner sum is over all integer matrices $D = (d_{ij})$ having size $m \times k$ for some $1 \leq m \leq k$, satisfying the following properties: No column of D vanishes identically; the entries of D have greatest common divisor equal to 1; and finally there exists a division $(\nu;\mu) = (\nu_1,\ldots,\nu_m;\mu_1,\ldots,\mu_{k-m})$ of the numbers $1,\ldots,k$ into two sequences $\nu_1,\ldots,\nu_m$ and $\mu_1,\ldots,\mu_{k-m}$, satisfying

$$\begin{aligned} &1 = \nu_1 < \nu_2 < \ldots < \nu_m \leq k,\\ &1 < \mu_1 < \mu_2 < \ldots < \mu_{k-m} \leq k,\\ &\nu_i \neq \mu_j, \text{ if } 1 \leq i \leq m,\ 1 \leq j \leq k-m, \end{aligned} \tag{2.2}$$

such that

$$\begin{aligned} &d_{i\nu_j} = q\delta_{ij}, \quad i = 1,\ldots,m,\ j = 1,\ldots,m,\\ &d_{i\mu_j} = 0, \quad \text{if } \mu_j < \nu_i,\ i = 1,\ldots,m,\ j = 1,\ldots,k-m. \end{aligned} \tag{2.3}$$

We call these matrices $\langle k,q\rangle$-*admissible*.[3] Finally $e_i = (\varepsilon_i, q)$, $i = 1,\ldots,m$, where $\varepsilon_1,\ldots,\varepsilon_m$ are the elementary divisors of the matrix D. We stress that the right-hand side of (2.1) is a *positive* infinite linear combination of integrals of ρ over certain linear subspaces of $(\mathbb{R}^n)^k$.

Remark 2.1. The formula (2.1) should be understood as an equality in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$; if either side of (2.1) is divergent, then so is the other side. By Schmidt, [24, Thm. 2], if ρ is bounded and of compact support then both sides of (2.1) are finite. Hence, under this restriction we may remove the assumption that ρ is non-negative, i.e. the formula (2.1) is in fact valid for any real-valued Borel measurable function ρ on $(\mathbb{R}^n)^k$ which is bounded and of compact support, with both sides of (2.1) being nicely absolutely convergent.

Remark 2.2. It follows from the conditions on the matrices D and [10, Thm. 14.5.1] that we always have $e_1 = 1$, and hence $\big(\frac{e_1}{q}\cdots\frac{e_m}{q}\big)^n \leq q^{-n}$.

We now state our new version of Rogers' mean value formula.

[3] Note that the only $\langle k,1\rangle$-admissible matrix with $m = k$ is the $k \times k$ identity matrix, and for $q > 1$ there are no $\langle k,q\rangle$-admissible matrices with $m = k$.

Theorem 2.3. *Let $n > k > 0$, and let $f_1, \ldots, f_k$ be real-valued Borel measurable functions on $\mathbb{R}^n$ which are bounded and of compact support. Define the functions $F_1, \ldots, F_k$ on X_n by*

$$F_j(L) := \sum_{\boldsymbol{m} \in L \setminus \{\boldsymbol{0}\}} f_j(\boldsymbol{m}) - \int_{\mathbb{R}^n} f_j(\boldsymbol{x})\, d\boldsymbol{x}. \tag{2.4}$$

Then

$$\mathbb{E}\Big(\prod_{j=1}^{k} F_j(L)\Big) = \sum_{q=1}^{\infty} \sum_{D}{}' \Big(\frac{e_1}{q} \cdots \frac{e_m}{q}\Big)^n \times \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f_1\Big(\sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{x}_i\Big) \cdots f_k\Big(\sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_m,$$

where $'$ indicates that the inner sum is over all $\langle k, q\rangle$-admissible matrices D with the property that there are at least two non-zero entries in each row.

We note that in the simple case $k = 1$, Theorem 2.3 states that

$$\mathbb{E}(F_1(L)) = 0.$$

This is in fact an immediate consequence of Siegel's mean value formula; see [26].

Proof. Let $K = \{1, \ldots, k\}$. Using (2.4) and (2.1), we get

$$\begin{aligned}
&\mathbb{E}\Big(\prod_{j=1}^{k} F_j(L)\Big) \\
&= \sum_{A \subset K} (-1)^{\#(K \setminus A)} \Big(\prod_{j \in K \setminus A} \int_{\mathbb{R}^n} f_j(\boldsymbol{x})\, d\boldsymbol{x}\Big) \mathbb{E}\Big(\prod_{j \in A} \Big(\sum_{\boldsymbol{m}_j \in L \setminus \{\boldsymbol{0}\}} f_j(\boldsymbol{m}_j)\Big)\Big) \\
&= \sum_{A \subset K} (-1)^{\#(K \setminus A)} \Big(\prod_{j \in K \setminus A} \int_{\mathbb{R}^n} f_j(\boldsymbol{x})\, d\boldsymbol{x}\Big) \sum_{q=1}^{\infty} \sum_{D} \Big(\frac{e_1}{q} \cdots \frac{e_m}{q}\Big)^n \\
&\qquad \times \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{\ell=1}^{a} f_{j_\ell}\Big(\sum_{i=1}^{m} \frac{d_{i\ell}}{q} \boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_m,
\end{aligned} \tag{2.5}$$

where A runs through all subsets of K, we write $a = \#A$ and $A = \{j_1, \ldots, j_a\}$ with $j_1 < \cdots < j_a$, and the inner sum is taken over all $\langle a, q\rangle$-admissible matrices D. As usual $m = m(D)$ denotes the number of rows of D. Note that all multiple sums and integrals appearing in (2.5) are absolutely convergent, because of our assumptions on $f_1, \ldots, f_k$; cf. Remark 2.1.

Given any $A = \{j_1, \ldots, j_a\}$, q and D appearing in the sum, we set $m' := m + k - a$ and write $K \setminus A = \{j'_1, \ldots, j'_{k-a}\}$ with $j'_1 < \cdots < j'_{k-a}$. We then let $D' = D'(A, D) = (d'_{ij})$ be the $m' \times k$ matrix which has $d'_{i,j_\ell} = d_{i,\ell}$ for $\langle i, \ell\rangle \in \{1, \ldots, m\} \times \{1, \ldots, a\}$, $d_{m+\ell,j'_\ell} = q$ for $\ell = 1, \ldots, k - a$, and all other entries equal to zero. Note that the matrix D' is typically not $\langle k, q\rangle$-admissible. Let $\varepsilon'_1, \ldots, \varepsilon'_{m'}$ be the elementary divisors of D' and set $e'_j = (\varepsilon'_j, q)$. Then $e'_1 \cdots e'_{m'} = q^{\#(K \setminus A)} e_1 \cdots e_m$ (cf., e.g., [19, Lemma 1]), and so $\frac{e_1}{q} \cdots \frac{e_m}{q} = \frac{e'_1}{q} \cdots \frac{e'_{m'}}{q}$. We may now rewrite each product of integrals in the right-hand side of (2.5) in terms of the matrices $D' = D'(A,D) = (d'_{ij})$:

$$\mathbb{E}\Big(\prod_{j=1}^{k} F_j(L)\Big) = \sum_{A\subset K} (-1)^{\#(K\setminus A)} \sum_{q=1}^{\infty}\sum_{D} \Big(\frac{e'_1}{q}\cdots\frac{e'_{m'}}{q}\Big)^n \times \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} \prod_{\ell=1}^{k} f_\ell\Big(\sum_{i=1}^{m'} \frac{d'_{i\ell}}{q}\boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1\ldots d\boldsymbol{x}_{m'}. \tag{2.6}$$

Note that any matrix $D' = D'(A,D)$ appearing in this sum can be brought, by a unique row permutation, into a $\langle k,q\rangle$-admissible matrix (this is easily seen by considering the admissibility conditions column by column, starting from the left). Conversely, given any $\langle k,q\rangle$-admissible matrix D', let $S(D')$ be the set of indices of those columns of D' which have the property that the column has a unique non-zero entry and this entry is also the only non-zero entry in its row. Then the matrix D' is attained as a row permutation of $D'(A,D)$ for exactly $2^{\#S(D')}$ pairs $\langle A,D\rangle$ appearing in the above sum, namely exactly once for each $B\subset S(D')$. Hence

$$\mathbb{E}\Big(\prod_{j=1}^{k} F_j(L)\Big) = \sum_{q=1}^{\infty}\sum_{D'} \Big(\sum_{B\subset S(D')} (-1)^{\#B}\Big) \Big(\frac{e'_1}{q}\cdots\frac{e'_{m'}}{q}\Big)^n \times \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} \prod_{\ell=1}^{k} f_\ell\Big(\sum_{i=1}^{m'} \frac{d'_{i\ell}}{q}\boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1\ldots d\boldsymbol{x}_{m'}, \tag{2.7}$$

where now the sum over D' is taken over all $\langle k,q\rangle$-admissible matrices. But here $\sum_{B\subset S(D')}(-1)^{\#B}$ equals 1 if $S(D')=\emptyset$ and equals 0 otherwise. Hence we obtain the formula stated in the theorem. □

Remark 2.4. Note the close connection between the formula in Theorem 2.3 and the formula in [30, Prop. 7.1].

Remark 2.5. Clearly the family of functions f_j admitted in Theorem 2.3 can be extended by approximation arguments. However the present family is more than sufficient for our purposes in this paper.

Remark 2.6. The formula in Theorem 2.3 is useful in the study of the Epstein zeta function $E_n(L,s)$. Recall from [30, Sect. 4] that, for $L\in X_n$ and $s\in\mathbb{C}\setminus\{0,\frac{n}{2}\}$, we have

$$\pi^{-s}\Gamma(s)E_n(L,s) = H_n(L,s) + H_n(L^*,\tfrac{n}{2}-s), \tag{2.8}$$

where L^* is the dual lattice of L,

$$H_n(L,s) := -\frac{1}{\frac{n}{2}-s} + \sum_{\boldsymbol{m}\in L\setminus\{\boldsymbol{0}\}} G\big(s,\pi|\boldsymbol{m}|^2\big),$$

and

$$G(s,x) := \int_1^\infty t^{s-1}e^{-xt}\,dt, \qquad \operatorname{Re} x > 0.$$

The connection between $E_n(L,s)$ and the present discussion comes from the relation

$$H_n(L,s) = \int_0^\infty G\Big(s,\pi\big(V_n^{-1}x\big)^{2/n}\Big)\, dR_{n,L}(x), \qquad 0<s<\tfrac{n}{2}$$

(cf. [30, Eq. (4.7)]). It follows that Theorem 2.3 can be used to calculate (truncated) moments of $H_n(L,s)$. Furthermore, since $H_n(L,s)$ dominates $H_n(L^*, \frac{n}{2}-s)$ in the interval $(\frac{1}{4}+\varepsilon)n < s < \frac{n}{2}$ ($\varepsilon > 0$ fixed) for most lattices $L \in X_n$ when n is large enough, we also find that the (truncated) moments of $H_n(L,s)$ are of apparent interest in the study of $E_n(L,s)$ in the limit as $n \to \infty$. We do not pursue this further here since we plan to give a detailed account of this topic elsewhere.

We close this section by giving a generalization of Theorem 2.3 which seems potentially useful, although it will not be used in the present paper.

Theorem 2.7. *Let $k, \ell > 0$ and $n > k\ell$. Let $g_j : (\mathbb{R}^n)^k \to \mathbb{R}$, $1 \le j \le \ell$, be Borel measurable functions which are bounded and of compact support. Consider the related functions $G_j : X_n \to \mathbb{R}$ defined by*

$$G_j(L) := \sum_{\boldsymbol{m}_1,\ldots,\boldsymbol{m}_k \in L\setminus\{\boldsymbol{0}\}} g_j(\boldsymbol{m}_1,\ldots,\boldsymbol{m}_k) - \mathbb{E}\Big(\sum_{\boldsymbol{m}_1,\ldots,\boldsymbol{m}_k \in L\setminus\{\boldsymbol{0}\}} g_j(\boldsymbol{m}_1,\ldots,\boldsymbol{m}_k)\Big).$$

Then

$$\mathbb{E}\Big(\prod_{j=1}^{\ell} G_j(L)\Big) = \sum_{q=1}^{\infty} \sum_{D}{}^{*} \Big(\frac{e_1}{q}\cdots\frac{e_m}{q}\Big)^n$$
$$\times \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} \prod_{j=0}^{\ell-1} g_{j+1}\Big(\sum_{i=1}^{m}\frac{d_{i,jk+1}}{q}\boldsymbol{x}_i, \sum_{i=1}^{m}\frac{d_{i,jk+2}}{q}\boldsymbol{x}_i,\cdots,\sum_{i=1}^{m}\frac{d_{i,jk+k}}{q}\boldsymbol{x}_i\Big)\,d\boldsymbol{x}_1\ldots d\boldsymbol{x}_m,$$

where * indicates that the inner sum is over all $\langle k\ell, q\rangle$-admissible matrices D with the property that there do not exist any $j \in \{0,\ldots,\ell-1\}$ and $1 \le i_1 \le i_2 \le m$ such that the submatrix at rows $i_1, i_1+1,\ldots,i_2$ and columns $jk+1, jk+2,\ldots,jk+k$ of D is a multiple of a $\langle k, q'\rangle$-admissible matrix for some $q' \mid q$, and all the remaining entries of these rows and columns of D are zero.

