V. SOME RESULTS ON THE UNIFORM EQUIDISTRIBUTION OF LONG CLOSED HOROCYCLES

Contents

1. Introduction 137
1.1. Acknowledgments 139
2. Equidistribution for $\beta - \alpha \geqq y^{1/2-\varepsilon}$ 139
2.1. Some preliminaries 139
2.2. Some fundamental estimates 144
2.3. Application of spectral decomposition 150
2.4. The case $A_1 \neq \{0\}$ 161
3. Explicit bounds on the error term 170
3.1. Two lemmas on the K-Bessel function 170
3.2. Uniform spectral bounds 171
4. Generic equidistribution for $\beta - \alpha \geqq y^{1-\varepsilon}$ 180
4.1. A genericity assumption 180
4.2. Some fundamental estimates 183
4.3. The incomplete Eisenstein series 186
4.4. The horocycle integral of $\theta_{\psi,k}$ 192
4.5. Proof of the main theorem 198
4.6. A result concerning the distribution of cusps 202
5. Alternative approach, using invariant measures 207
5.1. Preliminaries 207
5.2. The theorem 210
6. Appendix 1, proof of an approximation result 217
7. Appendix 2, some remarks on the Farey series 218
References 224

1. Introduction

Let Γ be a cofinite Fuchsian group acting on the Poincaré upper half-plane $\mathcal{H}$ with metric $ds = |dz|/y$. We assume that $\Gamma \setminus \mathcal{H}$ has at least one cusp. By an auxiliary conjugation, we may then assume that $\Gamma \setminus \mathcal{H}$ has one cusp located at $i\infty$, and that the isotropy group Γ_∞ is generated by the translation $S(z) = z + 1$.

Now for any $y > 0$, the curve $\{x + iy \mid x \in [0, 1]\}$ is a *closed horocycle* of length $1/y$ on $\Gamma \setminus \mathcal{H}$. When $y \to 0$, this curve is known to become *equidistributed* on $\Gamma \setminus \mathcal{H}$ with respect to the Poincaré area μ. (Recall that $d\mu = y^{-2}dxdy$.) Investigations related to this fact have been done by a number of people over the years including Selberg (unpublished), Zagier [Z1], Sarnak [S1] and Hejhal [H5, H6].

By elementary functional analysis, the equidistribution fact just mentioned is equivalent to the assertion that

$$\int_0^1 f(x + iy)\, dx \to \frac{1}{\mu(\Gamma \setminus \mathcal{H})} \int_{\Gamma \setminus \mathcal{H}} f(z)\, d\mu(z)$$

holds for every compactly supported function $f \in C(\Gamma \setminus \mathcal{H})$ as $y \to 0^+$.

It was proved in [H6] that the asymptotic equidistribution also takes place for *subsegments* "$[\alpha,\beta]+iy$" of the closed horocycle, so long as $\beta-\alpha$ is kept bigger than $y^{c(\Gamma)-\varepsilon}$. Here $c(\Gamma)$ is a certain positive constant $\leqq \frac{1}{3}$ which only depends on the group Γ. (One has $c(\Gamma)=\frac{1}{3}$ unless there are small eigenvalues $\lambda \in (0,\frac{3}{16})$ of the Laplacian on $\Gamma \setminus \mathcal{H}$.)

Our goal in the present paper is to strengthen and extend this result in several ways.

First of all, we will prove that the asymptotic equidistribution of subsegments holds already when $\beta-\alpha$ is kept bigger than y^{γ} *for any fixed exponent* $\gamma<\frac{1}{2}$, on *any* cofinite Fuchsian group Γ as above (*i.e.,* with no restrictions on small eigenvalues). In precise terms:

Let Γ *be given as above, and fix any number* $0<\gamma<\frac{1}{2}$. *Let* $f:\mathcal{H}\to\mathbb{C}$ *be any fixed bounded, continuous and* Γ*-invariant function. Then*

$$\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(x+iy)\,dx \to \frac{1}{\mu(\Gamma\setminus\mathcal{H})}\int_{\Gamma\setminus\mathcal{H}} f(z)\,d\mu(z) \tag{1.1}$$

uniformly as $y\to 0^{+}$ *so long as* $\beta-\alpha$ *remains bigger than* y^{γ}.

We will prove this theorem in Section 2 (under even milder assumptions on f). We remark that the restriction $\gamma<\frac{1}{2}$ is the best possible result (cf. §4.1). In Section 3 we will also obtain bounds which give *explicit information on the rate of convergence in (1.1)*, for a more restricted class of functions f. (A similar bound was proved in [H6] for exponents $\gamma<c(\Gamma)$.)

Next, in Section 4, we will prove that *if we keep* α *(or* β*) generic*, the asymptotic equidistribution actually takes place for *any exponent* $\gamma<1$. Specifically:

Let Γ *be given as above, and fix any number* $\frac{1}{2}<\gamma<1$. *Then, there exists a family of subsets* $\mathcal{G}(y)\subseteq\mathbb{R}$ *(which will be defined in a completely explicit way) such that*

$$\lim_{y\to 0^{+}} m\big([\mu,\nu]\cap\mathcal{G}(y)\big)=\nu-\mu \qquad \textit{for any}\ \ \mu<\nu$$

(where m *denotes ordinary Lebesgue measure), and, for any bounded, continuous and* Γ*-invariant function* $f:\mathcal{H}\to\mathbb{C}$,

$$\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(x+iy)\,dx \to \frac{1}{\mu(\Gamma\setminus\mathcal{H})}\int_{\Gamma\setminus\mathcal{H}} f(z)\,d\mu(z)$$

uniformly as $y\to 0^{+}$ *so long as* $\alpha\in\mathcal{G}(y)$ *and* $\beta-\alpha\geqq y^{\gamma}$.

As a consequence of the proofs in Section 4, we will also obtain (in §4.6) a result on the distribution of cusps on (short) subintervals of the boundary of $\mathcal{H}$. Here, to each cusp $\eta\in\mathbb{R}\subset\partial\mathcal{H}$ we attach, as a kind of "weight", the size of the unique horoball $\mathfrak{B}=\mathfrak{B}_{\eta}$ which is tangent to $\partial\mathcal{H}$ at η and for which $\mu(\Gamma_{\eta}\setminus\mathfrak{B})=1$. The result is, then:

Let Γ *be given as above, fix any number* $0<\gamma<\frac{1}{2}$, *and let* η_0 *be a fixed cusp. Denote by* $C(y,\alpha,\beta)$ *the number of cusps* $\eta\in[\alpha,\beta]$ *which are (*Γ*-)equivalent to* η_0 *and for which* $\mathfrak{B}_{\eta}$ *has Euclidean diameter* $\geqq y$. *We then have*

$$\frac{y}{\beta-\alpha}C(y,\alpha,\beta)\to\frac{1}{\pi\mu(\Gamma\setminus\mathcal{H})}$$

uniformly as $y\to 0^{+}$ *so long as* $\beta-\alpha\geqq y^{\gamma}$.

We will also obtain a similar result for any exponent $\frac{1}{2}<\gamma<1$, when α is kept generic.

Our proofs in Sections 2–4 are based on spectral-theoretic techniques, and the style is completely classical, as in [H6]. Nevertheless, our method to prove (1.1) differs a lot from the method used in [H6].

Finally, in Section 5, we will give alternative proofs of some of our main theorems, based on the characterization by Dani [D1, D2] of ergodic measures on $\Gamma \backslash \mathrm{SL}(2,\mathbb{R})$ invariant under the horocycle flow (see also Ratner, [R3]).

1.1. **Acknowledgments.** I am grateful to Dennis Hejhal for introducing me to these topics (in particular, for providing me with a copy of his manuscript containing the detailed proof of [H5, Thm. 1′], which later appeared in [H6] in strengthened form). I am also thankful to him for many inspiring and helpful discussions, and for his careful reading and valuable criticism of the present paper. I would also like to thank Jens Marklof for rewarding discussions.

2. EQUIDISTRIBUTION FOR $\beta - \alpha \geqq y^{1/2-\varepsilon}$

2.1. **Some preliminaries.** Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a cofinite Fuchsian group acting on the Poincaré upper half-plane $\mathcal{H}$. We assume that $\Gamma \setminus \mathcal{H}$ has at least one cusp.

Concerning the cusps and the fundamental domain, we will use the same notation as in [H3, p. 268] (cf. also [H3, pp. 282 item 2 ($\mathcal{F}_Y$), 291(3.24)]), except for some very minor differences. Specifically: we let $\mathcal{F} \subset \mathcal{H}$ be a canonical fundamental domain for $\Gamma \setminus \mathcal{H}$. We assume $\mathcal{F}$ to be *closed.* Thus $\cup_{T\in\Gamma} T(\mathcal{F}) = \mathcal{H}$, the set $\mathcal{F}$ equals the closure (in $\mathcal{H}$) of its interior $\overset{\circ}{\mathcal{F}}$, and the sets $T(\overset{\circ}{\mathcal{F}})$ (for $T \in \Gamma$) are pairwise disjoint.

We let $\eta_1, ..., \eta_\kappa$ (where $\kappa \geqq 1$) be the vertices of $\mathcal{F}$ along $\partial\mathcal{H} = \mathbb{R} \cup \{\infty\}$. Since $\mathcal{F}$ is canonical, $\eta_1, ..., \eta_\kappa$ are $\Gamma-$inequivalent.

We denote:

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For each $k \in \{1, ..., \kappa\}$ we choose $N_k \in \mathrm{PSL}(2,\mathbb{R})$ such that $N_k(\eta_k) = \infty$ and such that the stabilizer Γ_{η_k} is $[T_k]$, where $T_k := N_k^{-1} S^{-1} N_k$. Since $\mathcal{F}$ is canonical, by modifying N_k we can *also* ensure that

$$\begin{aligned} N_k(\mathcal{F}) \bigcap \{z \in \mathcal{H} \mid \mathrm{Im}\, z \geqq B\} \\ = \{z \in \mathcal{H} \mid 0 \leqq \mathrm{Re}\, z \leqq 1,\ \mathrm{Im}\, z \geqq B\}, \end{aligned} \tag{2.1}$$

for all B large enough. We *fix*, once and for all, a constant $B_0 > 1$ such that (2.1) holds for all $B \geqq B_0$ and all $k \in \{1, ..., \kappa\}$.

For $B \geqq B_0$, the corresponding cuspidal region in $\mathcal{F}$ is called $\mathcal{C}_{kB}$:

$$\mathcal{C}_{kB} = N_k^{-1}\big([0,1] \times [B, \infty)\big) \subset \mathcal{F} \tag{2.2}$$

We then define

$$\mathcal{F}_B = \mathcal{F} - \bigcup_{k=1}^{\kappa} \mathcal{C}_{kB}. \tag{2.3}$$

This is a bounded region.

Our assumption $B_0 > 1$ does in fact force the cuspidal regions $\mathcal{C}_{kB_0}$ to be pairwise disjoint. This is because of an observation that will also be useful for other purposes:

Lemma 2.1.1. *Let $j, k \in \{1, ..., \kappa\}$ be given. Take $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in N_k \Gamma N_j^{-1}$. Then:*

$$|c| \geqq 1, \quad \textit{unless } k = j \textit{ and } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S]. \tag{2.4}$$

Hence, for all $z \in \mathcal{H}$ and $W \in \Gamma$:

$$Im\ N_k W(z) \leqq (Im\ N_j(z))^{-1}, \quad \textit{unless } k = j \textit{ and } W \in [T_k]. \tag{2.5}$$

In particular, the cuspidal regions $\mathcal{C}_{kB_0}$ ($k \in \{1, ..., \kappa\}$) are pair-wise disjoint.

Proof. Our proof of (2.4) is taken from notes of D. Hejhal. We write $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = N_k W N_j^{-1}$ with $W \in \Gamma$, and consider the product TST^{-1}. Clearly,

$$TST^{-1} = N_k W N_j^{-1} S N_j W^{-1} N_k^{-1} \in N_k \Gamma N_k^{-1},$$

since $N_j^{-1} S N_j = T_j^{-1} \in \Gamma$. Notice that $N_k \Gamma N_k^{-1}$ is a Fuchsian group having ∞ as a cusp, with stabilizer $(N_k \Gamma N_k^{-1})_\infty = [S]$. Hence by Shimizu's lemma (cf. [S2, Lemma 4], or [M4, Lemma 1.7.3]) the lower left element in the matrix TST^{-1} has modulus $\geqq 1$, except if $TST^{-1} \in [S]$. We evaluate:

$$TST^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 - ac & * \\ -c^2 & 1 + ac \end{pmatrix}.$$

We thus conclude that either $|c| \geqq 1$ or $TST^{-1} \in [S]$. In the second case, we have $TST^{-1}(\infty) = \infty$, hence $T^{-1}(\infty)$ is a fix-point of S, and thus $T^{-1}(\infty) = \infty$, or equivalently $T(\infty) = \infty$. Using $T = N_k W N_j^{-1}$, we get $W(\eta_j) = \eta_k$. Since $\eta_1, ..., \eta_\kappa$ are pairwise inequivalent modulo Γ, we conclude that $k = j$, and $W \in \Gamma_{\eta_k} = [T_k]$, i.e. $T = N_k W N_k^{-1} \in [S]$. Now (2.4) is proved.