Outline of proof. Mimicking the beginning of the proof of Theorem 2.3, in particular expanding $\mathbb{E}\big(\prod_{j=1}^{\ell} G_j(L)\big)$ as much as possible using (2.1), we obtain the formula

$$\mathbb{E}\Big(\prod_{j=1}^{\ell} G_j(L)\Big) = \sum_{A\subset\{1,\ldots,\ell\}} (-1)^{\ell-\#A} \sum_{\{q_j\}}\sum_{\{D_j\}}\sum_{q=1}^{\infty}\sum_{D}\Big(\frac{e_1'}{q'}\cdots\frac{e_{m'}'}{q'}\Big)^n$$
$$\times \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\prod_{j=0}^{\ell-1} g_{j+1}\Big(\sum_{i=1}^{m'}\frac{d'_{i,jk+1}}{q'}\boldsymbol{x}_i,\cdots,\sum_{i=1}^{m'}\frac{d'_{i,jk+k}}{q'}\boldsymbol{x}_i\Big)\,d\boldsymbol{x}_1\cdots d\boldsymbol{x}_{m'}, \tag{2.9}$$

where the notation is as follows. As before, $a = \#A$ and $A = \{j_1,\ldots,j_a\}$ with $j_1 < \cdots < j_a$. In the sums, $\{q_j\}$ and $\{D_j\}$ are short-hands for $\{q_j\}_{j\in A^c}$ and $\{D_j\}_{j\in A^c}$, where A^c is the complement of A in $\{1,\ldots,\ell\}$; and $\{q_j\}$ runs through all $(\ell-a)$-tuples of positive integers while $\{D_j\}$ runs through all $(\ell-a)$-tuples of matrices such that D_j is $\langle k, q_j\rangle$-admissible for each $j \in A^c$. In the innermost sum, D runs through all $\langle ak, q\rangle$-admissible matrices. For any $A, \{q_j\}, \{D_j\}, q, D$ appearing in the multiple sum we let q' be the least common multiple of q and all the q_j's, and set $m' = m + \sum_{j\in A^c} m_j$, where m is the number of rows of D and m_j is the number of rows of D_j. Writing also $D_j = (d_{uv}^{(j)})$, $D = (d_{uv})$ and $\overline{m}_j := m + \sum_{\substack{j'\in A^c\\ j'<j}} m_{j'}$, we define $D' = D'(A, \{q_j\}, \{D_j\}, q, D) = (d'_{ij})$ to be the $m' \times k\ell$ matrix which has

$d'_{i,(j_u-1)k+v} = \dfrac{q'}{q} d_{i,(u-1)k+v}$ for all $i \in \{1,\ldots,m\}$, $u \in \{1,\ldots,a\}$, $v \in \{1,\ldots,k\}$, and $d_{\overline{m}_j+i,(j-1)k+v} = \dfrac{q'}{q_j} d^{(j)}_{iv}$ for all $j \in A^c$, $v \in \{1,\ldots,k\}$, $i \in \{1,\ldots,m_j\}$, and all other entries equal to zero. Finally $e'_j = (\varepsilon'_j, q)$, where $\varepsilon'_1,\ldots,\varepsilon'_{m'}$ are the elementary divisors of D'. This completes the description of the notation in (2.9).

One notes that each matrix D' which appears above can be brought, by a unique row permutation, into a $\langle k\ell, q'\rangle$-admissible matrix. The rest of the proof follows closely the proof of Theorem 2.3. □

3. Proofs of Theorem 1.3' and Theorem 1.2

Our first goal is to prove Theorem 1.3' (and thus also Theorem 1.3). Let f, S_n and Z_n be as in the statement of the theorem. Thus $f : \mathbb{Z}^+ \to \mathbb{R}^+$ is a function satisfying $\lim_{n\to\infty} f(n) = \infty$ and $f(n) = O_\epsilon(e^{\epsilon n})$ for every $\varepsilon > 0$; for each n, S_n is a Borel measurable subset of $\mathbb{R}^n$ which has volume $f(n)$ and which is symmetric about the origin (viz., $-S_n = S_n$), and finally

$$Z_n := \frac{\#(L \cap S_n \setminus \{\mathbf{0}\}) - f(n)}{\sqrt{2f(n)}}, \tag{3.1}$$

with L picked at random in (X_n, μ_n). It follows from Siegel's formula [26] that for each $n \geq 2$ we have $\mathbb{E}\big(\#(L \cap S_n \setminus \{\mathbf{0}\})\big) = f(n)$ and thus $\mathbb{E}(Z_n) = 0$. Using Theorem 2.3, we now determine the limits as $n \to \infty$ of the higher moments of Z_n.

Proposition 3.1. *For any fixed $k \in \mathbb{Z}^+$,*

$$\lim_{n\to\infty} \mathbb{E}\big(Z_n^k\big) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (k-1)!! & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let χ_n be the characteristic function of S_n. For any $n > k$, Theorem 2.3 gives

$$\begin{aligned}\mathbb{E}\big(Z_n^k\big) &= \frac{1}{(2f(n))^{k/2}}\, \mathbb{E}\bigg(\Big(\sum_{\boldsymbol{m}\in L\setminus\{\mathbf{0}\}} \chi_n(\boldsymbol{m}) - \int_{\mathbb{R}^n} \chi_n(\boldsymbol{x})\, d\boldsymbol{x}\Big)^k\bigg) \\ &= \frac{1}{(2f(n))^{k/2}} \sum_{q=1}^{\infty} \sideset{}{'}\sum_{D} \Big(\frac{e_1}{q}\cdots\frac{e_m}{q}\Big)^n \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} \prod_{j=1}^{k} \chi_n\Big(\sum_{i=1}^{m} \frac{d_{ij}}{q}\boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1\ldots d\boldsymbol{x}_m.\end{aligned} \tag{3.2}$$

We let

$$M_{k,n} := \sideset{}{''}\sum_{D} \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} \prod_{j=1}^{k} \chi_n\Big(\sum_{i=1}^{m} d_{ij}\boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1\ldots d\boldsymbol{x}_m, \tag{3.3}$$

where the sum is taken over all $\langle k, 1\rangle$-admissible matrices D having entries $d_{ij} \in \{0, \pm 1\}$, with at least two non-zero entries in each row and exactly one non-zero entry in each column. Let $R_{k,n}$ be the sum of all the terms in (3.2) that are not accounted for in $M_{k,n}$, so that

$$\mathbb{E}\big(Z_n^k\big) = (2f(n))^{-k/2}\big(M_{k,n} + R_{k,n}\big). \tag{3.4}$$

Now, let $\mathcal{P}'(k)$ denote the set of partitions of $\{1,\ldots,k\}$ containing no singleton sets. Using $S_n = -S_n$ and $\mathrm{vol}(S_n) = f(n)$, and then [27, Lemma 3], we have

$$M_{k,n} = \sum_D{}'' f(n)^m = \sum_{P\in\mathcal{P}'(k)} 2^{k-\#P} f(n)^{\#P}. \tag{3.5}$$

It remains to bound the term $R_{k,n}$ in (3.4). The summation condition in $\sum_D'$ implies that all matrices D appearing in $R_{k,n}$ have at most $k-1$ rows. Hence, an easy modification of the arguments in [20, Sect. 9] and [21, Sect. 4] (see also [27, Sect. 3]) gives that, for n sufficiently large,

$$0 \le R_{k,n} \ll \Big(\frac{3}{4}\Big)^{n/2} f(n)^{k-1}, \tag{3.6}$$

where the implied constant depends on k but not on n. If k is odd, then we may assume that $k \ge 3$ and in this situation we have $\#P \le (k-1)/2$ for every $P \in \mathcal{P}'(k)$. Recall that we are assuming $f(n) = O_\varepsilon(e^{\varepsilon n})$. Hence it follows from (3.4), (3.5) and (3.6) that, for any odd $k \ge 3$,

$$\lim_{n\to\infty} \mathbb{E}\big(Z_n^k\big) = 0.$$

On the other hand, if k is even, then (3.4), (3.5) and (3.6) imply that

$$\lim_{n\to\infty} \mathbb{E}\big(Z_n^k\big) = \#\big\{P \in \mathcal{P}'(k) : \#B = 2, \forall B \in P\big\} = (k-1)!!.$$

This completes the proof of the proposition. □

Remark 3.2. Note that the variance of Z_n can be controlled for a much larger class of functions f. Indeed, for *any* $f : \mathbb{Z}^+ \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} f(n) = \infty$, we have

$$\mathrm{Var}(Z_n) = 1 + O\big(\big(\tfrac{3}{4}\big)^{n/2}\big) \qquad \text{as } n \to \infty.$$

Cf. (3.4) and (3.6) and note that $k-1 = k/2 = 1$ for $k = 2$.

Proof of Theorem 1.3'. The desired convergence follows immediately from Proposition 3.1. □

Proof of Theorem 1.2. Let $\varepsilon > 0$ be given. It follows from Theorem 1.3 that there exist $x_0 > 0$ and $n_0 \in \mathbb{Z}^+$ such that for all $n \ge n_0$, $x \in [x_0, f(n)]$ and $r \in \mathbb{R}$,

$$\left|\mathrm{Prob}\Big(\frac{N_{n,L}(x) - x}{\sqrt{2x}} \le r\Big) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{r} e^{-t^2/2}\,dt\right| < \frac{\varepsilon}{2}. \tag{3.7}$$

(Indeed, otherwise there is a sequence of positive integers $n_1 < n_2 < \cdots$ and positive numbers $x_1, x_2, \ldots$ with $x_j \le f(n_j)$ and $\lim_{j\to\infty} x_j = \infty$, such that for each j, (3.7) fails for $n = n_j$, $x = x_j$ and some $r = r_j \in \mathbb{R}$. We then obtain a contradiction against Theorem 1.3 applied to the function $f_1 : \mathbb{Z}^+ \to \mathbb{R}^+$ given by $f_1(n_j) = x_j$ and, say, $f_1(n) = f(n)$ for $n \notin \{n_1, n_2, \ldots\}$.) Using also the fact that $\frac{2\mathcal{N}(x)-x}{\sqrt{2x}}$ tends in distribution to $N(0,1)$, and taking $r = \frac{2N-x}{\sqrt{2x}}$, it follows that after possibly increasing x_0, we have

$$\Big|\mathrm{Prob}(N_{n,L}(x) \le 2N) - \mathrm{Prob}(\mathcal{N}(x) \le N)\Big| < \varepsilon \tag{3.8}$$

for all $n \ge n_0$, $x \in [x_0, f(n)]$, $N \ge 0$. On the other hand it follows from Theorem 1.1 (or [21, Thm. 3]) that, after possibly increasing n_0, (3.8) also holds for all $n \ge n_0$, $x \in [0, x_0]$, $N \ge 0$.