We now consider any $z \in \mathcal{H}$ and $W \in \Gamma$, and again write $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = N_k W N_j^{-1}$. By (2.4) we then have either $k = j$ and $W \in [T_k]$, or else $|c| \geqq 1$ and

$$\begin{aligned} \text{Im } N_k W(z) = \text{Im } T N_j(z) = \frac{\text{Im } N_j(z)}{|c N_j(z) + d|^2} &\leqq \frac{\text{Im } N_j(z)}{c^2 (\text{Im } N_j(z))^2} \\ &\leqq (\text{Im } N_j(z))^{-1}. \end{aligned}$$

This proves (2.5).

Now if $k \neq j$ and $z \in \mathcal{C}_{jB_0}$ then $\text{Im } N_j(z) \geqq B_0 > 1$, and hence, by (2.5), $\text{Im } N_k(z) < 1 < B_0$, which implies $z \notin \mathcal{C}_{kB_0}$. Hence $\mathcal{C}_{jB_0} \cap \mathcal{C}_{kB_0} = \emptyset$. □

Closely related to the splitting (2.3) of the fundamental domain is the *invariant height* function, $\mathcal{Y}_\Gamma(z)$. This is defined through:

$$\mathcal{Y}_\Gamma(z) = \max_{k \in \{1, \ldots, \kappa\}} \max_{W \in \Gamma} \quad \text{Im } N_k W(z). \tag{2.6}$$

(Cf. [I1, (3.8)].) In the next lemma, we collect some of the basic properties of this function.

Lemma 2.1.2. *For each $z \in \mathcal{H}$, the maximum in* (2.6) *is attained for some $k \in \{1, ..., \kappa\}$, $W \in \Gamma$. The function $\mathcal{Y}_\Gamma(z)$ is continuous and Γ-invariant, and bounded from below by a positive constant $\mathcal{Y}_\Gamma^{(0)}$ that only depends on Γ. We have:*

$$\forall\, B \geqq B_0,\ k \in \{1, ..., \kappa\},\ z \in \mathcal{C}_{kB}: \qquad \mathcal{Y}_\Gamma(z) = Im\ N_k(z) \geqq B. \tag{2.7}$$

$$\forall\, B > B_0,\ z \in \mathcal{F}: \qquad z \in \mathcal{F}_B \Longleftrightarrow \mathcal{Y}_\Gamma(z) < B. \tag{2.8}$$

Finally, assume that $f(z)$ *is any* Γ*-invariant function such that, for some constants* $a \in \mathbb{R}$, $C_1, C_2 \geqq 0$ *and for all* $k \in \{1, ..., \kappa\}$*:*

$$|f(z)| \leqq C_1 \,(Im\ N_k(z))^a + C_2 \qquad when\ Im\ N_k(z) \geqq \mathcal{Y}_\Gamma^{(0)}. \tag{2.9}$$

Then:

$$|f(z)| \leqq C_1 \mathcal{Y}_\Gamma(z)^a + C_2 \qquad throughout\ z \in \mathcal{H}. \tag{2.10}$$

Proof. Most of this well-known or trivial from the definitions, but for maximal clarity, we present detailed proofs of all assertions.

To prove the first assertion, it suffices to show that for any fixed $z \in \mathcal{H}$ and $k \in \{1, ..., \kappa\}$, the maximum of Im $N_k W(z)$ is attained for some $W \in \Gamma$. By an auxiliary conjugation (and corresponding change of the point z), we may assume $N_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and thus $\eta_k = \infty$, $\Gamma_\infty = [S]$. Write $z = x + iy$. By (2.5) in Lemma 2.1.1 (with $j = k$) we have Im $W(z) \leqq \max(y, y^{-1})$ for all $W \in \Gamma$. Since Im $S^n W(z) =$ Im $W(z)$ for all $n \in \mathbb{Z}$, it follows that

$$\begin{aligned} &\max \{\text{Im } W(z) \ \mid \ W \in \Gamma\} \\ &= \max \left\{ \text{Im } W(z) \ \mid \ W \in \Gamma,\ W(z) \in [0,1] \times \left[y, \max\left(y, y^{-1}\right)\right] \right\}. \end{aligned}$$

But the set in the right hand side is *finite*, since Γ acts properly discontinuously on $\mathcal{H}$ (cf. [M4, p. 17]). Hence the maximum is attained for some $W \in \Gamma$.

To prove the continuity of $\mathcal{Y}_\Gamma(z)$, we first notice that for any two points $z = x+iy$ and $w = u + iv$ with hyperbolic distance $\rho = \rho(z, w)$, we have (cf. [K, Thm. 1.2.6(ii)])

$$\cosh \rho = 1 + \frac{(x-u)^2 + (y-v)^2}{2yv} \geqq 1 + \frac{(y-v)^2}{2yv} = \cosh \left| \log \frac{y}{v} \right|,$$

and hence $\rho \geqq |\log y/v|$, i.e.

$$e^{-\rho}\, \text{Im } z \leqq \text{Im } w \leqq e^{\rho}\, \text{Im } z. \tag{2.11}$$

Using this inequality, together with the definition (2.6) and the fact that each map $N_k W$ preserves the hyperbolic distance, we obtain

$$\forall z, w \in \mathcal{H}: \qquad e^{-\rho}\, \mathcal{Y}_\Gamma(z) \leqq \mathcal{Y}_\Gamma(w) \ \leqq e^{\rho} \mathcal{Y}_\Gamma(z).$$

This shows, in particular, that the function $\mathcal{Y}_\Gamma(z)$ is continuous.

The Γ-invariance of $\mathcal{Y}_\Gamma(z)$ is trivial from the definition.

We now prove (2.7). Take $B \geqq B_0$ and $z \in \mathcal{C}_{kB}$. Then Im $N_k(z) \geqq B > 1$. If $j = k$ and $W \in [T_k]$ then Im $N_j W(z) =$ Im $N_k(z)$. For all *other* $\langle j, W \rangle \in \{1, ..., \kappa\} \times \Gamma$ we have Im $N_j W(z) \leqq (\text{Im } N_k(z))^{-1} < 1$, by (2.5) in Lemma 2.1.1. Hence by (2.6), $\mathcal{Y}_\Gamma(z) =$ Im $N_k(z)$, and (2.7) is proved.

We next prove that $\mathcal{Y}_\Gamma(z)$ is bounded from below by a positive constant, i.e. that

$$\inf_{z \in \mathcal{H}} \mathcal{Y}_\Gamma(z) > 0. \tag{2.12}$$

By Γ-invariance, we may restrict attention to $z \in \mathcal{F}$. Because of (2.7), we may in fact assume $z \in \mathcal{F} - \cup_{k=1}^{\kappa} \mathcal{C}_{kB_0} = \mathcal{F}_{B_0}$. Since $\mathcal{F}_{B_0}$ is a bounded region, we have

$$\inf_{z \in \mathcal{F}_{B_0}} \mathcal{Y}_\Gamma(z) \geqq \inf_{z \in \mathcal{F}_{B_0}} \text{Im } N_1(z) > 0.$$

(2.12) follows from this.

We now prove (2.8). Take any $B > B_0$ and $z \in \mathcal{F}$. One direction of (2.8), "$\Leftarrow$", follows from (2.7) and (2.3). To prove the other direction, we assume that $\mathcal{Y}_\Gamma(z) \geqq B$, and show that this leads to $z \notin \mathcal{F}_B$. Take $k \in \{1, ..., \kappa\}$ and $W \in \Gamma$

such that Im $N_kW(z) = \mathcal{Y}_\Gamma(z) \geqq B$. (This is possible by the first statement of this lemma.) By an auxiliary conjugation we may assume that $N_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and thus Im $W(z) \geqq B$, $\eta_k = \infty$, $\Gamma_\infty = [S]$ and $\mathcal{C}_{kB} = [0,1] \times [B, \infty)$.

The region $\mathbb{R} \times [B_0, \infty) \subset \mathcal{H}$ is tessellated by the S^n-translates of the cuspidal regions $\mathcal{C}_{kB_0}$. It is clear from this that if we had $W \in \Gamma - [S]$ then $W(\mathcal{F})$ would be disjoint from $\mathbb{R} \times (B_0, \infty)$, which would contradict the fact Im $W(z) \geqq B > B_0$. Hence $W \in [S]$, and Im $z =$ Im $W(z) \geqq B$. By (2.1), (2.2), this gives $z \in \mathcal{C}_{kB}$, hence $z \notin \mathcal{F}_B$, and (2.8) is proved.

Finally, let $f(z)$ be any Γ-invariant function satisfying (2.9), for some $a \in \mathbb{R}$, $C_1, C_2 \geqq 0$. We will prove (2.10).

Take any $z \in \mathcal{H}$. Then, by the first statement of this lemma, there are some $k \in \{1, ..., \kappa\}$ and $W \in \Gamma$ such that Im $N_kW(z) = \mathcal{Y}_\Gamma(z)$. Also, by the definition of $\mathcal{Y}_\Gamma^{(0)}$, we have $\mathcal{Y}_\Gamma(z) \geqq \mathcal{Y}_\Gamma^{(0)}$. Using (2.9) and Γ-invariance, we then get:

$$|f(z)| = |f(W(z))| \leqq C_1(\text{Im } N_kW(z))^a + C_2 = C_1\mathcal{Y}_\Gamma(z)^a + C_2.$$

□

When the upper half plane $\mathcal{H}$ is tessellated by Γ-translates of $\mathcal{F}$, the cuspidal regions $\mathcal{C}_{kB}$ map onto a set of disjoint horoballs (see Lemma 2.1.3 below). It will be convenient to have a precise notation for these horoballs, particularly when we deal with generic equidistribution in Section 4.

Let $\eta \in \partial\mathcal{H} = \mathbb{R} \cup \{\infty\}$ be any cusp of Γ. We then have $\eta = U(\eta_k)$ for some $U \in \Gamma$, $k \in \{1, ..., \kappa\}$. We define:

$$\forall B > 0 : \qquad \mathfrak{B}_\eta^{(B)} := UN_k^{-1}\{z \in \mathcal{H} \mid \text{Im } z \geqq B\}. \tag{2.13}$$

One easily verifies that $\mathfrak{B}_\eta^{(B)}$ depends only on η, B, Γ, and *not* on the choice of the map U, or the cusp normalizing map N_k. In fact, $\mathfrak{B}_\eta^{(B)}$ can be defined more "intrinsically" as *the unique horoball* $\mathfrak{B}$ *tangent to* $\partial\mathcal{H}$ *at* η *for which* $\Gamma_\eta \setminus \mathfrak{B}$ *has hyperbolic area* $1/B$. To see this, first notice that $\Gamma_\eta = U\Gamma_{\eta_k}U^{-1} = [UT_kU^{-1}]$. Using (2.13) and $T_k = N_k^{-1}S^{-1}N_k$, we see that Γ_η maps $\mathfrak{B}_\eta^{(B)}$ onto itself, and that a fundamental region for the action of Γ_η on $\mathfrak{B}_\eta^{(B)}$ is given by $UN_k^{-1}\big([0,1] \times [B, \infty)\big)$. This region has area $1/B$, as asserted.

It is clear from the definition that, for any $T \in \Gamma$:

$$T(\mathfrak{B}_\eta^{(B)}) = \mathfrak{B}_{T(\eta)}^{(B)}. \tag{2.14}$$

Lemma 2.1.3. *Let* $B \geqq B_0$ *and* $k \in \{1, ..., \kappa\}$. *Then:*

$$\bigcup_{T \in \Gamma} T(\mathcal{C}_{kB}) = \bigcup_{\eta \sim \eta_k} \mathfrak{B}_\eta^{(B)} = \bigcup_{W_0 \in [S] \setminus N_k\Gamma} W_0^{-1}\{z \in \mathcal{H} \mid Im\ z \geqq B\} \tag{2.15}$$

(the second union is taken over all cusps η *equivalent to* η_k*). The last two unions in* (2.15) *are in fact taken over* exactly the same family of sets*; and this holds for any* $B > 0$. *Also, for any* $B > 1$, *the last two unions are both* disjoint *unions. In fact, for* any *two distinct cusps* η, η' *(be they equivalent or not), and any positive numbers* B, B' *satisfying* $BB' > 1$, *the two horoballs* $\mathfrak{B}_\eta^{(B)}$ *and* $\mathfrak{B}_{\eta'}^{(B')}$ *are disjoint.*

Proof. To prove the first equality in (2.15) we write $\Gamma = \sum U[T_k]$ and $T = UT_k^n$:

$$\begin{aligned}
\bigcup_{T\in\Gamma} T(\mathcal{C}_{kB}) &= \bigcup_{U\in\Gamma/[T_k]} \bigcup_{n\in\mathbb{Z}} UT_k^n(\mathcal{C}_{kB}) \\
&= \bigcup_U \bigcup_n UT_k^n N_k^{-1}\big([0,1]\times[B,\infty)\big) \qquad \{\text{cf. } (2.2)\} \\
&= \bigcup_U UN_k^{-1}\left(\bigcup_n S^{-n}\big([0,1]\times[B,\infty)\big)\right) \qquad \{\text{using } T_k^n N_k^{-1} = N_k^{-1}S^{-n}\} \\
&= \bigcup_U UN_k^{-1}\{z\in\mathcal{H} \mid \operatorname{Im} z \geqq B\} = \bigcup_{U\in\Gamma/[T_k]} \mathfrak{B}_{U(\eta_k)}^{(B)} \qquad \{\text{by } (2.13)\} \\
&= \bigcup_{\eta\sim\eta_k} \mathfrak{B}_\eta^{(B)},
\end{aligned}$$

since $U(\eta_k)$ runs through all cusps equivalent to η_k (without repetition) as U runs through $\Gamma/[T_k]$.

The second equality in (2.15) follows from the last two lines in the above computation, upon writing $W_0 = N_kU^{-1}$. (Then W_0 will run through $[S]\backslash N_k\Gamma$. To see this, notice that $\Gamma = \sum U[T_k]$ and $[T_k] = N_k^{-1}[S]N_k$ imply $\Gamma = \sum UN_k^{-1}[S]N_k$ and $\Gamma N_k^{-1} = \sum UN_k^{-1}[S]$. Applying inversion we get $N_k\Gamma = \sum[S]N_kU^{-1} = \sum[S]W_0$.) Quickly inspecting this proof we see that the last two unions in (2.15) are in fact taken over exactly the same family of sets, and that this is true for any $B > 0$.

We now prove the last assertion of the lemma. Clearly, this will imply that the last two unions in (2.15) are both disjoint unions, whenever $B > 1$.

We let η, η' be distinct cusps, and let $B, B' > 0$, $BB' > 1$. We then have $\eta = U(\eta_k)$ and $\eta' = T(\eta_j)$, for some $k, j \in \{1, ..., \kappa\}$, $U, T \in \Gamma$. Let z be any point in $\mathfrak{B}_{\eta'}^{(B')}$; then $\operatorname{Im} N_jT^{-1}(z) \geqq B'$, by (2.13). We write $w = T^{-1}(z)$ and $W = U^{-1}T$. Notice that we *cannot* have $\big[k = j \wedge W \in [T_k]\big]$, since this would imply $T \in U[T_k]$ and $\eta' = T(\eta_j) = T(\eta_k) = U(\eta_k) = \eta$, a contradiction to our assumptions. Hence, by (2.5) in Lemma 2.1.1:

$$\operatorname{Im} N_kU^{-1}(z) = \operatorname{Im} N_kW(w) \leqq \big(\operatorname{Im} N_j(w)\big)^{-1} \leqq (B')^{-1} < B.$$

In view of (2.13), this implies $z \notin \mathfrak{B}_\eta^{(B)}$.

Hence $\mathfrak{B}_\eta^{(B)} \cap \mathfrak{B}_{\eta'}^{(B')} = \emptyset$, as asserted. □

Remark 2.1.4. To make the last union in (2.15) even more explicit, notice that for any $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R})$ with $c \neq 0$, we have:

$$\begin{aligned}
(2.16) \qquad &\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}\big(\mathbb{R}\times[B,\infty)\big) \\
&= \left\{x+iy\in\mathcal{H} \;\middle|\; y \leqq \tfrac{1}{c^2B},\quad \left|x+\tfrac{d}{c}\right| \leqq \sqrt{\tfrac{y}{c^2B}-y^2}\right\}.
\end{aligned}$$

This is a horoball tangent to $\mathbb{R}$ at $-\frac{d}{c}$; it is a Euclidean circle with diameter $\frac{1}{c^2B}$.

If $c = 0$, then clearly $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}\big(\mathbb{R}\times[B,\infty)\big) = \mathbb{R}\times[\frac{d}{a}B,\infty)$, a horoball tangent to $\partial\mathcal{H}$ at ∞.

To prove (2.16), notice that $x+iy$ belongs to the set in the left hand side if and only if

$$B \leqq \operatorname{Im}\frac{a(x+iy)+b}{c(x+iy)+d} = \frac{y}{|c(x+iy)+d|^2} = \frac{y}{(cy)^2+(cx+d)^2},$$

i.e. $(cx+d)^2 \leqq \frac{y}{B} - (cy)^2$. This is clearly equivalent to $\left[y \leqq \frac{1}{c^2B} \ \wedge \ \left|x + \frac{d}{c}\right| \leqq \sqrt{\frac{y}{c^2B} - y^2} \right]$, and the proof is complete. □

Finally, as a tool to get explicit control on the union of "cuspidal horoballs" in (2.15) (cf. (2.16)), we introduce the following counting function:

$$\mathfrak{C}^k_{\mu\nu}(X) := \left| \left\{ W_0 = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in [S] \setminus N_k\Gamma \ \middle| \ 0 < |c| \leqq X, \ -\tfrac{d}{c} \in [\mu, \nu] \right\} \right|. \tag{2.17}$$

Here $k \in \{1, ..., \kappa\}$, and μ, ν, X are real numbers such that $\mu < \nu$ and $X > 0$.

Notice that by the second statement in Lemma 2.1.3 (with $B = 1$) and Remark 2.1.4,

$$\mathfrak{C}^k_{\mu\nu}(X) = \left| \left\{ \eta \in [\mu, \nu] \ \middle| \ \begin{array}{l} \eta \text{ is a cusp equivalent to } \eta_k, \\ \text{and } \mathfrak{B}^{(1)}_\eta \text{ intersects the line Im } z = X^{-2} \end{array} \right\} \right|. \tag{2.18}$$

Hence $\mathfrak{C}^k_{\mu\nu}(X)$ can be viewed as measuring the *distribution of all cusps* $\eta \sim \eta_k$ *over the real line*, when these η's are ordered by the size of $\mathfrak{B}^{(1)}_\eta$.

2.2. **Some fundamental estimates.** Throughout the rest of this paper, we will assume that

$$N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\text{and thus } \eta_1 = \infty, \ \Gamma_\infty = [S]). \tag{2.19}$$

We refer to this assumption by saying that "Γ has a normalized cusp". Notice that for an arbitrary cofinite, non-cocompact Fuchsian group, any cusp can be brought into such a normalized position, by first re-indexing the η_j's, and then applying an auxiliary conjugation to Γ.

In our first lemma we give some apriori bounds on the counting function $\mathfrak{C}^k_{\mu\nu}(X)$. (Cf. (2.17).)

Lemma 2.2.1. *Assume* (2.19). *Let* $k \in \{1, ..., \kappa\}$, $\mu < \nu$ *and* $X > 0$. *Then:*

$$\mathfrak{C}^k_{\mu\nu}(X) \leqq (\nu - \mu)X^2 + 1.$$

Also:

$$\mathfrak{C}^k_{\mu\nu}(X) = 0 \qquad \textit{for all } 0 < X < 1.$$

Remark 2.2.2. In Section 4.6, we will prove the following asymptotic formula:

$$\mathfrak{C}^k_{\mu\nu}(X) \sim \frac{1}{\pi\mu(\mathcal{F})} X^2(\nu - \mu). \tag{2.20}$$

This holds when $X \to \infty$, as long as $\nu - \mu$ does not decrease too rapidly. To compare to Lemma 2.2.1, recall that $\pi(\mathcal{F}) \geqq \pi/3$ as long as Γ has at least one cusp, cf. [K, p. 110, exercise 4.10].

Proof of Lemma 2.2.1. The second assertion follows directly from (2.4) in Lemma 2.1.1, with $j = 1$, $N_j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (cf. (2.19)).

We now prove the first assertion. (This proof is implicit in [I1, pp. 53–54].) Let M denote the set in the right hand side of (2.17), for our fixed k, μ, ν, X, and one fixed set of representatives $[S] \setminus N_k\Gamma$. Let us consider any two *distinct* elements $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $T' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ in M. Then $TT'^{-1} \notin [S]$. But from $T, T' \in N_k\Gamma$ it follows that $TT'^{-1} \in N_k\Gamma N_k^{-1}$, and this set is a Fuchsian group having $N_k(\eta_k) = \infty$ as a cusp, with stabilizer

$$\left(N_k\Gamma N_k^{-1}\right)_\infty = N_k\Gamma_{\eta_k}N_k^{-1} = [S].$$

Hence it follows from Shimizu's lemma (cf. [S2, Lemma 4], or [M4, Lemma 1.7.3]) that the lower left element in the matrix TT'^{-1} has modulus $\geqq 1$. Since $TT'^{-1} = \left(\begin{smallmatrix} * & * \\ cd'-dc' & * \end{smallmatrix}\right)$, we get:

$$|cd' - dc'| \geqq 1.$$

Now divide through by $|cc'|$. Since $0 < |c| \leqq X$ and $0 < |c'| \leqq X$, we get

$$\left|\frac{d'}{c'} - \frac{d}{c}\right| \geqq \frac{1}{|cc'|} \geqq \frac{1}{X^2}.$$

But all elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M$ have $-\frac{d}{c} \in [\mu, \nu]$. Hence, by ordering the elements in M by increasing quotients $\frac{d}{c}$, we see that

$$(|M| - 1) \cdot \frac{1}{X^2} \leqq \nu - \mu,$$

which is equivalent to the first statement in the lemma. □

Lemma 2.2.3. *Let real numbers $\mu < \nu$ and $1 \leqq A \leqq B$ be given, and let $\sum_{W_0}$ refer to a sum over a set of representatives $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$ restricted by $A \leqq |c| \leqq B$ and $-\frac{d}{c} \in [\mu, \nu]$. We then have the following bounds:*

$$\sum_{W_0} \frac{1}{|c|^\delta} \leqq \frac{2}{2-\delta}(\nu - \mu)B^{2-\delta} + A^{-\delta} \qquad \textit{for any } 0 < \delta < 2.$$

$$\sum_{W_0} \frac{1}{|c|^\delta} \leqq \frac{\delta}{\delta-2}(\nu - \mu)A^{2-\delta} + A^{-\delta} \qquad \textit{for any } \delta > 2.$$

$$\sum_{W_0} \frac{1}{|c|} \log\left(\frac{B}{|c|}\right) \leqq 2(\nu - \mu)B + \frac{1}{A}\log\left(\frac{B}{A}\right).$$

(Notice that the second bound also holds for "$B = \infty$", as follows by taking the limit $B \to \infty$.)

Proof. Take any $\varepsilon \in (0, 1)$. Keep $\delta > 0$ and $\delta \neq 2$. We then have:

$$\begin{aligned}\sum_{W_0} \frac{1}{|c|^\delta} &\leqq \int_{A-\varepsilon}^{B} \frac{1}{x^\delta}\, d\mathfrak{C}^k_{\mu\nu}(x) \\ &= \left[\frac{1}{x^\delta}\mathfrak{C}^k_{\mu\nu}(x)\right]_{x=A-\varepsilon}^{x=B} + \int_{A-\varepsilon}^{B} \frac{\delta}{x^{\delta+1}}\mathfrak{C}^k_{\mu\nu}(x)\, dx.\end{aligned}$$

Using Lemma 2.2.1, we see that this is

$$\begin{aligned}&\leqq (\nu - \mu)B^{2-\delta} + \frac{1}{B^\delta} + \int_{A-\varepsilon}^{B} \frac{\delta}{x^{\delta+1}}\left((\nu - \mu)x^2 + 1\right)\, dx \\ &\leqq (\nu - \mu)B^{2-\delta} + \frac{1}{B^\delta} + \left[\frac{\delta}{2-\delta}(\nu - \mu)x^{2-\delta} - x^{-\delta}\right]_{x=A-\varepsilon}^{x=B} \\ &= \frac{2}{2-\delta}(\nu - \mu)B^{2-\delta} + (A - \varepsilon)^{-\delta} - \frac{\delta}{2-\delta}(\nu - \mu)(A - \varepsilon)^{2-\delta}.\end{aligned}$$

From this, we obtain the first two inequalities in the lemma, by letting $\varepsilon \to 0$ and omitting negative terms.

Similarly, for any $\varepsilon \in (0,1)$, we get for the third sum:

$$\begin{aligned}
&\sum_{W_0} \frac{1}{|c|} \log\left(\frac{B}{|c|}\right) \leqq \int_{A-\varepsilon}^{B} \frac{1}{x} \log\left(\frac{B}{x}\right) \, d\mathfrak{C}^k_{\mu\nu}(x) \\
&= \left[\frac{1}{x} \log\left(\frac{B}{x}\right) \mathfrak{C}^k_{\mu\nu}(x)\right]_{x=A-\varepsilon}^{x=B} + \int_{A-\varepsilon}^{B} \left(\frac{1}{x^2} \log\left(\frac{B}{x}\right) + \frac{1}{x^2}\right) \mathfrak{C}^k_{\mu\nu}(x) \, dx.
\end{aligned}$$

Here the first term is $\leqq 0$, and thus, by Lemma 2.2.1, the whole expression is

$$\leqq \int_{A-\varepsilon}^{B} \left(\frac{1}{x^2} \log\left(\frac{B}{x}\right) + \frac{1}{x^2}\right) \big((\nu-\mu)x^2 + 1\big) \; dx.$$

We now substitute $x = B/u$, and obtain:

$$\begin{aligned}
&= (\nu-\mu)\, B \int_1^{B/(A-\varepsilon)} \frac{\log(u)+1}{u^2} \, du + \frac{1}{B} \int_1^{B/(A-\varepsilon)} \big(\log(u)+1\big) \, du \\
&\leqq (\nu-\mu)\, B \left[\frac{-\log(u)-2}{u}\right]_{u=1}^{u=\infty} + \frac{1}{B} \Big[u \log(u)\Big]_{u=1}^{u=B/(A-\varepsilon)} \\
&= 2\,(\nu-\mu)\, B + \frac{1}{A-\varepsilon} \log\left(\frac{B}{A-\varepsilon}\right).
\end{aligned}$$

The desired inequality again follows by letting $\varepsilon \to 0$. $\square$

The remaining estimates in this section concern the invariant height function, $\mathcal{Y}_\Gamma(z)$ (cf. (2.6) and Lemma 2.1.2). In later sections, we will several times have situations where some function on $\Gamma \setminus \mathcal{H}$ can be bounded by either $O(\mathcal{Y}_\Gamma(z))$ or $O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right)$. The following easy lemma will then be of basic importance. Given any constant $M \geqq 0$, we define the following cut-off function:

$$\lfloor x \rfloor_M := \begin{cases} x & \text{if } x \geqq M \\ 0 & \text{otherwise} \end{cases}. \tag{2.21}$$

Lemma 2.2.4. *For all* $M \geqq 0$*:*

$$\int_{\mathcal{F}} \sqrt{\lfloor \mathcal{Y}_\Gamma(z) \rfloor_M} \, d\mu(z) = O\left((M+1)^{-\frac{1}{2}}\right), \tag{2.22}$$

For all $B \geqq B_0$*:*

$$\int_{\mathcal{F}_B} \mathcal{Y}_\Gamma(z) \, d\mu(z) = O\left(\log(2B)\right). \tag{2.23}$$

In both these estimates, the implied constant depends only on Γ.