Hence we have proved that (1.4) holds uniformly with respect to all $N \geq 0$ and $0 \leq x \leq f(n)$. The extension to the remaining case, i.e. $x > f(n)$ and $N \leq f(n)$, is now straightforward: Applying what we have already proved to the function $n \mapsto 4f(n)$, it follows that the convergence in (1.4) holds uniformly with respect to all $N \geq 0$ and $0 \leq x \leq 4f(n)$; thus it only remains to consider the case when $x > 4f(n)$ and $N \leq f(n)$. However, for such x and N, we have

$$\text{Prob}_{\mu_n}(N_{n,L}(x) \leq 2N) \leq \text{Prob}_{\mu_n}(N_{n,L}(4f(n)) \leq 2f(n)) \tag{3.9}$$

and

$$\text{Prob}(\mathcal{N}(x) \leq N) \leq \text{Prob}(\mathcal{N}(4f(n)) \leq f(n)). \tag{3.10}$$

Here the right-hand side of (3.10) tends to zero as $n \to \infty$, and so by the convergence already established also the right-hand side of (3.9) tends to zero. Hence also the left-hand sides of (3.9) and (3.10) tend to zero as $n \to \infty$, uniformly over all $x > 4f(n)$ and $N \leq f(n)$. This concludes the proof. $\square$

4. Joint distribution for families of subsets, and proof of Theorem 1.5

Our main goal in this section is to prove Theorem 1.5. As a first step, we generalize Proposition 3.1 and Theorem 1.3' to finite families of disjoint subsets of $\mathbb{R}^n$. Specifically, let us again fix a function $f : \mathbb{Z}^+ \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} f(n) = \infty$ and $f(n) = O_\epsilon(e^{\epsilon n})$ for every $\varepsilon > 0$. Fix a positive integer r and positive real numbers $c_1, \ldots, c_r$. For each n, let $S_{1,n}, \ldots, S_{r,n}$ be Borel measurable subsets of $\mathbb{R}^n$ satisfying $\text{vol}(S_{j,n}) = c_j f(n)$, $-S_{j,n} = S_{j,n}$, and $S_{j,n} \cap S_{j',n} = \emptyset$ for all $j \neq j'$. In analogy with (3.1) we set

$$Z_{j,n} := \frac{\#(L \cap S_{j,n} \setminus \{\mathbf{0}\}) - c_j f(n)}{\sqrt{2f(n)}}, \tag{4.1}$$

with L picked at random in (X_n, μ_n).

Proposition 4.1. *In this situation, for any fixed $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{\geq 0}$,*

$$\lim_{n\to\infty} \mathbb{E}\Big(Z_{1,n}^{k_1} \cdots Z_{r,n}^{k_r}\Big) = \begin{cases} \prod_{j=1}^r \big(c_j^{k_j/2}(k_j-1)!!\big) & \text{if } k_1, \ldots, k_r \text{ are all even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Set $\widehat{k} = k_1 + \cdots + k_r$. Let $\chi_{j,n}$ be the characteristic function of $S_{j,n}$. For any $n > \widehat{k}$, Theorem 2.3 gives

$$\begin{aligned} \mathbb{E}\Big(Z_{1,n}^{k_1} \cdots Z_{r,n}^{k_r}\Big) &= (2f(n))^{-\widehat{k}/2} \sum_{q=1}^{\infty} \sideset{}{'}\sum_{D} \Big(\frac{e_1}{q} \cdots \frac{e_m}{q}\Big)^n \\ &\quad \times \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{r} \prod_{\ell_j=1}^{k_j} \chi_{j,n}\Big(\sum_{i=1}^m \frac{d_{i,k_1+\cdots+k_{j-1}+\ell_j}}{q} \boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_m, \end{aligned} \tag{4.2}$$

where the sum over $D = (d_{ij})$ runs through all $\langle \widehat{k}, q\rangle$-admissible matrices with the property that there are at least two non-zero entries in each row. As in the proof of Proposition 3.1, we divide the right-hand side into two parts as

$$\mathbb{E}\Big(Z_{1,n}^{k_1} \cdots Z_{r,n}^{k_r}\Big) = (2f(n))^{-\widehat{k}/2}\big(\widetilde{M}_{\boldsymbol{k},n} + \widetilde{R}_{\boldsymbol{k},n}\big),$$

where

$$(4.3)\qquad \widetilde{M}_{\boldsymbol{k},n} := \sum_D{}'' \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{r} \prod_{\ell_j=1}^{k_j} \chi_{j,n}\Big(\sum_{i=1}^{m} d_{i,k_1+\cdots+k_{j-1}+\ell_j}\, \boldsymbol{x}_i \Big)\, d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_m,$$

the sum being taken over all $\langle \widehat{k}, 1\rangle$-admissible matrices having entries $d_{ij} \in \{0, \pm 1\}$, with at least two non-zero entries in each row and exactly one non-zero entry in each column. Using the assumption that $S_{1,n}, \ldots, S_{r,n}$ are pairwise disjoint it follows that the terms in the right-hand side of (4.3) are zero unless, for each $i \in \{1, \ldots, m\}$, there is some $j \in \{1, \ldots, r\}$ such that the ith row of D has all its non-zero elements in columns corresponding to the fixed function $\chi_{j,n}$. The rest of the proof follows closely that of Proposition 3.1. □

Note that Proposition 4.1 immediately implies the following theorem, generalizing Theorem 1.3'.

Theorem 4.2. *Fix $r \in \mathbb{Z}^+$, $c_1, \ldots, c_r > 0$, and a function $f : \mathbb{Z}^+ \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} f(n) = \infty$ and $f(n) = O_\epsilon(e^{\epsilon n})$ for every $\varepsilon > 0$. For each n, let $S_{1,n}, \ldots, S_{r,n}$ be Borel measurable subsets of $\mathbb{R}^n$ which are pairwise disjoint, and which satisfy $\operatorname{vol}(S_{j,n}) = c_j f(n)$ and $-S_{j,n} = S_{j,n}$. Set*

$$Z_{j,n} := \frac{\#(L \cap S_{j,n} \setminus \{\mathbf{0}\}) - c_j f(n)}{\sqrt{2f(n)}},$$

with L picked at random in (X_n, μ_n). Then

$$\big(Z_{1,n}, \ldots, Z_{r,n}\big) \xrightarrow{\ \mathrm{d}\ } \big(N(0,c_1), N(0,c_2), \ldots, N(0,c_r)\big) \qquad \text{as } n \to \infty,$$

where the random vector in the right-hand side has independent coordinates.

We are now in position to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. To simplify notation, in this proof we write $\widetilde{Z}_n(t) := \widetilde{Z}_n^{(\mathrm{B})}(t)$. Given any fixed numbers $0 < t_1 < t_2 < \cdots < t_r \le 1$, by applying Theorem 4.2 with $S_{1,n}, \ldots, S_{r,n}$ as the annuli

$$S_{j,n} = \Big\{ \boldsymbol{x} \in \mathbb{R}^n \,:\, \Big(\frac{t_{j-1} f(n)}{V_n}\Big)^{1/n} < |\boldsymbol{x}| \le \Big(\frac{t_j f(n)}{V_n}\Big)^{1/n} \Big\}, \qquad j = 1, \ldots, r$$

(with $t_0 := 0$), we conclude that the random vector

$$\Big(\widetilde{Z}_n(t_1), \widetilde{Z}_n(t_2) - \widetilde{Z}_n(t_1), \ldots, \widetilde{Z}_n(t_r) - \widetilde{Z}_n(t_{r-1})\Big)$$

tends in distribution to

$$\big(N(0,t_1), N(0,t_2 - t_1), \ldots, N(0, t_r - t_{r-1})\big)$$

as $n \to \infty$. Note also that $\widetilde{Z}_n(0) = 0$ by definition. We have thus proved that the convergence in Theorem 1.5 holds on the level of finite dimensional distributions, and it now only remains to establish the tightness of the sequence P_n of probability measures on $\mathcal{D}[0,1]$.

By [3, Thm. 13.5 and (13.14)] (applied with $F(t) = C\sqrt{t}$ and $\beta = 1$), it suffices to prove that there exist $\alpha > \frac{1}{2}$ and $N \in \mathbb{N}$ such that

$$(4.4)\qquad \mathbb{E}\Big(\big(\widetilde{Z}_n(s) - \widetilde{Z}_n(r)\big)^2 \big(\widetilde{Z}_n(t) - \widetilde{Z}_n(s)\big)^2 \Big) \ll \big(\sqrt{t} - \sqrt{r}\big)^{2\alpha},$$

uniformly over all $0 \leq r \leq s \leq t \leq 1$ and $n \geq N$. We begin by noting that Proposition 4.1 implies that

$$\lim_{n\to\infty} \mathbb{E}\Big(\big(\widetilde{Z}_n(s)-\widetilde{Z}_n(r)\big)^2\big(\widetilde{Z}_n(t)-\widetilde{Z}_n(s)\big)^2\Big) = (t-s)(s-r) \leq (t-r)^2.$$

Hence, using also the fact that $t-r \leq 2(\sqrt{t}-\sqrt{r})$ for all $0 \leq r \leq t \leq 1$, we see that in the limit of large dimension n, (4.4) holds with $\alpha = 1$. In order to get a more uniform statement, note that by naively modifying Rogers' arguments in [20, Sect. 9] and [21, Sect. 4] as in the proofs of Propositions 3.1 and 4.1, we have

$$\begin{aligned}(4.5)\quad &\mathbb{E}\Big(\big(\widetilde{Z}_n(s)-\widetilde{Z}_n(r)\big)^2\big(\widetilde{Z}_n(t)-\widetilde{Z}_n(s)\big)^2\Big)\\ &\ll (t-r)^2 + \max\left(2^{-n}(t-r)f(n)^{-1}, \Big(\frac{3}{4}\Big)^{n/2}(t-r)^2, \Big(\frac{3}{4}\Big)^{n/2}(t-r)^3 f(n)\right)\end{aligned}$$

for all $n \geq 6$, where the implied constant is absolute.

The bound (4.5) is close but not quite sufficient for our purposes; the problematic term is $2^{-n}(t-r)f(n)^{-1}$. This term arises as a bound on the collected contribution of all $\langle 4, q\rangle$-admissible matrices D with $m = 1$ (and q arbitrary) in the expression that is obtained by applying Theorem 2.3 to the left-hand side of (4.5) (cf. (4.2)). Recall that $m = 1$ means that D has only one row. In order to improve the bound, note that any such matrix $D = (q, d_1, d_2, d_3)$ gives a contribution

$$(4.6)\qquad \frac{1}{4q^n f(n)^2}\int_{\mathbb{R}^n} \chi_1\big(V_n|\boldsymbol{x}|^n\big)\chi_1\Big(V_n\Big|\frac{d_1}{q}\boldsymbol{x}\Big|^n\Big)\chi_2\Big(V_n\Big|\frac{d_2}{q}\boldsymbol{x}\Big|^n\Big)\chi_2\Big(V_n\Big|\frac{d_3}{q}\boldsymbol{x}\Big|^n\Big)\,d\boldsymbol{x}$$

to the left-hand side of (4.5), where χ_1 and χ_2 are the characteristic functions of the open intervals $(rf(n), sf(n))$ and $(sf(n), tf(n))$, respectively. Let us temporarily assume that $r > 0$. Then, for the integral in (4.6) to be non-zero, we must have

$$1 < \Big|\frac{d_2}{q}\Big|^n, \Big|\frac{d_3}{q}\Big|^n < \frac{t}{r} = 1 + \frac{t-r}{r}.$$

Hence, since d_2 and d_3 are integers, we conclude that a (crude) necessary condition for (4.6) to be non-zero is

$$q^n > \frac{r}{t-r}.$$

Let Q be the smallest value of $q \in \mathbb{Z}^+$ satisfying this inequality. Then, for $n \geq 6$, the estimate [20, p. 246 (line 20)] with $\sum_{q=1}^{\infty}$ replaced by $\sum_{q=Q}^{\infty}$ gives

$$\begin{aligned}&\sum_{q=Q}^{\infty}\sideset{}{'}\sum_{\substack{D\\(m=1)}} \frac{1}{4q^n f(n)^2}\int_{\mathbb{R}^n} \chi_1\big(V_n|\boldsymbol{x}|^n\big)\chi_1\Big(V_n\Big|\frac{d_1}{q}\boldsymbol{x}\Big|^n\Big)\chi_2\Big(V_n\Big|\frac{d_2}{q}\boldsymbol{x}\Big|^n\Big)\chi_2\Big(V_n\Big|\frac{d_3}{q}\boldsymbol{x}\Big|^n\Big)\,d\boldsymbol{x}\\ (4.7)\qquad &\ll Q^{5-n}(t-r)f(n)^{-1}.\end{aligned}$$

Replacing the term $2^{-n}(t-r)f(n)^{-1}$ in (4.5) by the bound in (4.7) and using $Q \geq \max(1, (r/(t-r))^{1/n})$, we obtain, allowing now the implied constant to depend on

f:

$$\begin{aligned}
&\mathbb{E}\Big(\big(\widetilde{Z}_n(s)-\widetilde{Z}_n(r)\big)^2\big(\widetilde{Z}_n(t)-\widetilde{Z}_n(s)\big)^2\Big)\\
&\qquad \ll (t-r)^2+(t-r)\min\bigg(1,\Big(\frac{t-r}{r}\Big)^{1-\frac{5}{n}}\bigg)\\
&\qquad\qquad \ll (t-r)\min\bigg(1,\Big(\frac{t-r}{r}\Big)^{1-\frac{5}{n}}\bigg).
\end{aligned} \tag{4.8}$$

This bound is also valid when $r=0$, with the convention that $\min(1,\cdots)$ then equals 1.