Proof. Recall Lemma 2.1.2. To prove (2.22) we split $\mathcal{F}$ into the regions $\mathcal{F}_{B_0}$ and $\mathcal{C}_{kB_0}$, $k \in \{1, ..., \kappa\}$, cf. (2.3). The integral over $\mathcal{F}_{B_0}$ is $O(1)$, since $\sqrt{\lfloor \mathcal{Y}_\Gamma(z) \rfloor_M} \leqq \sqrt{\mathcal{Y}_\Gamma(z)}$, and $\sqrt{\mathcal{Y}_\Gamma(z)}$ is continuous. When $z \in \mathcal{C}_{kB_0}$ we have $\mathcal{Y}_\Gamma(z) = \text{Im } N_k(z)$ by (2.7) in Lemma 2.1.2. Hence, for all $M \geqq 0$:

$$\begin{aligned}
&\int_{\mathcal{F}} \sqrt{\lfloor \mathcal{Y}_\Gamma(z) \rfloor_M} \, d\mu(z) \leqq O(1) + \sum_{k=1}^{\kappa} \int_{\mathcal{C}_{kB_0}} \sqrt{\text{Im } N_k(z)} \, d\mu(z) \\
&= O(1) + \kappa \int_{B_0}^{\infty} \sqrt{y} \, \frac{dy}{y^2} = O(1).
\end{aligned}$$

Furthermore, if $M > B_0$, then $\lfloor \mathcal{Y}_\Gamma(z) \rfloor_M = 0$ for all $z \in \mathcal{F}_{B_0}$, by (2.8) in Lemma 2.1.2 (since $\mathcal{F}_{B_0} \subset \mathcal{F}_M$), and the above splitting gives:

$$\int_{\mathcal{F}} \sqrt{\lfloor \mathcal{Y}_\Gamma(z) \rfloor_M}\, d\mu(z) = 0 + \kappa \int_{B_0}^{\infty} \sqrt{\lfloor y \rfloor_M}\, \frac{dy}{y^2} = \kappa \int_M^{\infty} \frac{dy}{y^{\frac{3}{2}}} = O\left(M^{-\frac{1}{2}}\right).$$

Now (2.22) is proved.

Next, to prove (2.23), we split $\mathcal{F}_B$ into the regions $\mathcal{F}_{B_0}$ and $(\mathcal{C}_{kB_0} \cap \mathcal{F}_B)$, $k \in \{1, ..., \kappa\}$. As before, $\int_{\mathcal{F}_{B_0}} \mathcal{Y}_\Gamma(z)\, d\mu(z) = O(1)$. Notice that, by (2.3) and (2.2),

$$(\mathcal{C}_{kB_0} \cap \mathcal{F}_B) \subset \mathcal{C}_{kB_0} - \mathcal{C}_{kB} = N_k^{-1}\left([0,1] \times [B_0, B)\right).$$

Hence, using (2.7) in Lemma 2.1.2 as before:

$$\int_{\mathcal{F}_B} \mathcal{Y}_\Gamma(z)\, d\mu(z) = O(1) + \kappa \int_{B_0}^{B} y\, \frac{dy}{y^2} = O(\log(2B)),$$

since $B \geqq B_0 > 1$.

□

The next proposition is the main result of this section. It basically tells us that the function $\sqrt{\mathcal{Y}_\Gamma(z)}$ is a well-behaved majorant when studying the "horocycle integral"

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy)\, dx,$$

asymptotically as $y \to 0$, so long as $\beta - \alpha$ does not decrease too rapidly.

Proposition 2.2.5. *Let $\gamma \in (0, \frac{1}{2})$ be given. We then have*

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \sqrt{\mathcal{Y}_\Gamma(x + iy)}\, dx = O(1), \tag{2.24}$$

for all $0 < y < 1$ and all $\alpha, \beta \in \mathbb{R}$ such that $\beta - \alpha \geqq y^\gamma$. The implied constant depends only on Γ and γ. Furthermore, given any $M \geqq 10$ there is some $y_0 = y_0(M, \gamma) > 0$ such that

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \sqrt{\lfloor \mathcal{Y}_\Gamma(x + iy) \rfloor_M}\, dx \leqq \frac{25\kappa}{\sqrt{M}}, \tag{2.25}$$

for all $0 < y < y_0$ and all $\alpha, \beta \in \mathbb{R}$ such that $\beta - \alpha \geqq y^\gamma$.

We need two lemmas for the proof. The first one is a well-known computation.

Lemma 2.2.6. *Let $\psi : (0, \infty) \to \mathbb{C}$ be a piecewise continuous function such that $\psi(u) = O(u^\delta)$ as $u \to 0$, for some constant $\delta > \frac{1}{2}$. Then if $y > 0$, $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{R})$ and $c \neq 0$, we have:*

$$\int_{-\infty}^{\infty} \psi(Im\, W(x + iy))\, dx = \frac{\sqrt{y}}{|c|} \int_{c^2 y}^{\infty} \psi\left(\frac{1}{w}\right) \frac{dw}{\sqrt{w - c^2 y}}.$$

Proof.

$$\int_{-\infty}^{\infty} \psi(\text{Im } W(x+iy))\,dx = \int_{-\infty}^{\infty} \psi\left(\frac{y}{(cx+d)^2+(cy)^2}\right) dx$$

$$\left\{\text{Take } x = yu - \tfrac{d}{c}.\right\}$$

$$= y\int_{-\infty}^{\infty} \psi\left(\frac{y}{(cy)^2(u^2+1)}\right) du = 2y\int_{0}^{\infty} \psi\left(\frac{1}{c^2y}\cdot\frac{1}{u^2+1}\right) du$$

$$\left\{\text{Take } u = \sqrt{\tfrac{w}{c^2y}-1}.\right\}$$

$$= 2y\int_{c^2y}^{\infty} \psi\left(\frac{1}{w}\right)\frac{dw}{2c^2y\sqrt{w/(c^2y)-1}} = \frac{\sqrt{y}}{|c|}\int_{c^2y}^{\infty} \psi\left(\frac{1}{w}\right)\frac{dw}{\sqrt{w-c^2y}}.$$

Clearly, the integral is absolutely convergent, since $\psi(1/w) = O(w^{-\delta})$ as $w \to \infty$, with $\delta > \frac{1}{2}$. □

Lemma 2.2.7. *Let* $M > 0$, $\alpha < \beta$ *and* $k \in \{1, ..., \kappa\}$. *Then, for any* $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in N_k\Gamma$, $0 < y < M$ *and* $x \in [\alpha, \beta]$ *satisfying*

$$\textit{Im } W(x+iy) \geqq M,$$

we have:

$$1 \leqq |c| \leqq \frac{1}{\sqrt{My}} \qquad \textit{and} \qquad -\frac{d}{c} \in \left[\alpha - \frac{\sqrt{y}}{\sqrt{M}|c|}, \beta + \frac{\sqrt{y}}{\sqrt{M}|c|}\right].$$

If, furthermore, $\gamma \in (0, \frac{1}{2})$, $y < 1$ *and* $\beta - \alpha \geqq y^\gamma$, *then*

$$-\frac{d}{c} \in \left[\alpha - \frac{\beta-\alpha}{\sqrt{M}}, \beta + \frac{\beta-\alpha}{\sqrt{M}}\right].$$

Proof. Clearly, since $y < M \leqq \text{Im } W(x+iy)$, we cannot have $W \in [S]$. Hence by (2.4) in Lemma 2.1.1 (with $j = 1$), we have $|c| \geqq 1$.

Im $W(x+iy) \geqq M$ implies that $x+iy \in \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}\big(\mathbb{R} \times [M, \infty)\big)$, and we thus get by (2.16):

$$y \leqq \frac{1}{c^2M}, \qquad \text{i.e. } |c| \leqq \frac{1}{\sqrt{My}}, \tag{2.26}$$

and also:

$$\left|x + \frac{d}{c}\right| \leqq \sqrt{\frac{y}{c^2M} - y^2} \leqq \frac{\sqrt{y}}{\sqrt{M}|c|}. \tag{2.27}$$

Since $x \in [\alpha, \beta]$, this implies:

$$-\frac{d}{c} \in \left[\alpha - \frac{\sqrt{y}}{\sqrt{M}|c|}, \beta + \frac{\sqrt{y}}{\sqrt{M}|c|}\right]. \tag{2.28}$$

Finally, if $\gamma \in (0, \frac{1}{2})$, $y < 1$ and $\beta - \alpha \geqq y^\gamma$, then since $|c| \geqq 1$:

$$\frac{\sqrt{y}}{\sqrt{M}|c|} < \frac{\beta-\alpha}{\sqrt{M}}. \tag{2.29}$$

Together, (2.28) and (2.29) imply the last assertion of the lemma. □

Proof of Proposition 2.2.5. We first prove the second assertion, (2.25). Notice that, for all $z \in \mathcal{H}$:

$$\begin{aligned}\sqrt{\lfloor \mathcal{Y}_\Gamma(z) \rfloor_M} &= \max_{k\in\{1,\dots,\kappa\}} \max_{T\in\Gamma} \sqrt{\lfloor \mathrm{Im}\ N_k T(z) \rfloor_M} \\ &= \max_{k\in\{1,\dots,\kappa\}} \max_{W_0\in N_k\Gamma} \sqrt{\lfloor \mathrm{Im}\ W_0(z) \rfloor_M} \\ &\leqq \sum_{k=1}^{\kappa} \sum_{W_0\in[S]\backslash N_k\Gamma} \sqrt{\lfloor \mathrm{Im}\ W_0(z) \rfloor_M}. \end{aligned} \tag{2.30}$$

(Recall $[S]N_k = N_k[T_k]$.) Hence the left hand side in (2.25) is bounded from above by

$$\sum_{k=1}^{\kappa} \sum_{W_0\in[S]\backslash N_k\Gamma} \frac{1}{\beta-\alpha}\int_\alpha^\beta \sqrt{\lfloor \mathrm{Im}\ W_0(x+iy) \rfloor_M}\, dx. \tag{2.31}$$

We temporarily fix some $M \geqq 10$, some $y \in (0,1)$, and some α, β such that $\beta - \alpha \geqq y^\gamma$.

Take $k \in \{1, ..., \kappa\}$ and look at any $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$ which gives a nonzero contribution to (2.31). Then there is some $x \in [\alpha, \beta]$ for which Im $W_0(x + iy) \geqq M$, and hence, by Lemma 2.2.7:

$$1 \leqq |c| \leqq \frac{1}{\sqrt{My}}, \tag{2.32}$$

(in particular: $My \leqq 1$), and

$$-\frac{d}{c} \in [\mu, \nu], \qquad \text{where } \mu = \alpha - \frac{\beta-\alpha}{\sqrt{M}},\ \nu = \beta + \frac{\beta-\alpha}{\sqrt{M}}. \tag{2.33}$$

We also see, by Lemma 2.2.6,

$$\begin{aligned} &\int_\alpha^\beta \sqrt{\lfloor \mathrm{Im}\ W_0(x+iy) \rfloor_M}\, dx \leqq \int_{-\infty}^\infty \sqrt{\lfloor \mathrm{Im}\ W_0(x+iy) \rfloor_M}\, dx \\ &= \frac{\sqrt{y}}{|c|} \int_{c^2y}^\infty \sqrt{\lfloor 1/w \rfloor_M} \frac{dw}{\sqrt{w-c^2y}} \\ &= \frac{\sqrt{y}}{|c|} \int_{c^2y}^{1/M} \frac{dw}{\sqrt{w}\sqrt{w-c^2y}} \qquad \Big\{\text{Since } c^2y \leqq 1/M, \text{ by (2.32).}\Big\} \\ &= \frac{\sqrt{y}}{|c|} \int_1^{(Mc^2y)^{-1}} \frac{du}{\sqrt{u}\sqrt{u-1}} \qquad \Big\{\text{We substituted } w = c^2yu.\Big\}. \end{aligned}$$

But notice that

$$\frac{1}{\sqrt{u}\sqrt{u-1}} \leqq \begin{cases} \dfrac{1}{\sqrt{u-1}} & \text{for } 1 \leqq u \\ 2/u & \text{for } 2 \leqq u \text{ (or even for } 4/3 \leqq u). \end{cases}$$

Hence, for all $D \geqq 1$,

$$\int_1^D \frac{du}{\sqrt{u}\sqrt{u-1}} \leqq \int_1^2 \frac{du}{\sqrt{u-1}} + \int_1^D \frac{2}{u} du = 2 + 2\log D,$$

and we obtain:

$$\int_{-\infty}^\infty \sqrt{\lfloor \mathrm{Im}\ W_0(x+iy) \rfloor_M}\, dx \leqq \frac{2\sqrt{y}}{|c|}\left(1 + 2\log\left(\frac{1}{\sqrt{My}|c|}\right)\right). \tag{2.34}$$

Hence:

$$\sum_{W_0\in[S]\setminus N_k\Gamma} \frac{1}{\beta-\alpha}\int_\alpha^\beta \sqrt{\lfloor \mathrm{Im}\, W_0(x+iy)\rfloor_M}\,dx \tag{2.35}$$
$$\leqq \frac{1}{\beta-\alpha}\sum_{W_0}\frac{2\sqrt{y}}{|c|}\left(1+2\log\left(\frac{1}{\sqrt{My}|c|}\right)\right),$$

where the sum in the right hand side is taken over a set of representatives $W_0 = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in [S]\setminus N_k\Gamma$ restricted by (2.32) and (2.33). If $My > 1$ then the sum is empty, i.e. we get 0. If $My \leqq 1$ we get, by Lemma 2.2.3:

$$\leqq \frac{2\sqrt{y}}{\beta-\alpha}\left[\frac{2(\nu-\mu)}{\sqrt{My}}+1+\frac{4(\nu-\mu)}{\sqrt{My}}+2\log\left(\frac{1}{\sqrt{My}}\right)\right].$$

From (2.33) and $M \geqq 10$ it follows that $\nu-\mu \leqq 2(\beta-\alpha)$. Using $\beta-\alpha\geqq y^\gamma$, we finally conclude[1]:

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \sqrt{\lfloor \mathcal{Y}_\Gamma(x+iy)\rfloor_M}\,dx \tag{2.36}$$
$$\leqq \frac{24\kappa}{\sqrt{M}} + 2\kappa\, y^{\frac{1}{2}-\gamma}\left(1+2\log^+\left(\frac{1}{\sqrt{My}}\right)\right).$$

This holds for all $0<y<1$ and all $\alpha,\beta\in\mathbb{R}$ such that $\beta-\alpha\geqq y^\gamma$. Clearly, since $\gamma<\frac{1}{2}$, there is some $y_0 = y_0(M,\gamma)$ such that the right hand side of (2.36) is less than $25\kappa/\sqrt{M}$ anytime $0<y<y_0$. This proves the second assertion, (2.25), in Proposition 2.2.5.

The first assertion in Proposition 2.2.5 follows from (2.36) with $M = 10$ and the inequality

$$\sqrt{\mathcal{Y}_\Gamma(z)} \leqq \sqrt{10}+\sqrt{\lfloor \mathcal{Y}_\Gamma(z)\rfloor_{10}}, \qquad \forall\, z\in\mathcal{H}.$$

□

2.3. **Application of spectral decomposition.** We will now state our first main theorem.

Let D denote the non-Euclidean Laplacian and let $\phi_0,\phi_1,...$ be the discrete eigenfunctions of $-D$, taken to be orthonormal and to have increasing eigenvalues $0=\lambda_0<\lambda_1\leqq\lambda_2\leqq ...$ In general, we do not know if the set $\{\phi_0,\phi_1,...\}$ is infinite or not. For any $\lambda\geqq 0$, we let $A(\lambda)$ be the finite dimensional subspace of $L_2(\Gamma\setminus\mathcal{H})$ spanned by $\{\phi_n \mid \lambda_n=\lambda\}$.

As in [H3, p. 282, item 5] we define $A_0(\lambda)$ to be the subspace spanned by the *cusp forms* in $A(\lambda)$, and $A_1(\lambda)$ as its orthogonal complement in $A(\lambda)$. We also write $A_0 := \oplus_\lambda A_0(\lambda)$ and $A_1 := \oplus_{\lambda>0}A_1(\lambda)$. It is known that $\lambda\geqq\frac{1}{4}\implies A_1(\lambda)=0$.

---

[1] We use the following standard notation: $\log^+(x)=\log(x)$ if $x\geqq 1$, otherwise $\log^+(x)=0$.

Theorem 2.3.1. *Let Γ be a cofinite Fuchsian group with a normalized cusp (see (2.19)).* Assume $A_1 = \{0\}$. *Let $\gamma \in (0, \frac{1}{2})$ be given, and let $f : \mathcal{H} \to \mathbb{C}$ be any continuous, Γ-invariant function satisfying the growth condition*

$$|f(z)| \leqq C\sqrt{\mathcal{Y}_\Gamma(z)} \qquad \forall z \in \mathcal{H}, \tag{2.37}$$

for some positive constant C. Then:

$$\frac{1}{\beta - \alpha}\int_\alpha^\beta f(x+iy)\,dx \to \frac{1}{\mu(\mathcal{F})}\int_{\mathcal{F}} f(z)d\mu(z) \tag{2.38}$$

uniformly as $y \to 0$ so long as $\beta - \alpha$ remains bigger than y^γ. (Here Γ, γ, f are kept fixed.)

(We will see later, in Section 2.4, that the assumption $A_1 = \{0\}$ can be disposed of in Theorem 2.3.1. For a proper understanding of the growth condition (2.37), cf. Lemma 2.1.2, especially (2.9), (2.10).)

The basic idea in the proof of Theorem 2.3.1 is to apply spectral decomposition to $f(z)$, and then study the "horocycle integral" (i.e. the left hand side of (2.38)) separately for each spectral component. To make this work, we first have to restrict attention to functions $f(z)$ satisfying

$$f \in C^2(\mathcal{H}) \cap L_2(\Gamma \setminus \mathcal{H}), \quad \text{and } Df \in L_2(\Gamma \setminus \mathcal{H}). \tag{2.39}$$

We will then pass to more general functions $f(z)$ as in (2.37) in the end, using approximation. (It may be of interest to point out here that any function $f(z)$ satisfying (2.39) also satisfies (2.37); cf. Remark 2.3.4 below.)

According to [H3, p. 317, Prop. 5.3, and p. 733, note 5], any function $f(z)$ as in (2.39) can be decomposed as

$$f(z) = \sum_{n \geqq 0} d_n \phi_n(z) + \sum_{k=1}^{\kappa} \int_0^\infty g_k(R) E_k\left(z, \tfrac{1}{2} + iR\right)\,dR, \tag{2.40}$$

with uniform and absolute convergence over $z \in \mathcal{H}$-compacta. In this expansion, $E_k(z,s)$ is the Eisenstein series associated to the cusp η_k, cf. [H3, pp. 280(Def. 3.5), 130(Def. 11.7, Thm. 11.8, esp. (E)), 296–297]. Also, $d_n = \langle f, \phi_n \rangle$, and "$g_k(R) = \frac{1}{2\pi}\int_{\mathcal{F}} f(z)\overline{E_k(z, \frac{1}{2} + iR)}\,d\mu(z)$" (this has to be properly considered as a limit in $L_2(0,\infty)$–norm), cf. [H3, p. 242(Prop. 2.3), p. 317(line 4)].

The proof in [H3] of the beautiful convergence in (2.40) starts by considering the spectral expansion (in L_2-sense) of the function $Df + a(1-a)f \in L_2(\Gamma \setminus \mathcal{H})$, for some fixed number $a > 1$; this is then integrated against the Green's function $G_a(z,w)$. It is seen in this proof that

$$\sum_{n \geqq 0} |d_n|^2 \left(a(1-a) - \lambda_n\right)^2 + 2\pi \sum_{k=1}^{\kappa} \int_0^\infty |g_k(R)|^2 \left(a(1-a) - \tfrac{1}{4} - R^2\right)^2 dR \tag{2.41}$$
$$= \int_{\mathcal{F}} |Df(z) + a(1-a)f(z)|^2 < \infty.$$

Cf. [H3, pp. 91(9.36), 244–245, 291(3.23)].

We will need some *uniform* bound on the rate of convergence in (2.40) for all $z \in \mathcal{H}$. This is obtained in the next two lemmas.

Lemma 2.3.2. *Let $\delta > 2$ be given. Define the numbers $r_0, r_1, \dots$ through $\lambda_n = \frac{1}{4} + r_n^2$, $r_n \in i[-1/2, 0] \cup [0, \infty)$. We then have, for all $T \geqq 1$, $z \in \mathcal{H}$:*

$$\sum_{r_n > T} r_n^{-\delta} |\phi_n(z)|^2 + \sum_{k=1}^{\kappa} \int_T^\infty R^{-\delta} \left| E_k(z, \tfrac{1}{2} + iR) \right|^2 dR = O\left(T^{2-\delta} \mathcal{Y}_\Gamma(z) \right), \tag{2.42}$$

and $$\sum_{r_n > T} r_n^{-\delta} = O\left(T^{2-\delta} \right).$$

(The implied constants depend only on δ and Γ.)

Proof. We write

$$S(X) = \sum_{0 \leqq r_n \leqq X} |\phi_n(z)|^2 + \sum_{k=1}^{\kappa} \int_0^X |E_k(z, \tfrac{1}{2} + iR)|^2 \, dR. \tag{2.43}$$

We then have, by Bessel's inequality [I1, Prop. 7.2]:

$$S(X) = O\left(X^2 + X \, \mathcal{Y}_\Gamma(z) \right) = O\left(X^2 \, \mathcal{Y}_\Gamma(z) \right), \tag{2.44}$$

for all $X \geqq 1$. The implied constant depends here only on Γ. (We used the fact that $\mathcal{Y}_\Gamma(z)$ is bounded from below by a positive constant, cf. Lemma 2.1.2.) Now the left hand side in (2.42) equals

$$\begin{aligned} \int_T^\infty R^{-\delta} \, dS(R) &= \Big[R^{-\delta} S(R) \Big]_T^\infty + \delta \int_T^\infty R^{-\delta-1} S(R) \, dR \\ &\leqq \int_T^\infty O\left(R^{1-\delta} \mathcal{Y}_\Gamma(z) \right) dR = O\left(T^{2-\delta} \mathcal{Y}_\Gamma(z) \right). \end{aligned}$$

(Convergence is clear from (2.44) and $\delta > 2$.)

The second bound, $\sum_{r_n > T} r_n^{-\delta} = O\left(T^{2-\delta} \right)$, follows by a similar computation using $\left| \{ 0 \leqq r_n \leqq X \} \right| = O\left(X^2 \right)$ for $X \geqq 1$ (cf. [H3, p. 315(ii)]). □

Lemma 2.3.3. *Let $f(z)$ be as in (2.39), (2.40). We then have, uniformly over $T \geqq 1$, $z \in \mathcal{H}$:*

$$\begin{aligned} &\left| f(z) - \sum_{\lambda_n \leqq 1/4 + T^2} d_n \phi_n(z) - \sum_{k=1}^{\kappa} \int_0^T g_k(R) E_k(z, \tfrac{1}{2} + iR) \, dR \right| \\ &\leqq O\left(T^{-1} \sqrt{\mathcal{Y}_\Gamma(z)} \right). \end{aligned} \tag{2.45}$$

(The implied constant depends only on $f(z)$ and Γ.)

Proof. By (2.40), the difference in the left hand side of (2.45) equals

$$\sum_{\lambda_n > 1/4 + T^2} d_n \phi_n(z) + \sum_{k=1}^{\kappa} \int_T^\infty g_k(R) E_k(z, \tfrac{1}{2} + iR) \, dR. \tag{2.46}$$

By (2.41), the sum and the κ integrals

$$\sum_{\lambda_n > 1/4 + 1} |d_n|^2 \lambda_n^2; \qquad \int_1^\infty |g_k(R)|^2 R^4 \, dR \quad (k = 1, 2, \dots, \kappa)$$

are all bounded from above by some finite constant. Hence by writing $d_n \phi_n(z) = (d_n \lambda_n) \cdot (\phi_n(z)/\lambda_n)$ and $g_k(R) E_k(\dots) = (g_k(R) R^2) \cdot (E_k(\dots)/R^2)$ in (2.46), and then applying Cauchy's inequality, we get that the modulus of (2.46) is bounded from above by

$$O(1)\sqrt{\sum_{\lambda_n>1/4+T^2} \frac{|\phi_n(z)|^2}{\lambda_n^2}} + O(1)\sum_{k=1}^{\kappa} \sqrt{\int_T^\infty \frac{|E_k(z,\frac{1}{2}+iR)|^2}{R^4}\,dR}$$

$$= O(1)\sqrt{\sum_{r_n>T} \frac{|\phi_n(z)|^2}{r_n^4} + \sum_{k=1}^{\kappa} \int_T^\infty \frac{|E_k(z,\frac{1}{2}+iR)|^2}{R^4}\,dR}.$$

Now (2.45) follows from Lemma 2.3.2. □

Remark 2.3.4. Using Lemma 2.3.3, it is easy to prove that any function $f(z)$ satisfying the L_2-condition (2.39) also satisfies the growth condition (2.37). To see this, notice that by Lemma 2.1.2 ("(2.9)⇒(2.10)") and [H3, 271(Def. 3.4), 282(3.9)], each of the finitely many eigenfunctions $\phi_n(z)$ with $\lambda_n \leqq 1/4+1$ is bounded by $O(\sqrt{\mathcal{Y}_\Gamma(z)})$ on all of $z \in \mathcal{H}$. Similarly, for each $k \in \{1,...,\kappa\}$ we have $E_k(z,\frac{1}{2}+iR) = O(\sqrt{\mathcal{Y}_\Gamma(z)})$ uniformly over $0 \leqq R \leqq 1$ and $z \in \mathcal{H}$; cf. [H3, p. 297(F)]. Combining these facts with Lemma 2.3.3 (where we take $T=1$), we see that any function $f(z)$ satisfying (2.39) is bounded by $O(\sqrt{\mathcal{Y}_\Gamma(z)})$ on all of $z \in \mathcal{H}$, as asserted. □

For each eigenfunction $\phi_n \in A_0$, we will now consider its Fourier expansion at the cusp $\eta_1 = \infty$. (Cf. (2.47) below, and [H3, pp. 271 (Def. 3.4), 282 (3.9), item 5].) We first recall a result by Hafner, [H1, Thm. 3][2], which gives good upper bounds on certain *linear sums* of the corresponding Fourier coefficients. We will review the proof in Section 2.4 (see pp. 162ff), where we also extend the result to A_1-eigenfunctions.