Now fix the constant $\frac{1}{2}<\alpha<1$ in an arbitrary manner, and then take $N\geq 6$ so large that $1-\alpha-\frac{5}{N}>0$. We then claim that

$$(t-r)\min\bigg(1,\Big(\frac{t-r}{r}\Big)^{1-\frac{5}{n}}\bigg)\ll\big(\sqrt{t}-\sqrt{r}\big)^{2\alpha}, \tag{4.9}$$

uniformly over all $n\geq N$ and $0\leq r\leq t\leq 1$. Indeed, if $t\geq 2r$ then (4.9) is clear from $(\sqrt{t}-\sqrt{r})^{2\alpha}\asymp(\sqrt{t})^{2\alpha}=t^\alpha$. In the remaining case, i.e. when $0<r\leq t<2r$, we have $\sqrt{t}-\sqrt{r}\asymp(t-r)/\sqrt{r}$ and (4.9) is equivalent to $t-r\ll r^{(1-\alpha-\frac{5}{n})/(2-2\alpha-\frac{5}{n})}$, which is true since $(1-\alpha-\frac{5}{n})/(2-2\alpha-\frac{5}{n})<1$ and $t<2r$. This completes the proof of (4.9), and in view of (4.8) we thus obtain (4.4), completing the proof of Theorem 1.5. □

5. Moment bounds for exponentially growing volumes

Our goal in this section is to prove Theorem 1.7. Thus, for each n we assume given a Borel subset S_n of $\mathbb{R}^n$ satisfying $\mathrm{vol}(S_n)=f(n)$ and $S_n=-S_n$. Throughout the section we let χ_n denote the characteristic function of S_n. Our task is to go back to the proof of Proposition 3.1 and improve the bound on $R_{k,n}$, i.e. the sum of those terms in (3.2) which come from $\langle k,q\rangle$-admissible matrices D with at least two non-zero entries in each row and such that either $q\geq 2$, or some column contains more than one non-zero entry, or some entry has absolute value $|d_{ij}|\geq 2$. It will turn out that the dominating contribution to $R_{k,n}$ comes from $\langle k,1\rangle$-admissible matrices D of the form

$$D=\begin{pmatrix}1&0&\cdots&0&\pm1\\0&1&\cdots&0&\pm1\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&1&\pm1\end{pmatrix}\qquad(\text{thus }\ m=m(D)=k-1). \tag{5.1}$$

5.1. Auxiliary lemmas.

In our first lemma, by repeated use of an integral inequality of Rogers, [22, Theorem 1], we bound $\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\prod_{j=1}^k\chi_n\big(\sum_{i=1}^m\frac{d_{ij}}{q}\boldsymbol{x}_i\big)\,d\boldsymbol{x}_1\ldots d\boldsymbol{x}_m$ from above by a product of integrals of the following form:

$$J_a^{(n)}[c_1,\ldots,c_a]:=\int_{(\mathbb{R}^n)^a}I\bigg(|\boldsymbol{x}_i|<1\,(i=1,\ldots,a),\ \Big|\sum_{i=1}^a c_i\boldsymbol{x}_i\Big|<1\bigg)\,d\boldsymbol{x}_1\cdots d\boldsymbol{x}_a. \tag{5.2}$$

Here $n\geq a\geq 1$ and $c_1,\ldots,c_a\in\mathbb{R}_{>0}$, and $I(\cdot)$ is the indicator function. We extend the definition to the case $a=0$ by setting $J_0^{(n)}[\,]:=1$ for all n.

Let D be a $\langle k,q\rangle$-admissible matrix of size $m\times k$, having at least two non-zero entries in each row. Set $r=k-m$, let $(\nu;\mu)=(\nu_1,\ldots,\nu_m;\mu_1,\ldots,\mu_r)$ be as in

Section 2, and let $\mu_1',\ldots,\mu_r'$ be an arbitrary permutation of $\mu_1,\ldots,\mu_r$. For $j=1,\ldots,r$, we set

$$\overline{A}_j=\big\{i\in\{1,\ldots,m\}\,:\,d_{i,\mu_j'}\neq 0\big\};\qquad A_j=\overline{A}_j\setminus(\cup_{\ell<j}\overline{A}_\ell),\quad\text{and}\quad a_j=\#A_j.$$

Since D has at least two non-zero entries in each row, the sets $A_1,\ldots,A_r$ form a partition of $\{1,\ldots,m\}$, possibly with $A_j=\emptyset$ for some j's. Hence $\sum_{j=1}^r a_j=m$.

Lemma 5.1. *For D as above,*

(5.3)
$$\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\prod_{j=1}^k\chi_n\Big(\sum_{i=1}^m\frac{d_{ij}}{q}\boldsymbol{x}_i\Big)\,d\boldsymbol{x}_1\ldots d\boldsymbol{x}_m\leq V_n^{-m}f(n)^m\prod_{j=1}^r J_{a_j}^{(n)}\big[\big(|d_{i,\mu_j'}|/q\big)_{i\in A_j}\big].$$

Proof. We express the left-hand side of (5.3) as an iterated integral in the following way. For each $j\in\{1,\ldots,r\}$ we write $\boldsymbol{x}^{(j)}:=(\boldsymbol{x}_i)_{i\in A_j}\in(\mathbb{R}^n)^{a_j}$ and $d\boldsymbol{x}^{(j)}:=\prod_{i\in A_j}d\boldsymbol{x}_i$. (If $a_j=0$ then we understand $(\mathbb{R}^n)^0$ and $d\boldsymbol{x}^{(j)}$ to be the singleton set $\{\mathbf{0}\}$ with its unique probability measure.) Let $F_r(\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(r)})$ be the constant function 1, and set, iteratively for $j=r,r-1,\ldots,1$,

$$F_{j-1}(\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(j-1)})$$
$$:=\int_{(\mathbb{R}^n)^{a_j}}\Big(\prod_{i\in A_j}\chi_n(\boldsymbol{x}_i)\Big)\chi_n\Big(\sum_{i=1}^m\frac{d_{i,\mu_j'}}{q}\boldsymbol{x}_i\Big)F_j(\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(j)})\,d\boldsymbol{x}^{(j)}. \tag{5.4}$$

Then the left-hand side of (5.3) equals F_0. (The sum $\sum_{i=1}^m(d_{i,\mu_j'}/q)\boldsymbol{x}_i$ appearing in the right-hand side of (5.4) is well-defined since $d_{i,\mu_j'}=0$ for all $i\in\{1,\ldots,m\}\setminus(A_1\cup\cdots\cup A_j)$.)

Now let B be the closed ball of volume $f(n)$ centered at the origin in $\mathbb{R}^n$, and let χ_B be its characteristic function. Using (5.4) and [22, Theorem 1], we have

$$\begin{aligned}F_{j-1}(\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(j-1)})&\leq\big(\sup F_j\big)\int_{(\mathbb{R}^n)^{a_j}}\Big(\prod_{i\in A_j}\chi_n(\boldsymbol{x}_i)\Big)\chi_n\Big(\sum_{i=1}^m\frac{d_{i,\mu_j'}}{q}\boldsymbol{x}_i\Big)\,d\boldsymbol{x}^{(j)}\\&\leq\big(\sup F_j\big)\int_{(\mathbb{R}^n)^{a_j}}\Big(\prod_{i\in A_j}\chi_B(\boldsymbol{x}_i)\Big)\chi_B\Big(\sum_{i\in A_j}\frac{d_{i,\mu_j'}}{q}\boldsymbol{x}_i\Big)\,d\boldsymbol{x}^{(j)},\end{aligned}$$

since χ_B is the spherical symmetrization both of χ_n and of $\boldsymbol{y}\mapsto\chi_n(\boldsymbol{y}+\boldsymbol{z})$ for any fixed $\boldsymbol{z}\in\mathbb{R}^n$. Hence, since B has radius $V_n^{-1/n}f(n)^{1/n}$, we conclude

$$\sup F_{j-1}\leq V_n^{-a_j}f(n)^{a_j}J_{a_j}^{(n)}\big[\big(|d_{i,\mu_j'}|/q\big)_{i\in A_j}\big]\cdot\sup F_j.$$

Using this bound for $j=1,\ldots,r$, together with $\sum_{j=1}^r a_j=m$, we obtain (5.3). □

We say that a function $F:(\mathbb{R}^n)^m\to\mathbb{R}$ $(1\leq m\leq n)$ is $\mathrm{O}(n)$-invariant if $F(k\boldsymbol{x}_1,\ldots,k\boldsymbol{x}_m)=F(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m)$ for all $k\in\mathrm{O}(n)$. When this holds, we define $\overline{F}:(\mathbb{R}^m)^m\to\mathbb{R}$ through $\overline{F}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m)=F(\iota(\boldsymbol{x}_1),\ldots,\iota(\boldsymbol{x}_m))$, where ι is any fixed Euclidean isometry of $\mathbb{R}^m$ into $\mathbb{R}^n$. Note that $\overline{F}$ is independent of the choice of ι. Given any $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m\in\mathbb{R}^m$, we denote by $[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m]$ the volume of the parallelotope in $\mathbb{R}^m$ spanned by $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m$. Finally, we write $\omega_n:=nV_n$ for the volume of the $(n-1)$-sphere.

Lemma 5.2. *Let $1 \leq m \leq n$ and let $F : (\mathbb{R}^n)^m \to \mathbb{R}_{\geq 0}$ be a non-negative Borel measurable function which is* $\mathrm{O}(n)$*-invariant. Then*

$$\int_{(\mathbb{R}^n)^m} F(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)\, d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_m$$

$$= \frac{\prod_{j=n-m+1}^{n} \omega_j}{\prod_{j=1}^{m} \omega_j} \int_{(\mathbb{R}^m)^m} \overline{F}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)\, [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m]^{n-m}\, d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_m. \tag{5.5}$$

Proof. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the standard unit vectors in $\mathbb{R}^n$. Passing to polar coordinates and then performing the same substitution as in [28, p. 754], the left-hand side of (5.5) becomes

$$\Big(\prod_{j=n-m+1}^{n} \omega_j\Big) \int_{(\mathbb{R}_{>0})^m} \int_{(0,\pi)^M} F(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \prod_{1 \leq i < j \leq m} (\sin \phi_{i,j})^{n-i-1} \prod_{j=1}^{m} r_j^{n-1}\, d\boldsymbol{\phi}\, d\boldsymbol{r},$$

where $M = \binom{m}{2}$, $\boldsymbol{r} = (r_1, \ldots, r_m)$, $\boldsymbol{\phi} = (\phi_{i,j})_{1 \leq i < j \leq m}$, and $d\boldsymbol{r}$ and $d\boldsymbol{\phi}$ denote Lebesgue measure on $\mathbb{R}^m$ and $\mathbb{R}^M$, respectively, and

$$\boldsymbol{x}_j = r_j \bigg(\sum_{1 \leq i < j} \Big(\prod_{i' < i} \sin \phi_{i',j} \Big) (\cos \phi_{i,j}) \boldsymbol{e}_i + \Big(\prod_{i' < j} \sin \phi_{i',j} \Big) \boldsymbol{e}_j \bigg). \tag{5.6}$$

(In particular $\boldsymbol{x}_1 = r_1 \boldsymbol{e}_1$.) We view $\mathbb{R}^m$ as a subspace of $\mathbb{R}^n$ through $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. Then all the $\boldsymbol{x}_j$ in (5.6) lie in $\mathbb{R}^m$. The desired formula now follows by performing the same substitutions backwards, in $\mathbb{R}^m$ instead of in $\mathbb{R}^n$, and using $(\prod_{j=1}^{m} r_j) \prod_{i<j} \sin \phi_{i,j} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m]$. □

Applying Lemma 5.2 to the integral in (5.2), we see that the asymptotics of $J_a^{(n)}[c_1, \ldots, c_a]$ as $n \to \infty$ depends mainly on the quantity

$$\mathcal{V}_a[c_1, \ldots, c_a] := \sup\Big\{ [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a] \;:\; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_a \in \mathbb{R}^a,\ |\boldsymbol{x}_j| \leq 1\ (\forall j),$$
$$|c_1 \boldsymbol{x}_1 + \cdots + c_a \boldsymbol{x}_a| \leq 1 \Big\}. \tag{5.7}$$

Lemma 5.3. *For any $1 \leq a \leq n$ and $c_1, \ldots, c_a > 0$,*

$$J_a^{(n)}[c_1, \ldots, c_a] \ll_a n^{a(a+3)/4} V_n^a\, \mathcal{V}_a[c_1, \ldots, c_a]^{n-a}. \tag{5.8}$$

On the other hand, for any fixed $c_1, \ldots, c_a \in \mathbb{R}_{>0}$ and $\mathcal{V} \in (0, \mathcal{V}_a[c_1, \ldots, c_a])$, we have $\lim_{n \to \infty} \mathcal{V}^{-n} V_n^{-a} J_a^{(n)}[c_1, \ldots, c_a] = \infty$.