Lemma 2.3.5. *Let $\phi(z)$ be a cusp form on $\Gamma \setminus \mathcal{H}$, with eigenvalue $\lambda = s(1-s)$. Let its Fourier expansion at the cusp $\eta_1 = \infty$ be:*

$$\phi(z) = \sum_{n\neq 0} c_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y)\, e(nx). \tag{2.47}$$

We then have, uniformly over $N \geqq 1$ and $\alpha \in \mathbb{R}$:

$$\sum_{n=1}^{N} c_n e(n\alpha) = O\left(\sqrt{N}(\log 2N)^2\right).$$

(The implied constant depends on Γ and $\phi(z)$.) The same bound holds for the sum $\sum_{n=-N}^{-1} c_n e(n\alpha)$.

Using this bound, we will now prove that *for cusp forms*, the horocycle integral converges to what it should *for any exponent $\gamma < 1$:*

Proposition 2.3.6. *Let $\gamma \in (0,1)$, and let $\phi(z)$ be a cusp form on $\Gamma \setminus \mathcal{H}$, with eigenvalue $\lambda = s(1-s)$. We then have*

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(x+iy)\,dx \to 0 \tag{2.48}$$

uniformly as $y \to 0$ so long as $\beta-\alpha$ remains bigger than y^γ.

---

[2] We present the result in a slightly modified form.

Proof. Assuming $\phi(z)$ to have Fourier expansion as in (2.47), the left hand side in (2.48) equals

$$\frac{1}{\beta-\alpha}\sum_{n\neq 0} c_n\sqrt{y}K_{s-\frac{1}{2}}\left(2\pi|n|y\right)\frac{e(n\beta)-e(n\alpha)}{2\pi i n}. \tag{2.49}$$

(for recall from [H3, pp. 23–25] that the expansion (2.47) is uniformly convergent over any half plane $\operatorname{Im} z \geqq Y > 0$, hence in particular over any fixed horocycle segment $[\alpha+iy,\beta+iy]$.)

We will encounter similar sums several times in this paper, and we state a general lemma which applies to all these cases:

Lemma 2.3.7. *Let C_1, C_2, C_3, a be any positive numbers, and let $0 \leqq h < b < h+1$. Let $F(x)$ be a function in $C^1(\mathbb{R}^+)$ satisfying the inequalities*

$$\begin{array}{lll} |F(x)| \leqq C_1 x^{-h}, & |F'(x)| \leqq C_1 x^{-h-1}, & \text{for } 0 < x \leqq 1; \\ |F(x)| \leqq C_2 e^{-ax}, & |F'(x)| \leqq C_2 e^{-ax} & \text{for } 1 \leqq x. \end{array}$$

Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that

$$S_\alpha(X) := \sum_{1\leqq n\leqq X} c_n e(\alpha n) \quad \textit{satisfies} \quad |S_\alpha(X)| \leqq C_3 X^b$$

for all $X \geqq 0$, $\alpha \in \mathbb{R}$.

We then have

$$\frac{1}{\beta-\alpha}\sum_{n=1}^{\infty} c_n\sqrt{y}F(ny)\frac{e(n\beta)-e(n\alpha)}{2\pi i n} = O\left(y^{1/2-h}(\beta-\alpha)^{h-b}\right),$$

for all $0 < y \leqq 1$ and all real numbers α, β satisfying $\beta - \alpha \geqq y$. (The implied constant depends on C_1, C_2, C_3, a, h, b, but not *on $F(x)$, $\{c_n\}$, α, β, y.)*

Proof. We write $\delta = \beta - \alpha$. We first assume $\delta \leqq 2$, and split the given sum at $N = [2\delta^{-1}] \geqq 1$ (where $[x]$ denotes the largest integer $\leqq x$). The first part is then

$$\begin{aligned} &= \frac{\sqrt{y}}{2\pi i\delta}\sum_{n=1}^{N} c_n e(\alpha n)F(ny)\frac{e(n\delta)-1}{n} \\ &= \frac{\sqrt{y}}{2\pi i\delta}\int_{1/2}^{N} F(xy)\frac{e(x\delta)-1}{x}\,dS_\alpha(x) \\ &= \frac{\sqrt{y}}{2\pi i\delta}\left[F(xy)\frac{e(x\delta)-1}{x}S_\alpha(x)\right]_{x=1/2}^{x=N} - \frac{\sqrt{y}}{2\pi i\delta}\int_{1/2}^{N}\Bigg(yF'(xy)\frac{e(x\delta)-1}{x} \\ &\qquad\qquad + F(xy)\frac{2\pi i\delta\cdot e(x\delta)\cdot x-(e(x\delta)-1)}{x^2}\Bigg)S_\alpha(x)\,dx. \end{aligned}$$

Notice that $Ny = [2\delta^{-1}]\,y \leqq 2y^{-1}y = 2$. Now use the given bounds on $F(x)$, $F'(x)$ and $S_\alpha(x)$ (noticing $e^{-ax} = O(x^{-h-1}) = O(x^{-h})$ for $1 \leqq x \leqq 2$), together with $e(x) = O(1)$, $e(x)-1 = O(x)$ (true for all $x \in \mathbb{R}$). We get:

$$\begin{aligned} &O\left(\frac{\sqrt{y}}{\delta}(Ny)^{-h}\frac{N\delta}{N}N^b\right) + O\left(\frac{\sqrt{y}}{\delta}\int_{1/2}^{N}\left(y(xy)^{-h-1}\frac{x\delta}{x}+(xy)^{-h}\frac{x\delta}{x^2}\right)x^b\,dx\right) \\ &= O\left(y^{1/2-h}N^{b-h}\right) + O\left(y^{1/2-h}\int_{1/2}^{N} x^{b-h-1}\,dx\right) \\ &= O\left(y^{1/2-h}N^{b-h}\right) = O\left(y^{1/2-h}\delta^{h-b}\right). \end{aligned}$$

In the last steps we used $b - h > 0$ and $N = [2\delta^{-1}] \leqq 2\delta^{-1}$.

To treat the $\sum_{n>N}$-part of the given sum, we split it as follows:

$$\frac{1}{\delta} \sum_{n>N} c_n \sqrt{y} F(ny) \frac{e(n\beta)}{2\pi i n} - \frac{1}{\delta} \sum_{n>N} c_n \sqrt{y} F(ny) \frac{e(n\alpha)}{2\pi i n}. \tag{2.50}$$

Here the first sum equals

$$\begin{aligned} \frac{\sqrt{y}}{2\pi i \delta} \int_N^\infty \frac{F(xy)}{x} \, dS_\beta(x) &= \frac{\sqrt{y}}{2\pi i \delta} \left[\frac{F(xy)}{x} S_\beta(x) \right]_{x=N}^{x=\infty} \\ &\quad - \frac{\sqrt{y}}{2\pi i \delta} \int_N^\infty \frac{yF'(xy)x - F(xy)}{x^2} S_\beta(x) \, dx. \end{aligned} \tag{2.51}$$

(Convergence follows easily from the given bound on $S_\beta(x)$, and the exponential decay of $|F(x)|$, $|F'(x)|$.) Recall that $Ny \leqq 2$; to start with, we will assume that $Ny < 1$. We use the given bounds on $F(xy)$, $F'(xy)$ and $S_\beta(x)$. Since the bounds on $F(xy)$, $F'(xy)$ differ according to whether $xy \leqq 1$ or $xy \geqq 1$, we have to split the integral at the point $x = 1/y$. We get:

$$\begin{aligned} &O\left(\frac{\sqrt{y}}{\delta} \frac{(Ny)^{-h}}{N} N^b \right) + O\left(\frac{\sqrt{y}}{\delta} \right) \int_N^{1/y} \frac{y(xy)^{-h-1}x + (xy)^{-h}}{x^2} x^b \, dx \\ &\qquad + O\left(\frac{\sqrt{y}}{\delta} \right) \int_{1/y}^\infty \frac{xye^{-axy}}{x^2} x^b \, dx \\ &= O\left(y^{1/2-h} N^{-1-h+b} \delta^{-1} \right) + O\left(y^{1/2-h} \delta^{-1} \int_N^{1/y} x^{-2-h+b} \, dx \right) \\ &\qquad + O\left(y^{3/2} \delta^{-1} \int_{1/y}^\infty x^{-1+b} e^{-ayx} \, dx \right) \\ &\left\{ \begin{array}{l} \text{In the second term, use } -2-h+b < -1. \\ \text{In the last integral, substitute } x = u/(ay). \end{array} \right\} \\ &= O\left(y^{1/2-h} N^{-1-h+b} \delta^{-1} \right) + O\left(y^{3/2-b} \delta^{-1} \int_a^\infty u^{-1+b} e^{-u} \, du \right) \\ &= O\left(y^{1/2-h} \delta^{h-b} \right) + O\left(y^{3/2-b} \delta^{-1} \right). \end{aligned}$$

In the last step we used $-1 - h + b < 0$ and $N = [2\delta^{-1}] > \delta^{-1}$ for $\delta \leqq 2$. Exactly the same bound on (2.51) also holds in the other case, i.e. when $1 \leqq Ny \leqq 2$. This is proved by a similar computation. (This computation is easier, though, since we do not have to split the integral.)

The second sum in (2.50) is entirely similar.

Notice that $y^{1/2-h}\delta^{h-b} \geqq y^{3/2-b}\delta^{-1}$ necessarily holds, since this equivalent to $\delta^{1+h-b} \geqq y^{1+h-b} \iff \delta \geqq y$. Hence the total sum is:

$$O\left(y^{1/2-h} \delta^{h-b} \right) + O\left(y^{3/2-b} \delta^{-1} \right) = O\left(y^{1/2-h} \delta^{h-b} \right).$$

This concludes the proof for the case $\delta \leqq 2$.

For the remaining case, $\beta - \alpha = \delta > 2$, the proof is now very easy. We write $\beta' = \beta - m$, where $m \in \{1, 2, 3, ...\}$ is such that $\beta' - \alpha \in [1, 2)$. We then have, for all $0 < y \leqq 1$:

$$\begin{aligned}&\frac{1}{\beta-\alpha}\sum_{n=1}^{\infty} c_n\sqrt{y}F(ny)\frac{e(n\beta)-e(n\alpha)}{2\pi in}\\ &=\frac{\beta'-\alpha}{\beta-\alpha}\cdot\frac{1}{\beta'-\alpha}\sum_{n=1}^{\infty} c_n\sqrt{y}F(ny)\frac{e(n\beta')-e(n\alpha)}{2\pi in}\\ &=O\left(\frac{1}{\beta-\alpha}\cdot y^{\frac{1}{2}-h}(\beta'-\alpha)^{h-b}\right)=O\left(\frac{1}{\beta-\alpha}y^{\frac{1}{2}-h}\right)=O\left(y^{\frac{1}{2}-h}(\beta-\alpha)^{h-b}\right),\end{aligned}$$

since $-1 < h-b < 0$. □

Proof of Proposition 2.3.6 (continued). We split the sum (2.49) as follows:

$$\frac{1}{\beta-\alpha}\sum_{n=1}^{\infty} c_n\sqrt{y}K_{s-\frac{1}{2}}(2\pi ny)\frac{e(n\beta)-e(n\alpha)}{2\pi in}.$$
$$+\frac{1}{\beta-\alpha}\sum_{n=1}^{\infty} c_{-n}\sqrt{y}K_{s-\frac{1}{2}}(2\pi ny)\frac{e(n(-\alpha))-e(n(-\beta))}{2\pi in}.$$

For each of these two sums, we now apply Lemma 2.3.7, with

$$F(x)=K_{s-\frac{1}{2}}(2\pi x);\qquad a=2\pi;\qquad h=\mathrm{Re}\,(s)-\tfrac{1}{2}+\varepsilon;\qquad b=\tfrac{1}{2}+\varepsilon,$$

for some arbitrary number $\varepsilon > 0$. To see that all assumptions in Lemma 2.3.7 are then fulfilled, recall $\mathrm{Re}\,(s) \in [\frac{1}{2}, 1)$, and cf. Lemma 2.3.5 and [W, pp. 77(2),78(6),80(14)] (for $0 < x \leqq 1$), [W, pp. 79(2),202(1)] (for $1 \leqq x$). The conclusion from Lemma 2.3.7 is that for $y \leqq 1$ and $\beta-\alpha \geqq y$, the left hand side in (2.48) is:

$$O\left(y^{1-\mathrm{Re}\,(s)-\varepsilon}(\beta-\alpha)^{\mathrm{Re}\,(s)-1}\right). \tag{2.52}$$

Since $\mathrm{Re}\,(s)-1 < 0$ and $\beta-\alpha \geqq y^{\gamma}$, we get:

$$O\left(y^{(1-\mathrm{Re}\,(s))(1-\gamma)-\varepsilon}\right).$$

Because of $\gamma < 1$, the exponent here is *positive*, provided ε was taken small enough. Hence the expression tends to 0 when $y \to 0$. □

We now turn to the Eisenstein series $E_k(z, \frac{1}{2}+iR)$. Again, we will consider the Fourier expansion at the cusp $\eta_1 = \infty$, cf. (2.53) below, and [H3, pp. 280(Prop. 3.7), 297(F)]. We will prove a Rankin-Selberg type bound on the Fourier coefficients.