Proof. Let B be the open unit ball in $\mathbb{R}^a$ centered at the origin. Then Lemma 5.2 gives

$$J_a^{(n)}[c_1, \ldots, c_a] = \frac{\prod_{j=n-a+1}^{n} \omega_j}{\prod_{j=1}^{a} \omega_j} \int_{B^a} I\bigg(\Big| \sum_{i=1}^{a} c_i \boldsymbol{x}_i \Big| < 1 \bigg) [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a]^{n-a}\, d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_a$$
$$\leq \frac{\prod_{j=n-a+1}^{n} \omega_j}{\prod_{j=1}^{a} \omega_j} V_a^a \mathcal{V}_a[c_1, \ldots, c_a]^{n-a}. \tag{5.9}$$

Furthermore, by Stirling's formula,

$$\omega_j = j V_j = \frac{2\pi^{j/2}}{\Gamma(j/2)} \asymp_a n^{1+(n-j)/2} V_n \tag{5.10}$$

for all $j \in \{n-a+1, n-a+2, \ldots, n\}$. These two bounds imply (5.8).

Next, let $c_1,\ldots,c_a\in\mathbb{R}_{>0}$ and $\mathcal{V}$ be given as in the statement of the lemma. It is clear from (5.7) that there exist non-empty open subsets $U_1,\ldots,U_a$ of B such that all $(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_a)\in U_1\times\cdots\times U_a$ satisfy both $|c_1\boldsymbol{x}_1+\cdots+c_a\boldsymbol{x}_a|<1$ and $[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_a]>\mathcal{V}$. Using the first equality in (5.9), it follows that

$$J_a^{(n)}[c_1,\ldots,c_a]\geq\frac{\prod_{j=n-a+1}^{n}\omega_j\prod_{j=1}^{a}\operatorname{vol}(U_j)}{\prod_{j=1}^{a}\omega_j}\mathcal{V}^{n-a}.$$

Using this and (5.10), the second claim of the lemma follows. □

The next lemma gives a bound on the product $\frac{e_1}{q}\cdots\frac{e_m}{q}$ appearing in (3.2). Recall that $e_i=(\varepsilon_i,q)$, where $\varepsilon_1,\ldots,\varepsilon_m$ are the elementary divisors of the matrix D.

Lemma 5.4. *For any D as in Lemma 5.1,*

$$\frac{e_1}{q}\cdots\frac{e_m}{q}\leq\prod_{j=1}^{r}\frac{g_j}{q},\qquad\textit{with }\ g_j=\gcd\big(\{q\}\cup\{d_{i,\mu_j'}\,:\,i\in A_j\}\big).\tag{5.11}$$

(Note that if $A_j=\emptyset$ then $g_j=q$, giving a factor 1 in the product in (5.11).)

Proof. By [19, Lemma 1],

$$e_1\cdots e_m=N(D,q):=\#\Big\{(x_1,\ldots,x_m)\in(\mathbb{Z}/q\mathbb{Z})^m\,:\,\sum_{i=1}^{m}d_{ij}x_i\equiv0\bmod q\ \ (\forall j)\Big\}.$$

As a preliminary step, note that for any integers $c,d_1,\ldots,d_\ell$,

$$\#\Big\{(x_1,\ldots,x_\ell)\in(\mathbb{Z}/q\mathbb{Z})^\ell\,:\,\sum_{j=1}^{\ell}d_jx_j\equiv c\bmod q\Big\}\leq q^{\ell-1}\gcd(q,d_1,\ldots,d_\ell).\tag{5.12}$$

Indeed, this is immediate when q is a prime power, and the general case can be reduced to this case using the Chinese Remainder Theorem. We now set $\widetilde{A}_0:=\emptyset$ and $\widetilde{A}_j:=A_1\cup\cdots\cup A_j=\overline{A}_1\cup\cdots\cup\overline{A}_j$ for $j\geq1$. For any $j\in\{1,\ldots,r\}$ and any given $(x_i)_{i\in\widetilde{A}_{j-1}}$ in $(\mathbb{Z}/q\mathbb{Z})^{\#\widetilde{A}_{j-1}}$, it follows from (5.12) that the number of tuples $(x_i)_{i\in A_j}\in(\mathbb{Z}/q\mathbb{Z})^{a_j}$ satisfying $\sum_{i=1}^{m}d_{i,\mu_j'}x_i\equiv0\bmod q$ is less than or equal to $q^{a_j-1}g_j$. Using this fact for each $j=1,\ldots,r$, we obtain

$$N(D,q)\leq\prod_{j=1}^{r}\big(q^{a_j-1}g_j\big)=q^m\prod_{j=1}^{r}\frac{g_j}{q}.$$

This completes the proof of the lemma. □

5.2. **Some basic properties of $\mathcal{V}_a[c_1,\ldots,c_a]$.** Recall that, for any integer $a\geq1$ and real numbers $c_1,\ldots,c_a>0$,

$$\mathcal{V}_a[c_1,\ldots,c_a]:=\sup\Big\{[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_a]\,:\,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_a\in\mathbb{R}^a,\ |\boldsymbol{x}_j|\leq1\,(\forall j),\ |c_1\boldsymbol{x}_1+\cdots+c_a\boldsymbol{x}_a|\leq1\Big\},\tag{5.13}$$

where $[\boldsymbol{x}_1,\ldots,\boldsymbol{x}_a]$ denotes the volume of the parallelotope in $\mathbb{R}^a$ spanned by $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_a$. Note that $0<\mathcal{V}_a[c_1,\ldots,c_a]\leq1$, and $\mathcal{V}_a[c_1,\ldots,c_a]$ is invariant under any permutation of $c_1,\ldots,c_a$.

Lemma 5.5. $\mathcal{V}_a[c_1,\ldots,c_a]=c_1^{-1}\mathcal{V}_a[c_1^{-1},c_1^{-1}c_2,\ldots,c_1^{-1}c_a]$, *for any $c_1,\ldots,c_a>0$.*

Proof. Set $\boldsymbol{d} := c_1\boldsymbol{x}_1 + \ldots + c_a\boldsymbol{x}_a$ and note that $\boldsymbol{x}_1 = c_1^{-1}(\boldsymbol{d} - \sum_{j=2}^{a} c_j\boldsymbol{x}_j)$ and $[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a] = c_1^{-1}[\boldsymbol{d}, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_a]$. Hence the lemma follows by substituting $\boldsymbol{x}_1 = \boldsymbol{d}^{(old)}$ and $\boldsymbol{x}_j = -\boldsymbol{x}_j^{(old)}$ $(j \geq 2)$ in the definition of $\mathcal{V}_a[c_1^{-1}, c_1^{-1}c_2, \ldots, c_1^{-1}c_a]$. □

Lemma 5.6. *If $c_1^2 + \cdots + c_a^2 \leq 1$, then $\mathcal{V}_a[c_1, \ldots, c_a] = 1$. Furthermore, we have $\mathcal{V}_a[c_1, \ldots, c_a] \leq c_\ell^{-1}$ for each $\ell \in \{1, \ldots, a\}$, and if $c_\ell^2 \geq 1 + \sum_{j \neq \ell} c_j^2$ then $\mathcal{V}_a[c_1, \ldots, c_a] = c_\ell^{-1}$.*

Proof. The first statement is clear by taking $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a$ to be an ON-basis in the definition of $\mathcal{V}_a[c_1, \ldots, c_a]$. The remaining statements follow from the first statement of the lemma, combined with the general bound $\mathcal{V}_a[c_1, \ldots, c_a] \leq 1$, Lemma 5.5, and the invariance of $\mathcal{V}_a[c_1, \ldots, c_a]$ under permutations of $c_1, \ldots, c_a$. □

Remark 5.7. For $a = 1$ we have $\mathcal{V}_1[c] = \min(1, c^{-1})$. This is clear directly from the definition, or from Lemma 5.6.

Lemma 5.8. *For any $c_1, \ldots, c_a > 0$ and $c_1', \ldots, c_a' > 0$,*

$$\mathcal{V}_a[c_1', \ldots, c_a'] \geq \Big(1 + \sum_{j=1}^{a} |c_j - c_j'|\Big)^{-a} \mathcal{V}_a[c_1, \ldots, c_a].$$

In particular $\mathcal{V}_a$ is a continuous function on $(\mathbb{R}_{>0})^a$.

Proof. Set $\delta = (1 + \sum_{j=1}^{a} |c_j - c_j'|)^{-1} \leq 1$. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a$ be vectors which achieve the supremum in (5.13). Then

$$|c_1'\boldsymbol{x}_1 + \cdots + c_a'\boldsymbol{x}_a| \leq |c_1\boldsymbol{x}_1 + \cdots + c_a\boldsymbol{x}_a| + \sum_{j=1}^{a} |c_j - c_j'| \leq \delta^{-1}.$$

Hence the vectors $\delta\boldsymbol{x}_1, \ldots, \delta\boldsymbol{x}_a$ are admissible in the supremum defining $\mathcal{V}_a[c_1', \ldots, c_a']$, so that $\mathcal{V}_a[c_1', \ldots, c_a'] \geq [\delta\boldsymbol{x}_1, \ldots, \delta\boldsymbol{x}_a] = \delta^a \mathcal{V}_a[c_1, \ldots, c_a]$. □

The following technical lemma gives the key input both to a monotonicity property of $\mathcal{V}_a$ which we will need (Lemma 5.10), and to the explicit determination of $\mathcal{V}_a[c_1, \ldots, c_a]$ in the case $c_1 = \cdots = c_a$ (Lemma 5.11).

Lemma 5.9. *Assume $c_1, \ldots, c_a > 0$, $c_1^2 + \cdots + c_a^2 > 1$ and $c_j^2 < 1 + \sum_{\ell \neq j} c_\ell^2$ for each j. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a$ be vectors which achieve the supremum in* (5.13)*. Let $\boldsymbol{d} := c_1\boldsymbol{x}_1 + \ldots + c_a\boldsymbol{x}_a$, and for each $j \in \{1, \ldots, a\}$, let δ_j be the length of the orthogonal projection of $\boldsymbol{d}$ onto the subspace $U_j = \operatorname{Span}\{\boldsymbol{x}_\ell : \ell \in \{1, \ldots, a\} \setminus \{j\}\}$. Then, for each $j \in \{1, \ldots, a\}$,*

(i) there is $\varepsilon > 0$ such that $c_j' \in (c_j - \varepsilon, c_j) \Rightarrow \mathcal{V}_a[c_1, \ldots, c_j', \ldots, c_a] > \mathcal{V}_a[c_1, \ldots, c_a]$;

(ii) $\delta_j^2 + c_j^2 > 1$, and the number $\delta_j^2 - (\delta_j - \delta_j^3)(\delta_j^2 + c_j^2 - 1)^{-1/2}$ is independent of j.