Lemma 2.3.8. *Let T be a fixed, positive number, and let $k \in \{1, ..., \kappa\}$. Consider the Fourier expansion of the Eisenstein series $E_k\left(z, \frac{1}{2}+iR\right)$ at the cusp $\eta_1 = \infty$:*

$$\begin{aligned}E_k\left(z,\tfrac{1}{2}+iR\right)=\delta_{k1}y^{\frac{1}{2}+iR}+\varphi_{k1}\left(\tfrac{1}{2}+iR\right)y^{\frac{1}{2}-iR}\\ +\sum_{n\neq 0} c_n\sqrt{y}K_{iR}(2\pi|n|y)\,e(nx).\end{aligned} \tag{2.53}$$

(Thus, our c_n's depend on R.) We then have, uniformly over $N \geqq 1$ and $0 \leqq R \leqq T$:

$$\sum_{1\leqq|n|\leqq N}|c_n|^2 = O\left(N\log 2N\right)$$

(The implied constant depends only on T and Γ.)

Proof. The basic method in the following proof is the same as in [I1, Thm. 3.2]. However, some modifications are necessary, since $E_k(\cdot, \frac{1}{2} + iR)$ is not in $L_2(\Gamma \setminus \mathcal{H})$.

We keep $0 < Y < H$ (we will later choose Y to be a *small* number, depending on N) and try to find a bound from above for the following integral:

$$J = \int_{\mathcal{D}} \left| E_k\left(z, \tfrac{1}{2} + iR\right)\right|^2 d\mu(z), \qquad \text{where} \quad \mathcal{D} = (0,1) \times (Y, H).$$

We tessellate $\mathcal{D}$ by translates of the fundamental region, i.e. we write $\mathcal{D} = \cup_{T\in\Gamma}(\mathcal{D} \cap T(\mathcal{F}))$, an essentially disjoint union. Using the automorphy of $E_k(z, \frac{1}{2} + iR)$ we then get

$$J = \sum_{T\in\Gamma} \int_{\mathcal{F}} I\big[T(z) \in \mathcal{D}\big] \cdot \left| E_k\left(z, \tfrac{1}{2} + iR\right)\right|^2 d\mu(z), \tag{2.54}$$

where $I[\cdot]$ is the indicator function;

$$I[P] = \begin{cases} 1 & \text{if statement } P \text{ is true} \\ 0 & \text{if statement } P \text{ is false} \end{cases}. \tag{2.55}$$

Recall our definition of B_0 just below (2.1), and relations (2.2), (2.3) defining $\mathcal{C}_{jB}$ and $\mathcal{F}_B$. We let

$$B = \max\left(B_0, H, \frac{1}{Y}\right). \tag{2.56}$$

We then claim that in (2.54), the integrand is zero for all $z \in \mathcal{F} - \mathcal{F}_B$ and all $T \in \Gamma$. Indeed, for any $z \in \mathcal{F} - \mathcal{F}_B$ there is some $j \in \{1, ..., \kappa\}$ such that $z \in \mathcal{C}_{jB}$, i.e. Im $N_j(z) \geqq B$. We then get, by Lemma 2.1.1, for each $T \in \Gamma$:

$$\begin{gathered} \operatorname{Im} T(z) = \operatorname{Im} N_1 T(z) \leqq (\operatorname{Im} N_j(z))^{-1} \leqq B^{-1} \leqq Y \\ \textit{or } j = 1,\ T \in [S], \ \text{ and } \operatorname{Im} T(z) = \operatorname{Im} z \geqq B \geqq H. \end{gathered}$$

In both cases, we get $T(z) \notin \mathcal{D}$, i.e. $I[T(z) \in \mathcal{D}] = 0$. This proves our claim, and as a result we see that *in* (2.54), *we may replace* $\mathcal{F}$ *by* $\mathcal{F}_B$.

Furthermore, since $\mathcal{F}_B$ and $\mathcal{D}$ are both bounded regions, we have

$$\#\{T \in \Gamma \mid T(\mathcal{F}_B) \cap \mathcal{D} \neq \emptyset\} < \infty,$$

(cf., e.g., [M4, p. 17]). Hence there are only finitely many $T \in \Gamma$ for which the integrand in (2.54) does not vanish identically. We may thus (with a 100% elementary justification) change order between summation and integration, so as to obtain:

$$J = \int_{\mathcal{F}_B} \#\Big\{T \in \Gamma \mid T(z) \in \mathcal{D}\Big\} \cdot \left| E_k\left(z, \tfrac{1}{2} + iR\right)\right|^2 d\mu(z).$$

But we have by [I1, Lemma 2.10]:

$$\begin{aligned} &\#\Big\{T \in \Gamma \;\Big|\; T(z) \in \mathcal{D}\Big\} \leqq \#\Big\{W_0 \in [S] \setminus \Gamma \;\Big|\; \operatorname{Im} W_0(z) > Y\Big\} \\ &= 1 + O\left(Y^{-1}\right), \end{aligned}$$

where the implied constant depends only on Γ, i.e. the bound is uniform over all $z \in \mathcal{H}$ and $Y > 0$. This implies:

$$J = O\big(1 + Y^{-1}\big) \int_{\mathcal{F}_B} \left| E_k\left(z, \tfrac{1}{2} + iR\right)\right|^2 d\mu(z). \tag{2.57}$$

By [H3, pp. 280 (Prop. 3.7), 297(F)] and Lemma 2.1.2 ("(2.9)⇒(2.10)"), we have $\left|E_k\left(z, \frac{1}{2} + iR\right)\right| = O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right)$ uniformly over $R \in [0, T]$, $z \in \mathcal{H}$. Hence by Lemma 2.2.4,

$$J = O\Big(\big(1 + Y^{-1}\big)\ \log(2B)\Big). \tag{2.58}$$

On the other hand, substituting (2.53) directly in the definition of J, and then applying Parseval's formula, we get:

$$\begin{aligned} J &= \int_Y^H \left(\left|\delta_{k1} y^{\frac{1}{2}+iR} + \varphi_{k1}\left(\tfrac{1}{2} + iR\right) y^{\frac{1}{2}-iR}\right|^2 + \sum_{n\neq 0} |c_n|^2 y K_{iR}\left(2\pi|n|y\right)^2\right) \frac{dy}{y^2} \\ &\geqq \sum_{n\neq 0} |c_n|^2 \int_Y^H K_{iR}(2\pi|n|y)^2\, \frac{dy}{y} = \sum_{n\neq 0} |c_n|^2 \int_{2\pi|n|Y}^{2\pi|n|H} K_{iR}(y)^2\, \frac{dy}{y}. \end{aligned}$$

We now take $H = 1$ and $Y = (2\pi N)^{-1}$. With this choice we have $[1, 2\pi] \subset \big[2\pi|n|Y, 2\pi|n|H\big]$ whenever $1 \leqq |n| \leqq N$, and hence

$$\sum_{1\leqq|n|\leqq N} |c_n|^2 \leqq C^{-1}\, J, \tag{2.59}$$

where C is the following positive number:

$$C = C(T) = \inf_{R\in[0,T]} \int_1^{2\pi} K_{iR}(y)^2\, \frac{dy}{y}.$$

We also get $B = \max\left(B_0, 2\pi N\right)$ (cf. (2.56)), and the lemma follows from (2.58) and (2.59). □

Remark 2.3.9. At least for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, *the* $O(N \log 2N)$*-bound in Lemma 2.3.8 is the best possible.* In fact, for *fixed* $R > 0$, if we take the Fourier expansion of $E(z, \frac{1}{2} + iR)$ on $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ to be as in (2.53), we have

$$\sum_{n=1}^{N} |c_n|^2 \sim \frac{24}{\pi^2} \cosh(\pi R) \cdot N \log N \qquad \text{as } N \to \infty.$$

This can be proved using the formulas [H3, p. 76(line 10)] and [T1, (1.3.3), with $a = -b = 2iR$], and standard techniques from analytic number theory, similar to [I2, Ch. II].

(Notice carefully that the above asymptotic relation does *not* hold uniformly as $R \to 0$. In fact, $E(z, \frac{1}{2}) \equiv 0$ for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$; hence, when $R = 0$, all coefficients c_n are 0.)

Proposition 2.3.10. *Let* $\gamma \in (0, \frac{1}{2})$*,* $T > 0$ *and* $k \in \{1, ..., \kappa\}$*. We then have:*

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta E_k\left(x + iy, \tfrac{1}{2} + iR\right)\, dx \to 0 \tag{2.60}$$

uniformly as $y \to 0$ *so long as* $R \in [0, T]$ *and* $\beta - \alpha$ *remains bigger than* y^γ*.*

Proof. Let the Fourier expansion of $E_k\left(z, \frac{1}{2} + iR\right)$ be as in (2.53). Then the left hand side in (2.60) equals

$$\begin{aligned} &\delta_{k1} y^{\frac{1}{2}+iR} + \varphi_{k1}\left(\tfrac{1}{2} + iR\right) y^{\frac{1}{2}-iR} \\ &\qquad + \frac{1}{\beta - \alpha} \sum_{n\neq 0} c_n \sqrt{y} K_{iR}\left(2\pi|n|y\right) \frac{e(n\beta) - e(n\alpha)}{2\pi i n}. \end{aligned} \tag{2.61}$$

To treat the sum in (2.61), we split it into two parts exactly as in the proof of Proposition 2.3.6, and then apply Lemma 2.3.7, with

$$F(x) = K_{iR}(2\pi x); \qquad a = 2\pi; \qquad h = 2\varepsilon; \qquad b = 1 + \varepsilon.$$

We claim that for any given $\varepsilon \in (0, \frac{1}{10})$, say, it is possible to find constants $C_1, C_2, C_3 > 0$ such that the assumptions in Lemma 2.3.7 hold true for all $R \in [0, T]$. To see this, first notice that Lemma 2.3.8 implies, by Cauchy's inequality,

$$\sum_{n=1}^{N} |c_n| = O\left(N\sqrt{\log 2N}\right),$$

uniformly over $0 \leqq R \leqq T$, and similarly for $\sum_{n=1}^{N} |c_{-n}|$. Next, a quick way to obtain the desired bounds on $|F(x)|$ and $|F'(x)|$ for $0 < x \leqq 1$ is to first use [W, pp. 77(2),78(6)] to obtain $|K_v(2\pi x)| \leqq C_1 x^{-2\varepsilon}$ and $|2\pi K'_v(2\pi x)| \leqq C_1 x^{-2\varepsilon-1}$ for all $x \in (0,1]$ and all v on the boundary of the rectangle $[2\varepsilon, 2\varepsilon] \times [-T, T]$ in the complex plane, and then, for each fixed $x \in (0,1]$, apply the maximum modulus principle in the v-variable. Finally, the bounds for $x \geqq 1$ follows from [W, pp. 79(2), 202(1)][3].

The application of Lemma 2.3.7 shows that for any $y \leqq 1$ and any α, β such that $\beta - \alpha \geqq y$, the left hand side in (2.60) (cf. (2.61)) is:

$$O\left(y^{\frac{1}{2}}\right) + O\left(y^{\frac{1}{2}-2\varepsilon}(\beta - \alpha)^{\varepsilon - 1}\right) \tag{2.62}$$

Since $\varepsilon - 1 < 0$ and $\beta - \alpha \geqq y^{\gamma}$, we get:

$$O\left(y^{\frac{1}{2}}\right) + O\left(y^{\frac{1}{2}-\gamma-(2-\gamma)\varepsilon}\right).$$

Because of $\gamma < \frac{1}{2}$, the exponent $\frac{1}{2} - \gamma - (2 - \gamma)\varepsilon$ is *positive*, provided that ε was taken small enough. Hence the expression tends to 0 when $y \to 0$. □

We remark that in the above situation, the proof of Lemma 2.3.7 could have been simplified a lot, since our bound actually holds for $\sum_{n=1}^{N} |c_n|$. Cf. the first part of the proof of Lemma 3.2.2 below.

Finally, we are ready to complete the proof of the first main theorem:

Proof of Theorem 2.3.1. We first give the proof under the assumption that f satisfies (2.39). Let $\varepsilon > 0$ be given. Let $C_0 = C_0(\Gamma, \gamma)$ be the implied constant in (2.24) in Proposition 2.2.5 (p. 147). By Lemma 2.3.3, we can find some number $T = T(\Gamma, \gamma, f, \varepsilon) \geqq 1$ such that, for all $z \in \mathcal{H}$,

$$\left| f(z) - \sum_{\lambda_n \leqq 1/4 + T^2} d_n \phi_n(z) - \sum_{k=1}^{\kappa} \int_0^T g_k(R) E_k\left(z, \tfrac{1}{2} + iR\right) dR \right| \leqq C_0^{-1} \varepsilon \sqrt{\mathcal{Y}_\Gamma(z)}.$$

---

[3] Notice that [W, p. 202(1)] holds uniformly for "$v \in$ compacta". This is easily seen by inspecting the proof on pp. 197–198 in [W].