Proof. For each j, $\boldsymbol{x}_j \notin U_j$ since $\mathcal{V}_a[c_1, \ldots, c_a] > 0$; we let $\boldsymbol{e}_j$ be the unique unit vector in $\mathbb{R}^a$ which is orthogonal to U_j and satisfies $\boldsymbol{x}_j \cdot \boldsymbol{e}_j > 0$. Let $\boldsymbol{d}_j$ be the orthogonal projection of $\boldsymbol{d}$ onto U_j; thus $\delta_j = |\boldsymbol{d}_j|$.

Let us fix j temporarily, and set $\boldsymbol{y} = \boldsymbol{d} - c_j\boldsymbol{x}_j \in U_j$ and $y = |\boldsymbol{y}|$. The optimality property of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a$ implies in particular that among all $\boldsymbol{x}_j' \in \mathbb{R}^a$ satisfying $|\boldsymbol{x}_j'| \leq 1$ and $|c_j\boldsymbol{x}_j' + \boldsymbol{y}| \leq 1$, the vector $\boldsymbol{x}_j' = \boldsymbol{x}_j$ has maximal distance from U_j. By a straightforward analysis one deduces from this fact (and $\boldsymbol{x}_j \cdot \boldsymbol{e}_j > 0$) that

$$\boldsymbol{x}_j = -\alpha\boldsymbol{y} + \beta\boldsymbol{e}_j, \tag{5.14}$$

with

$$\begin{cases} \alpha = 0 \text{ and } \beta = 1 & \text{if } y^2 \leq 1 - c_j^2, \\ \alpha = \beta = c_j^{-1} & \text{if } y^2 \leq c_j^2 - 1, \\ \alpha = (2c_j y^2)^{-1}(y^2 + c_j^2 - 1) \text{ and } \beta = \sqrt{1 - (\alpha y)^2} & \text{if } y^2 > |c_j^2 - 1|. \end{cases} \tag{5.15}$$

Let us first assume that $y^2 < 1 - c_j^2$. Then $\boldsymbol{x}_j = \boldsymbol{e}_j$ by (5.15) and $|\boldsymbol{d}| = |\boldsymbol{y} + c_j \boldsymbol{x}_j| = (y^2 + c_j^2)^{1/2} < 1$, and so the optimality property of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a$ forces $\{\boldsymbol{x}_\ell : \ell \neq j\}$ to be an orthonormal basis of U_j. Hence $\sum_{\ell \neq j} c_\ell^2 = y^2 < 1 - c_j^2$, which contradicts our assumption that $c_1^2 + \cdots + c_a^2 > 1$. This shows that $y^2 < 1 - c_j^2$ cannot hold.

Similarly, $y^2 < c_j^2 - 1$ is impossible. Indeed, $[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a] = c_j^{-1}[\boldsymbol{d}, \boldsymbol{x}_1, \ldots, \widehat{\boldsymbol{x}}_j, \ldots, \boldsymbol{x}_a]$ (where $\widehat{\boldsymbol{x}}_j$ denotes omission of $\boldsymbol{x}_j$ in the list), and hence the optimality property of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a$ can be rephrased as saying that the a vectors $\boldsymbol{d}, \boldsymbol{x}_1, \ldots, \widehat{\boldsymbol{x}}_j, \ldots, \boldsymbol{x}_a$ maximize $[\boldsymbol{d}, \boldsymbol{x}_1, \ldots, \widehat{\boldsymbol{x}}_j, \ldots, \boldsymbol{x}_a]$ subject to $|\boldsymbol{d}| \leq 1$, $|\boldsymbol{x}_\ell| \leq 1$ (all $\ell \neq j$) and $|\boldsymbol{d} - \sum_{\ell \neq j} c_\ell \boldsymbol{x}_\ell| \leq c_j$. Assume now $y^2 < c_j^2 - 1$. Then (5.15) gives $\boldsymbol{d} = \boldsymbol{y} + c_j \boldsymbol{x}_j = \boldsymbol{e}_j$ and $|\boldsymbol{x}_j|^2 = (y/c_j)^2 + (1/c_j)^2 < 1$, and so the optimality property just noted forces $\{\boldsymbol{x}_\ell : \ell \neq j\}$ to again be an orthonormal basis of U_j. Therefore $\sum_{\ell \neq j} c_\ell^2 = y^2 < c_j^2 - 1$, contradicting our assumption that $c_j^2 < 1 + \sum_{\ell \neq j} c_\ell^2$.

In conclusion, $y^2 \geq |c_j^2 - 1|$ must hold. Let us also assume $y > 0$. Then one verifies that the formulas for α and β in the third line of (5.15) hold true (viz., they remain valid even when $y^2 = |c_j^2 - 1|$). These formulas imply $|\boldsymbol{x}_j| = |\boldsymbol{d}| = 1$. Using $\boldsymbol{d}_j = (1 - c_j \alpha)\boldsymbol{y}$ and the formula for α, we obtain $\delta_j = (y^2 + 1 - c_j^2)/(2y)$ and $0 \leq \delta_j \leq y$. Solving for y gives $\delta_j^2 + c_j^2 \geq 1$ and

$$y = \delta_j + \tau_j, \quad \text{with } \tau_j := (\delta_j^2 + c_j^2 - 1)^{1/2}. \tag{5.16}$$

Eliminating $\boldsymbol{y}$ from $\boldsymbol{x}_j = -\alpha \boldsymbol{y} + \beta \boldsymbol{e}_j$ and $\boldsymbol{d} = \boldsymbol{y} + c_j \boldsymbol{x}_j$ gives $(1 - c_j \alpha)\boldsymbol{x}_j = -\alpha \boldsymbol{d} + \beta \boldsymbol{e}_j$, and here $1 - c_j \alpha = \delta_j / y$. Hence $c_j \delta_j \boldsymbol{x}_j = c_j y(\beta \boldsymbol{e}_j - \alpha \boldsymbol{d})$. Using (5.16), we obtain $c_j \alpha y = \tau_j$ and $c_j \beta = (1 - \delta_j^2)^{1/2}$. Therefore

$$c_j \delta_j \boldsymbol{x}_j = (\delta_j + \tau_j)(1 - \delta_j^2)^{1/2} \boldsymbol{e}_j - \tau_j \boldsymbol{d}. \tag{5.17}$$

We take note of two more facts. First:

$$\boldsymbol{d} \cdot \boldsymbol{e}_j = (c_j \boldsymbol{x}_j + \boldsymbol{y}) \cdot \boldsymbol{e}_j = c_j \beta = (1 - \delta_j^2)^{1/2} > 0. \tag{5.18}$$

Second:

$$\tau_j = 0 \Rightarrow \boldsymbol{x}_j = \boldsymbol{e}_j. \tag{5.19}$$

Indeed, $\tau_j = 0$ implies $y = \delta_j = (y^2 + 1 - c_j^2)/(2y)$ by (5.16); thus $y^2 = 1 - c_j^2$, giving $\boldsymbol{x}_j = \boldsymbol{e}_j$.

In the remaining case $y = 0$, we have $c_j = 1$ (since $y^2 \geq |c_j^2 - 1|$) and $\boldsymbol{x}_j = \boldsymbol{e}_j$ (by (5.15), (5.14)); thus also $\boldsymbol{d} = \boldsymbol{e}_j$, $\delta_j = \tau_j = 0$, and all of (5.16)–(5.19) are still valid.

We now prove the first half of (ii), which asserts that in fact $\tau_j > 0$ must hold for all j. Assume $\tau_i = 0$ for some i. Then $\boldsymbol{x}_i = \boldsymbol{e}_i$ by (5.19), and now for every $j \neq i$ we have $\boldsymbol{e}_j \cdot \boldsymbol{e}_i = \boldsymbol{e}_j \cdot \boldsymbol{x}_i = 0$, since $\boldsymbol{x}_i \in U_j$. Similarly $\boldsymbol{x}_j \cdot \boldsymbol{e}_i = 0$. Therefore $\tau_j \boldsymbol{d} \cdot \boldsymbol{e}_i = 0$, by (5.17); but $\boldsymbol{d} \cdot \boldsymbol{e}_i > 0$ (cf. (5.18)); hence $\tau_j = 0$. It follows that $\tau_j = 0$ and $\boldsymbol{x}_j = \boldsymbol{e}_j$ for *all* j; hence $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a$ is an orthonormal basis of $\mathbb{R}^a$. Then $1 = |\boldsymbol{d}|^2 = c_1^2 + \cdots + c_a^2$, which contradicts one of our assumptions. Hence indeed $\tau_j > 0$ for all j.

Next, for any $i \neq j$ in $\{1, \dots, a\}$, we compute $c_i c_j \delta_i \delta_j \boldsymbol{x}_i \cdot \boldsymbol{x}_j$ in two different ways. On the one hand, using (5.17) and $\boldsymbol{x}_i \cdot \boldsymbol{e}_j = 0$, we have

$$\begin{aligned} c_i c_j \delta_i \delta_j \boldsymbol{x}_i \cdot \boldsymbol{x}_j = c_i \delta_i \boldsymbol{x}_i \cdot (-\tau_j \boldsymbol{d}) &= \big((\delta_i + \tau_i)(1-\delta_i^2)^{1/2} \boldsymbol{e}_i - \tau_i \boldsymbol{d}\big) \cdot (-\tau_j \boldsymbol{d}) \\ &= \tau_j(\tau_i \delta_i^2 + \delta_i^3 - \delta_i), \end{aligned} \tag{5.20}$$

where in the last equality we used $|\boldsymbol{d}| = 1$ and $\boldsymbol{e}_i \cdot \boldsymbol{d} = (1-\delta_i^2)^{1/2}$. On the other hand, by symmetry, the same formula holds with i and j interchanged. Thus

$$\tau_j(\tau_i \delta_i^2 + \delta_i^3 - \delta_i) = \tau_i(\tau_j \delta_j^2 + \delta_j^3 - \delta_j). \tag{5.21}$$

This holds for all $i \neq j$, and dividing through with $\tau_i \tau_j$, we have proved (ii).

Let t be the number $\delta_j^2 - \tau_j^{-1}(\delta_j - \delta_j^3)$, which is independent of j. Let us first assume that $\delta_\ell = 0$ for some ℓ. Then $t = 0$, and also $\boldsymbol{d} \cdot \boldsymbol{e}_\ell = 1$ by (5.18), and since $|\boldsymbol{d}| = 1$ this forces $\boldsymbol{d} = \boldsymbol{e}_\ell$. For each $j \neq \ell$, we have $U_j \neq U_\ell$ and thus $\delta_j > 0$. For any $i \neq j$, the right-hand side of (5.20) vanishes, since $t = 0$, and if further $i, j \neq \ell$ then we may divide through with $\delta_i \delta_j$ to conclude that $\boldsymbol{x}_i \cdot \boldsymbol{x}_j = 0$. Hence $\{\boldsymbol{e}_\ell\} \cup \{\boldsymbol{x}_j : j \neq \ell\}$ is an orthonormal basis of $\mathbb{R}^a$. Now, from $c_\ell \boldsymbol{x}_\ell = \boldsymbol{d} - \sum_{j \neq \ell} c_j \boldsymbol{x}_j = \boldsymbol{e}_\ell - \sum_{j \neq \ell} c_j \boldsymbol{x}_j$ it follows that $c_\ell^2 = 1 + \sum_{j \neq \ell} c_j^2$, which contradicts our assumption that $c_\ell^2 < 1 + \sum_{j \neq \ell} c_j^2$. Hence we conclude that $\delta_j > 0$ must hold for all j. Expanding $1 = |\boldsymbol{d}|^2 = |\sum_j c_j \boldsymbol{x}_j|^2$ using (5.20), we now obtain

$$1 = \sum_{j=1}^{a} c_j^2 + 2 \sum_{i<j} \frac{\tau_i \tau_j}{\delta_i \delta_j} t.$$

In view of our assumption $\sum c_j^2 > 1$, this forces $t < 0$. Hence $\tau_j \delta_j < 1 - \delta_j^2$, or equivalently $c_j^2 > (\delta_j + \tau_j)^2 - 1$, for all j.

Now fix j again, and write $\boldsymbol{y} = \boldsymbol{d} - c_j \boldsymbol{x}_j$ and $y = |\boldsymbol{y}|$ as before; note that $y > 0$ since $\tau_j > 0$. By (5.16), $c_j^2 > (\delta_j + \tau_j)^2 - 1$ means that $c_j^2 > y^2 - 1$, and this is easily seen to imply that there is some $\varepsilon > 0$ such that the function $c \mapsto (y^2 + c^2 - 1)/(2yc)$ is strictly increasing in the interval $c \in [c_j - \varepsilon, c_j]$. We have $y^2 > 1 - c_j^2$ since $\tau_j > 0$; hence, by shrinking ε if necessary, we may also assume that $(y^2 + c^2 - 1)/(2yc) > 0$ for all $c \in [c_j - \varepsilon, c_j]$. In particular, taking α, β as in (5.15), and setting, for any given $c_j' \in (c_j - \varepsilon, c_j)$,

$$\alpha' = (2c_j' y^2)^{-1}(y^2 + {c_j'}^2 - 1) \quad \text{and} \quad \beta' = \sqrt{1 - (\alpha' y)^2},$$

we have $0 < y\alpha' < y\alpha < 1$, and hence $\beta' > \beta > 0$. Now set $\boldsymbol{x}_j' = -\alpha' \boldsymbol{y} + \beta' \boldsymbol{e}_j$. Then $|\boldsymbol{x}_j'| = 1$ since $(\alpha' y)^2 + \beta'^2 = 1$, and $|\sum_{i \neq j} c_i \boldsymbol{x}_i + c_j' \boldsymbol{x}_j'| = |\boldsymbol{y} + c_j' \boldsymbol{x}_j'| = 1$ since $(1 - c_j' \alpha')^2 y^2 + {c_j'}^2 \beta'^2 = 1$. Hence

$$\begin{aligned} \mathcal{V}_a[c_1, \dots, c_j', \dots, c_a] \geq [\boldsymbol{x}_1, \dots, \boldsymbol{x}_j', \dots, \boldsymbol{x}_a] = \frac{\beta'}{\beta}[\boldsymbol{x}_1, \dots, \boldsymbol{x}_a] &> [\boldsymbol{x}_1, \dots, \boldsymbol{x}_a] \\ &= \mathcal{V}_a[c_1, \dots, c_a], \end{aligned}$$

which concludes the proof of (i). □

We next establish a monotonicity property of the function $\mathcal{V}_a[c_1, \dots, c_a]$.

Lemma 5.10. *If $c_j \geq c_j' > 0$ for $j = 1, \dots, a$, then $\mathcal{V}_a[c_1, \dots, c_a] \leq \mathcal{V}_a[c_1', \dots, c_a']$.*

Proof. It suffices to prove that for any fixed $c_2, \ldots, c_a > 0$, $\mathcal{V}_a[c_1, c_2, \ldots, c_a]$ is a decreasing function of $c_1 > 0$. Without loss of generality, we assume that $c_2 \geq c_j$ for $j \geq 3$. Set

$$\alpha = \max\Big(0, 1 - \sum_{j\geq 2} c_j^2,\ c_2^2 - 1 - \sum_{j\geq 3} c_j^2\Big)^{1/2} \quad \text{and} \quad \beta = \Big(1 + \sum_{j\geq 2} c_j^2\Big)^{1/2}.$$

Then for $\alpha < c_1 < \beta$, Lemma 5.9 applies, and part (i) of that lemma, together with the continuity of $\mathcal{V}_a$ (cf. Lemma 5.8), implies that $c_1 \mapsto \mathcal{V}_a[c_1, c_2, \ldots, c_a]$ is strictly decreasing for $\alpha < c_1 < \beta$. In fact this is valid for $\alpha \leq c_1 \leq \beta$, again by continuity. Finally, Lemma 5.6 implies that $c_1 \mapsto \mathcal{V}_a[c_1, c_2, \ldots, c_a]$ is decreasing for $0 < c_1 \leq \alpha$ and for $c_1 \geq \beta$, and the proof is complete. □

The following lemma gives the exact value of $\mathcal{V}_a[c_1, \ldots, c_a]$ when $c_1 = \cdots = c_a$.

Lemma 5.11. *For $a \geq 1$ and $0 < c \leq 1$,*

$$\widetilde{\mathcal{V}}_{a,c} := \mathcal{V}_a[c, \ldots, c] = \begin{cases} \sqrt{\dfrac{c^{-2}(a^2 - c^{-2})^{a-1}}{a^a(a-1)^{a-1}}} & \text{if } c > a^{-1/2}, \\ 1 & \text{if } c \leq a^{-1/2}. \end{cases}$$

Proof. The case $c \leq a^{-1/2}$ follows from Lemma 5.6; hence we now assume $c > a^{-1/2}$ (and $a \geq 2$). Then Lemma 5.9 applies. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a$ and $\delta_1, \ldots, \delta_a$ be as in the statement of that lemma. Set $\gamma := 1 - c^2 \in [0, 1)$. One verifies by differentiation that $\frac{\delta - \delta^3}{(\delta^2 - \gamma)^{1/2}}$ is a strictly decreasing function of δ in the interval $\sqrt{\gamma} < \delta \leq 1$; hence, a fortiori, $\delta^2 - \frac{\delta - \delta^3}{(\delta^2 - \gamma)^{1/2}}$ is strictly increasing in that interval. Hence Lemma 5.9 (ii) implies $\delta_1 = \cdots = \delta_a > \sqrt{\gamma}$. Using this in the formula (5.20) (wherein $\tau_i = (\delta_i^2 + c^2 - 1)^{1/2}$), it follows that the scalar product $\boldsymbol{x}_i \cdot \boldsymbol{x}_j$ takes one and the same value for all choices of $i \neq j$. Call this value s. It was also seen in the proof of Lemma 5.9 that $|\boldsymbol{x}_j| = 1$ for all j, and $|\sum_{j=1}^a c\boldsymbol{x}_j| = 1$. Squaring and expanding the last relation gives $c^2(a + a(a-1)s) = 1$. We have thus proved

$$\boldsymbol{x}_i \cdot \boldsymbol{x}_j = s = \frac{c^{-2} - a}{a(a-1)}, \qquad \text{for all } i \neq j.$$

Hence

$$\mathcal{V}_a[c, \ldots, c] = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_a] = \sqrt{D_{a,s}}, \qquad \text{with } D_{a,s} := \begin{vmatrix} 1 & s & \cdots & s \\ s & 1 & \cdots & s \\ \vdots & & \ddots & \vdots \\ s & s & \cdots & 1 \end{vmatrix}.$$

Subtracting s times the first row from each of the other rows, we get

$$D_{a,s} = \begin{vmatrix} 1-s^2 & s-s^2 & \cdots & s-s^2 \\ s-s^2 & 1-s^2 & \cdots & s-s^2 \\ \vdots & & \ddots & \vdots \\ s-s^2 & s-s^2 & \cdots & 1-s^2 \end{vmatrix} = (1-s^2)^{a-1} D_{a-1, s/(1+s)},$$

and from this one proves by induction that $D_{a,s} = (as - s + 1)(1-s)^{a-1}$. This gives the formula stated in the lemma. □

The case $c = 1$ will turn out to be of special importance, and we set

$$\widetilde{\mathcal{V}}_a := \widetilde{\mathcal{V}}_{a,1} = \sqrt{\frac{(a+1)^{a-1}}{a^a}} \quad (a \geq 1); \qquad \widetilde{\mathcal{V}}_0 := 1. \tag{5.22}$$

5.3. **Proof of Theorem 1.7.** For $k = 2$, the statement of Theorem 1.7 follows from Remark 3.2. Hence, from now on we fix k to be an integer ≥ 3. We also fix c and f as in Theorem 1.7; thus $0 < c < c_k$, $\lim_{n\to\infty} f(n) = \infty$ and $f(n) = O(e^{cn})$.

The following lemma takes care of all except finitely many terms in (3.2); it is proved using the same bounds as in Rogers, [20, pp. 245–246], which were also used in the proof of Proposition 3.1 above.

Lemma 5.12. *The total contribution to* (3.2) *from all D which satisfy* $\max\{|d_{ij}|\} \geq \widetilde{\mathcal{V}}_{k-1}^{-1}$ *(the maximum being taken over all entries of D) tends to zero as $n \to \infty$.*

Remark 5.13. If $k \leq 10$ then $\widetilde{\mathcal{V}}_{k-1}^{-1} < 2$, so that Lemma 5.12 in fact takes care of all D except those which have $q = 1$ and all entries $d_{ij} \in \{-1, 0, 1\}$.

Proof. We fix $m \in \{1, \ldots, k-1\}$, and consider the contribution from all D as in the lemma with the further requirement that D is of size $m \times k$. Set $\Delta := \max\{|d_{ij}|\}$. Then, by [27, Remark 1] and [20, (72)],

$$\Big(\frac{e_1}{q} \cdots \frac{e_m}{q}\Big)^n \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{k} \chi_n\Big(\sum_{i=1}^{m} \frac{d_{ij}}{q} \boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_m \leq f(n)^m \Delta^{-n}.$$

Note that the number of $\langle k, q\rangle$-admissible matrices of size $m \times k$ and with given values of q and Δ, is less than $\binom{k-1}{m-1}(3\Delta)^{m(k-m)}$, and there are *no* such matrices with $\Delta < q$. Hence, if we let v_k be the smallest integer $\geq \widetilde{\mathcal{V}}_{k-1}^{-1}$ (thus $v_k \geq 2$), and assume that $n \geq m(k-m) + 3$, then the total contribution to (3.2) from all D with $q \geq \widetilde{\mathcal{V}}_{k-1}^{-1}$ is

$$\leq \binom{k-1}{m-1} f(n)^{m-k/2} \sum_{q=v_k}^{\infty} \sum_{\Delta \geq q} (3\Delta)^{m(k-m)} \Delta^{-n} \ll_k f(n)^{m-k/2} v_k^{-n}.$$

Similarly, assuming $n \geq m(k-m) + 2$, the total contribution to (3.2) from all D satisfying $q < \widetilde{\mathcal{V}}_{k-1}^{-1}$ and $\Delta \geq \widetilde{\mathcal{V}}_{k-1}^{-1}$ (viz., $q < v_k$ and $\Delta \geq v_k$) is

$$\leq \binom{k-1}{m-1} f(n)^{m-k/2} \sum_{q=1}^{v_k - 1} \sum_{\Delta = v_k}^{\infty} (3\Delta)^{m(k-m)} \Delta^{-n} \ll_k f(n)^{m-k/2} v_k^{-n}.$$

Finally, using $\lim_{n\to\infty} f(n) = \infty$ and $f(n) = O(e^{cn})$ with $0 < c < c_k$, the desired convergence is seen to follow from the fact that

$$c\Big(m - \frac{k}{2}\Big) - \log v_k < c_k\Big(\frac{k}{2} - 1\Big) + \log \widetilde{\mathcal{V}}_{k-1} = 0,$$

cf. (1.7) and (5.22). □

In the next three lemmas, we let D be any fixed $\langle k, q\rangle$-admissible matrix appearing in the sum in (3.2). (We could assume that D does not satisfy the condition in Lemma 5.12, but we won't need this.) Let $m, r, (\mu'_j)_{j=1}^r, (\overline{A}_j)_{j=1}^r, (A_j)_{j=1}^r, (a_j)_{j=1}^r$ be as in Section 5.1.

Lemma 5.14. *If* $n \geq \max(a_1, \ldots, a_r)$, *then*

$$\Big(\frac{e_1}{q} \cdots \frac{e_m}{q}\Big)^n \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{k} \chi_n\Big(\sum_{i=1}^{m} \frac{d_{ij}}{q} \boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_m$$

$$\ll_m n^{m(m+3)/4} f(n)^m \Big(\prod_{j=1}^{r} \widetilde{\mathcal{V}}_{a_j}\Big)^n. \tag{5.23}$$

Proof. By Lemmas 5.1, 5.3, 5.4, and using $\sum_{j=1}^r a_j = m$ (thus $\sum_{j=1}^r a_j^2 \leq m^2$), the left-hand side of (5.23) is

$$\ll_m n^{m(m+3)/4} f(n)^m \prod_{j=1}^{r} \Big(c_j^n\, \mathcal{V}_{a_j}\big[\big(|d_{i,\mu'_j}|/q\big)_{i \in A_j}\big]^{n-a_j}\Big), \tag{5.24}$$

where $c_j := q^{-1} \gcd\big(\{q\} \cup \{d_{i,\mu'_j} : i \in A_j\}\big)$, and we use the convention that $\mathcal{V}_0[\,] := 1$. Using Lemma 5.10 and the fact that $|d_{i,\mu'_j}| \geq qc_j$ for all $i \in A_j$, we have $\mathcal{V}_{a_j}[(|d_{i,\mu'_j}|/q)_{i\in A_j}] \leq \widetilde{\mathcal{V}}_{a_j,c_j}$ for each j. Note that $0 < c_j \leq 1$ by definition, and thus $\widetilde{\mathcal{V}}_{a_j,c_j} \geq \widetilde{\mathcal{V}}_{a_j} \gg_m 1$. Also, inspecting the formula in Lemma 5.11, one notes that for any fixed $a \geq 1$, $c\,\widetilde{\mathcal{V}}_{a,c}$ is a strictly increasing function of $c \in (0,1]$; on the other hand, for each j with $a_j = 0$ we have $c_j = 1$ and $\widetilde{\mathcal{V}}_{a_j,c_j} = 1$. Using these facts, we see that for each $j \in \{1, \ldots, r\}$,

$$c_j^n\, \mathcal{V}_{a_j}\big[\big(|d_{i,\mu'_j}|/q\big)_{i\in A_j}\big]^{n-a_j} \ll_m \big(c_j\, \widetilde{\mathcal{V}}_{a_j,c_j}\big)^n \leq \widetilde{\mathcal{V}}_{a_j}^n. \tag{5.25}$$

Now (5.23) follows from (5.24) and (5.25). □

Lemma 5.15. *Let* D *be as above, and assume furthermore that* D *has some column containing more than one non-zero element. Then the contribution from* D *to* (3.2) *tends to zero as* $n \to \infty$.

Proof. Recall that Lemma 5.14 is valid for $\mu'_1, \ldots, \mu'_r$ an arbitrary permutation of $\mu_1, \ldots, \mu_r$. We now fix the choice of $\mu'_1, \ldots, \mu'_r$ so that the number of non-zero elements in column number μ'_1 is as large as possible. Then $a_1 = \#A_1 = \#\overline{A}_1 \geq \#\overline{A}_j \geq a_j$ for all $j \in \{1, \ldots, r\}$, and $a_1 \geq 2$ by our assumption on D.

Now note that $\log(\widetilde{\mathcal{V}}_x)$, which we take to be defined for arbitrary real $x \geq 1$ through the formula (5.22), is a strictly decreasing and strictly convex function of $x \geq 1$. This is easily verified by differentiation. It follows that for any $j \geq 2$, if $a_j \geq 2$ (and thus $a_1 \geq a_j \geq 2$), the product $\prod_{j=1}^r \widetilde{\mathcal{V}}_{a_j}$ *increases* if we simultaneously replace a_1 by $a_1 + 1$ and a_j by $a_j - 1$. Repeating this operation for as long as possible, and recalling $\widetilde{\mathcal{V}}_1 = \widetilde{\mathcal{V}}_0 = 1$, we conclude that $\prod_{j=1}^r \widetilde{\mathcal{V}}_{a_j} \leq \widetilde{\mathcal{V}}_a$ for some integer $a \geq a_1 \geq 2$ satisfying $a + r - 1 \geq m$, i.e. $a \geq 2m - k + 1$. Hence, applying Lemma 5.14 and dividing through by $f(n)^{k/2}$, we conclude that the contribution from D to (3.2) is

$$\ll_m n^{m(m+3)/4} f(n)^{m-k/2} \widetilde{\mathcal{V}}_a^n.$$

If $m \leq k/2$, then this bound obviously tends to zero as $n \to \infty$, since $\widetilde{\mathcal{V}}_a < 1$ and $f(n) \to \infty$; hence from now on we assume that $m > k/2$. Then, using the assumption $f(n) = O(e^{cn})$ and the fact that $\widetilde{\mathcal{V}}_a$ is a decreasing function of a, we see that our term is $\ll_m n^{m(m+3)/4} \exp\big((c(m-k/2) + \log \widetilde{\mathcal{V}}_{2m-k+1})n\big)$, and hence to complete the proof of the lemma it suffices to prove that

$$c < \frac{-2\log \widetilde{\mathcal{V}}_{2m-k+1}}{2m-k}. \tag{5.26}$$

However, by what we noted above, $-2\log \widetilde{\mathcal{V}}_{x+1}$ is a strictly concave function of $x \geq 0$, taking the value 0 at $x = 0$. Also $2m-k \leq k-2$. Hence

$$\frac{-2\log \widetilde{\mathcal{V}}_{2m-k+1}}{2m-k} \geq \frac{-2\log \widetilde{\mathcal{V}}_{k-1}}{k-2} = c_k$$

(cf. (1.7) and (5.22)), and so (5.26) follows from the assumption that $c < c_k$. □

The matrices D not covered by Lemma 5.15 are very easy to handle:

Lemma 5.16. *Let D be a matrix appearing in* (3.2) *with exactly one non-zero element in each column. Then either D is accounted for in $M_{k,n}$ (cf.* (3.3)*) or else the contribution from D to* (3.2) *tends to zero as $n \to \infty$.*

Proof. Let $\Delta_i := \max(|d_{i1}|, |d_{i2}|, \ldots, |d_{ik}|)$ for $i = 1, \ldots, m$. Then, using [27, Remark 1], we obtain

$$\frac{1}{(2f(n))^{k/2}} \Big(\frac{e_1}{q} \cdots \frac{e_m}{q}\Big)^n \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{k} \chi_n\Big(\sum_{i=1}^{m} \frac{d_{ij}}{q}\boldsymbol{x}_i\Big)\, d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_m$$

$$\leq f(n)^{-k/2} q^{-n} \prod_{i=1}^{m} \Big(\int_{\mathbb{R}^n} \chi_n\Big(\frac{\Delta_i}{q}\boldsymbol{x}_i\Big)\, d\boldsymbol{x}_i\Big) = q^{-n} f(n)^{m-k/2} \prod_{i=1}^{m} \Big(\frac{q}{\Delta_i}\Big)^n. \tag{5.27}$$

Now note that $k \geq 2m$, since D has exactly one non-zero element in each column but at least two non-zero entries in each row. Hence, if we keep n so large that $f(n) \geq 1$, we have $f(n)^{m-k/2} \leq 1$. Note also that $\Delta_i \geq q$ for each i, since D is $\langle k, q\rangle$-admissible. Furthermore, assuming that D is *not* accounted for in $M_{k,n}$, we have either $q \geq 2$ or $q = 1$ at the same time as $\Delta_i > 1$ for some i. Hence the bound in (5.27) is $\leq 2^{-n}$, and the lemma is proved. □

Proof of Theorem 1.7. Taken together, Lemma 5.12 and Lemmas 5.14–5.16 show that the total contribution from all D in (3.2) which are not accounted for in $M_{k,n}$ tends to zero as $n \to \infty$. On the other hand, the treatment of $M_{k,n}$ in the proof of Proposition 3.1 applies verbatim in the present situation with a more general function f, and shows that $\lim_{n\to\infty}(2f(n))^{-k/2}M_{k,n}$ exists and equals 0 for k odd and $(k-1)!!$ for k even. Hence (1.8) holds.

We now turn to the second statement of Theorem 1.7. Thus assume that $k \geq 3$ and $c > c_k$; let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ be a function satisfying $f(n) \gg e^{cn}$ as $n \to \infty$, and consider (3.2) with χ_n being the characteristic function of the closed ball of volume $f(n)$ centered at the origin. Then the contribution from any matrix D as in (5.1) to the sum in (3.2) equals

$$2^{-k/2} f(n)^{\frac{k}{2}-1} V_n^{1-k} J_{k-1}^{(n)}[1, \ldots, 1]. \tag{5.28}$$

Now $c > c_k$ implies that $e^{(1-\frac{k}{2})c} < \widetilde{\mathcal{V}}_{k-1}$ (cf. (1.7) and (5.22)); hence by the second part of Lemma 5.3, the expression in (5.28) tends to ∞ as $n \to \infty$. This completes the proof of Theorem 1.7. □

References

[1] V. Bentkus, F. Götze, On the lattice point problem for ellipsoids, Acta Arith. **80** (1997), no. 2, 101–125.

[2] V. Bentkus, F. Götze, Lattice point problems and distribution of values of quadratic forms, Ann. of Math. **150** (1999), no. 3, 977–1027.

[3] P. Billingsley, *Convergence of probability measures*, second edition, Wiley Series in Probability and Statistics, John Wiley & Sons Inc., New York, 1999.

[4] P. Bleher, On the distribution of the number of lattice points inside a family of convex ovals, Duke Math. J. **167** (1992), no. 3, 461–481.

[5] F. Götze, Lattice point problems and the central limit theorem in Euclidean spaces, Proceedings of the ICM 1998, pp. 245–255.

[6] F. Götze, Lattice point problems and values of quadratic forms, Invent. Math. **157** (2004), no. 1, 195–226.

[7] G. H. Hardy, The average order of the arithmetical functions $P(x)$ and $\Delta(x)$, Proc. London Math. Soc. s2-15 (1917), no. 1, 192–213.

[8] D. R. Heath-Brown, The distribution and moments of the error term in the Dirichlet divisor problem, Acta Arith. **60** (1992), no. 4, 389–415.

[9] S. Holmin, The number of points from a random lattice that lie inside a ball, preprint 2013, arXiv:1311.2865.

[10] L. K. Hua, *Introduction to number theory*, Translated from the Chinese by Peter Shiu, Springer-Verlag, 1982.

[11] C. P. Hughes, Z. Rudnick, On the distribution of lattice points in thin annuli, Int. Math. Res. Not. (2004), no. 13, 637–658.

[12] M. N. Huxley, Exponential sums and lattice points. III, Proc. London Math. Soc. **87** (2003), no. 3, 591–609.

[13] D. Kelmer, On the mean square of the remainder for the Euclidean lattice point counting problem, preprint 2015, arXiv:1512.06012.

[14] S. Kim, On the distribution of lengths of short vectors in a random lattice, Math. Z. **282** (2016), no. 3-4, 1117–1126.

[15] S. Kim, Random lattice vectors in a set of size $O(n)$, preprint 2016, arXiv:1611.02880.

[16] E. Landau, Über die Anzahl der Gitterpunkte in gewissen Bereichen, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1924 (1924), 137–150.

[17] M. Peter, Almost periodicity and the remainder in the ellipsoid problem, Michigan Math. J. **49** (2001), no. 2, 331–351.

[18] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972.

[19] C. A. Rogers, Mean values over the space of lattices, Acta Math. **94** (1955), 249–287.

[20] C. A. Rogers, The moments of the number of points of a lattice in a bounded set, Phil. Trans. R. Soc. Lond. A. **248** (1955), 225–251.

[21] C. A. Rogers, The number of lattice points in a set, Proc. London Math. Soc. (3) **6** (1956), 305–320.

[22] C. A. Rogers, A single integral inequality, J. London Math. Soc. **32** (1957), 102–108.

[23] P. Sarnak and A. Strömbergsson, Minima of Epstein's Zeta Function and Heights of Flat Tori, Invent. Math. **165** (2006), 115–151.

[24] W. Schmidt, On the convergence of mean values over lattices, Canad. J. Math. **10** (1958), 103–110.

[25] W. M. Schmidt, A metrical theorem in geometry of numbers, Trans. Amer. Math. Soc. **95** (1960), 516–529.

[26] C. L. Siegel, A mean value theorem in geometry of numbers, Ann. of Math. **46** (1945), 340–347.

[27] A. Södergren, On the Poisson distribution of lengths of lattice vectors in a random lattice, Math. Z. **269** (2011), no. 3-4, 945–954.

[28] A. Södergren, On the distribution of angles between the N shortest vectors in a random lattice, J. London Math. Soc. (2) **84** (2011), no. 3, 749–764.

[29] A. Södergren, On the value distribution and moments of the Epstein zeta function to the right of the critical strip, J. Number Theory **131** (2011), 1176–1208.

[30] A. Södergren, On the value distribution of the Epstein zeta function in the critical strip, Duke Math. J. **162** (2013), no. 1, 1–48.
[31] A. Terras, *Harmonic analysis on symmetric spaces and applications, II*, Springer-Verlag, Berlin, 1988.
[32] I. Wigman, Statistics of lattice points in thin annuli for generic lattices, Doc. Math. **11** (2006), 1–23.

Department of Mathematics, Box 480, Uppsala University, 751 06 Uppsala, Sweden
astrombe@math.uu.se

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark
Present address: School of Science and Technology, Örebro University, 701 82 Örebro, Sweden
anders.sodergren@oru.se