It then follows from (2.24) and our choice of C_0 that

$$\begin{aligned}&\Bigg|\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy)\,dx-\sum_{\lambda_n\leqq 1/4+T^2} d_n\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi_n(x+iy)\,dx\\ &-\sum_{k=1}^{\kappa}\int_0^T g_k(R)\frac{1}{\beta-\alpha}\int_\alpha^\beta E_k\left(x+iy,\tfrac12+iR\right)\,dx\,dR\Bigg|\leqq\varepsilon \end{aligned}\tag{2.63}$$

whenever $0<y<1$ and $\beta-\alpha\geqq y^\gamma$. *Because of our assumption $A_1=\{0\}$,* we know that each eigenfunction ϕ_n $(n\geqq 1)$ is a *cusp form.* Also, $g_k(R)\in L_1(0,T)$, since $g_k(R)\in L_2(0,\infty)$. Hence by Proposition 2.3.6 and Proposition 2.3.10 there is some $y_0=y_0(\Gamma,\gamma,f,\varepsilon)\in(0,1)$ such that

$$\begin{aligned}&\sum_{\substack{\lambda_n\leqq 1/4+T^2\\(n\neq 0)}}|d_n|\cdot\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta\phi_n(x+iy)\,dx\right|\\ &+\sum_{k=1}^{\kappa}\int_0^T|g_k(R)|\,dR\cdot\sup_{R\in[0,T]}\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta E_k\left(x+iy,\tfrac12+iR\right)\,dx\right|\leqq\varepsilon\end{aligned}$$

anytime $0<y<y_0$ and $\beta-\alpha\geqq y^\gamma$. Using this in (2.63) we get

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy)\,dx-d_0\frac{1}{\beta-\alpha}\int_\alpha^\beta\phi_0(x+iy)\,dx\right|\leqq 2\varepsilon.\tag{2.64}$$

But $\phi_0(z)\equiv 1/\sqrt{\mu(\mathcal{F})}$ and $d_0=\langle f,\phi_0\rangle$, hence we have

$$d_0\frac{1}{\beta-\alpha}\int_\alpha^\beta\phi_0(x+iy)\,dx=\frac{1}{\mu(\mathcal{F})}\int_{\mathcal{F}}f(z)d\mu(z)$$

in (2.64). Since ε was arbitrary, we obtain the desired result, (2.38).

To complete the proof, we now have to show how to treat arbitrary functions $f\in C(\Gamma\setminus\mathcal{H})$, restricted only by (2.37). The main tools will be (2.22) in Lemma 2.2.4 and (2.25) in Proposition 2.2.5. By the first of these two inequalities, there is some positive constant C_1 which only depends on Γ, such that

$$\frac{1}{\mu(\mathcal{F})}\int_{\mathcal{F}}\sqrt{\lfloor\mathcal{Y}_\Gamma(z)\rfloor_M}\,d\mu(z)\leqq\frac{C_1}{\sqrt{M}}\qquad\forall\, M\geqq 1.\tag{2.65}$$

Consider any $f\in C(\Gamma\setminus\mathcal{H})$ satisfying (2.37), and let $\varepsilon>0$ be given. We will approximate f in two steps. Recall our definition of B_0 just below (2.1). We *fix* M to be a positive number satisfying

$$M>\max(B_0,10)\qquad\text{and}\qquad\frac{C_1C}{\sqrt{M}}+\frac{25\kappa C}{\sqrt{M}}\leqq\varepsilon,\tag{2.66}$$

where C and C_1 are as in (2.37) and (2.65), respectively. We then let $F:\mathbb{R}^+\to[0,1]$ be a continuous function such that $F(y)=1$ for $0<y\leqq M$ and $F(y)=0$ for $M+1\leqq y$. The first step is to approximate $f(z)$ by the function

$$f_0(z)=F(\mathcal{Y}_\Gamma(z))\cdot f(z).\tag{2.67}$$

This is a continuous and Γ-invariant function. $f_0(z)$ has *compact support* in $\Gamma\setminus\mathcal{H}$, since by (2.8) in Lemma 2.1.2, $f_0(z)$ vanishes for all $z\in\mathcal{F}-\mathcal{F}_{M+1}$. Using (2.37), we see that $|f(z)-f_0(z)|\leqq C\sqrt{\lfloor\mathcal{Y}_\Gamma(z)\rfloor_M}$ for all $z\in\mathcal{H}$.

The second step is to find a function $f_1(z)\in C^\infty(\mathcal{H})\cap C_c(\Gamma\setminus\mathcal{H})$ (where the index "c" means "of compact support") such that $|f_0(z)-f_1(z)|\leqq\varepsilon$ for all $z\in\mathcal{H}$.

This can be achieved by a standard convolution argument. (We give a detailed proof in Appendix 1.)

By the triangle inequality, we now have:

$$|f(z) - f_1(z)| \leqq \varepsilon + C\sqrt{\lfloor \mathcal{Y}_\Gamma(z) \rfloor_M} \qquad \forall z \in \mathcal{H}. \tag{2.68}$$

Since $f_1(z) \in C^\infty(\mathcal{H}) \cap C_c(\Gamma \setminus \mathcal{H})$, the function $f_1(z)$ certainly satisfies the L^2-conditions (2.39). Hence, by what we have proved, $f_1(z)$ also satisfies the conclusion (2.38) in Theorem 2.3.1. In particular, there is some $y_1 = y_1(\Gamma, \gamma, f_1, \varepsilon) > 0$ such that

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta f_1(x + iy)\, dx - \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} f_1(z) d\mu(z) \right| \leqq \varepsilon \tag{2.69}$$

anytime $0 < y < y_1$ and $\beta - \alpha \geqq y^\gamma$.

But by (2.68) and (2.25) in Proposition 2.2.5 there is some $y_0 = y_0(M, \gamma) > 0$ such that

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy)\, dx - \frac{1}{\beta - \alpha} \int_\alpha^\beta f_1(x + iy)\, dx \right| \leqq \varepsilon + \frac{25\kappa C}{\sqrt{M}} \tag{2.70}$$

whenever $0 < y < y_0$ and $\beta - \alpha \geqq y^\gamma$. Also, by (2.65) and (2.68) we have

$$\left| \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} f_1(z) d\mu(z) - \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} f(z) d\mu(z) \right| \leqq \varepsilon + \frac{C_1 C}{\sqrt{M}}. \tag{2.71}$$

Adding up (2.69), (2.70) and (2.71), using the triangle inequality and (2.66), we get

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy)\, dx - \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} f(z) d\mu(z) \right| \leqq 4\varepsilon,$$

anytime $0 < y < \min(y_0, y_1)$ and $\beta - \alpha \geqq y^\gamma$. Since ε was arbitrary, this concludes the proof of Theorem 2.3.1. □ □ □

2.4. **The case $A_1 \neq \{0\}$.** The aim of this section is to prove:

Theorem 2.4.1. *Theorem 2.3.1 remains true in the case* $A_1 \neq \{0\}$.

By reviewing the proof of Theorem 2.3.1 in last section, we see that Theorem 2.4.1 will be proved *once we manage to extend Proposition 2.3.6 to the case* $\phi(z) \in A_1$, $\gamma \in (0, \frac{1}{2})$. This will be carried out below, in Proposition 2.4.4.

The main ingredient of the proof is the following upper bound on linear sums of the Fourier coefficients of an A_1-function. Recall that $A_1(\lambda) = \{0\}$ anytime $\lambda \geqq \frac{1}{4}$. Also recall [H3, pp. 271 (Def. 3.4), 282 (3.9), item 5].

Proposition 2.4.2. *Let Γ be a cofinite Fuchsian group with a normalized cusp (see (2.19)). Let $\phi(z) \in A_1(\lambda)$, with $\lambda = s(1 - s)$, $s \in (\frac{1}{2}, 1)$. Let the Fourier expansion of $\phi(z)$ at the cusp $\eta_1 = \infty$ be:*

$$\phi(z) = c_0 y^{1-s} + \sum_{n \neq 0} c_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y)\, e(nx). \tag{2.72}$$

We then have, uniformly over $N \geqq 1$, $\alpha \in \mathbb{R}$:

$$\sum_{n=1}^{N} c_n e(n\alpha) = O\left(N^{3/2-s}\right).$$

(The implied constant depends on Γ and $\phi(z)$.) The same bound holds for the sum $\sum_{n=-N}^{-1} c_n e(n\alpha)$.

To prepare for the proof of Proposition 2.4.2, we first give the proof of Lemma 2.3.5, viz., the corresponding bound for the case where $\phi(z)$ is a *cusp form.*

Proof of Lemma 2.3.5. (This proof is taken from [H1, Thm. 3], except for some trivial modifications.)

The main idea is to study the following integral, for $N \geqq 1$ and $\delta \in [0,1)$:

$$I = \int_0^\infty \int_0^1 \phi(x+iy) \left(\sum_{m=1}^N e(m(\alpha - x)) \right) \frac{dxdy}{y^\delta} \tag{2.73}$$

To see that we have beautiful absolute convergence, we need just notice that since $\phi(z)$ is a cusp form, it is a bounded function on $\mathcal{H}$, and also, $\phi(x+iy) = O\left(e^{-2\pi y}\right)$ for $y \geqq 1$ (cf. (2.47) and [H3, p. 25(bottom)]). In fact, this leads to the following upper bound:

$$\begin{aligned}
|I| &\leqq \left(\int_0^1 O(1) \frac{dy}{y^\delta} + \int_1^\infty O\left(e^{-2\pi y}\right) \frac{dy}{y^\delta} \right) \int_0^1 \left| \sum_{m=1}^N e(m(\alpha - x)) \right| dx \\
&= O\left(\frac{1}{1-\delta}\right) \int_{-1/2}^{1/2} \left| \sum_{m=1}^N e(mx) \right| dx \\
&= O\left(\frac{1}{1-\delta}\right) \int_{-1/2}^{1/2} \left| e(x) \frac{e(Nx)-1}{e(x)-1} \right| dx \\
&= O\left(\frac{1}{1-\delta}\right) \int_{-1/2}^{1/2} O\left(\frac{\min\left(N|x|, 1\right)}{|x|} \right) dx \\
&= O\left(\frac{1}{1-\delta}\right) \left(\int_0^{1/2N} N\, dx + \int_{1/2N}^{1/2} \frac{1}{x}\, dx \right) = O\left(\frac{\log 2N}{1-\delta} \right), \qquad (2.74)
\end{aligned}$$

uniformly over $N \geqq 1$, $\alpha \in \mathbb{R}$, $\delta \in [0,1)$.

On the other hand we can evaluate the inner integral in (2.73) using the Fourier expansion of $\phi(z)$, (2.47), thus obtaining:

$$\begin{aligned}
I &= \int_0^\infty \sum_{m=1}^N c_m \sqrt{y} K_{s-\frac{1}{2}}\left(2\pi m y\right) e(m\alpha) \frac{dy}{y^\delta} \qquad (2.75) \\
&= \sum_{m=1}^N c_m e(m\alpha) (2\pi m)^{\delta - \frac{3}{2}} \int_0^\infty K_{s-\frac{1}{2}}(u) u^{\frac{1}{2}-\delta}\, du \\
&= \sum_{m=1}^N c_m e(m\alpha) (2\pi m)^{\delta - \frac{3}{2}} 2^{-\frac{1}{2}-\delta}\, \Gamma\left(\frac{2-s-\delta}{2} \right) \Gamma\left(\frac{1+s-\delta}{2} \right).
\end{aligned}$$

For the last equality, cf. [W, p. 388(8)]. We now combine this with (2.74). Since the gamma factors are bounded away from 0, uniformly over $\delta \in [0,1)$ (for *fixed* s), we get

$$\sum_{m=1}^N c_m e(m\alpha) m^{\delta - 3/2} = O\left(\frac{\log 2N}{1-\delta} \right),$$

uniformly over $N \geqq 1$, $\alpha \in \mathbb{R}$, $\delta \in [0,1)$. We now apply partial summation. We write

$$S(X) := \sum_{1 \leqq n \leqq X} c_n e(n\alpha) n^{\delta - 3/2} = O\left(\frac{\log 2X}{1-\delta} \right) \qquad \text{for } X \geqq \frac{1}{2}, \quad \delta \in [0,1).$$

Then:

$$\begin{aligned}\sum_{n=1}^{N} c_n e(n\alpha) &= \int_{1/2}^{N} x^{3/2-\delta}\, dS(x)\\ &= \left[x^{3/2-\delta} S(x)\right]_{x=1/2}^{x=N} - \int_{1/2}^{N} \left(\tfrac{3}{2}-\delta\right) x^{1/2-\delta} S(x)\, dx\\ &= O\left(\frac{N^{3/2-\delta}\log 2N}{1-\delta}\right) + \int_{1/2}^{N} O\left(\frac{x^{1/2-\delta}\log 2N}{1-\delta}\right) dx\\ &= O\left(\frac{N^{3/2-\delta}\log 2N}{1-\delta}\right),\end{aligned}$$

uniformly over $N \geqq 1$, $\alpha \in \mathbb{R}$, $\delta \in [0,1)$. Here take $\delta = 1 - 1/\log(10N)$; we then get

$$\sum_{n=1}^{N} c_n e(n\alpha) = O\left(\sqrt{N}\,(\log 2N)^2\right),$$

as asserted. By replacing the inner sum in (2.73) by $\sum_{m=-N}^{-1} e(m(\alpha - x))$, and then arguing as above, we also obtain:

$$\sum_{n=-N}^{-1} c_n e(n\alpha) = O\left(\sqrt{N}\,(\log 2N)^2\right).$$

☐

Proof of Proposition 2.4.2. The proof is again based on a study of the integral I, (2.73), where we now *fix one choice of the exponent* δ, subject only to the condition

$$s < \delta < 1. \tag{2.76}$$

This time, we cannot assert that the double integral I is absolutely convergent, but only that

$$\int_0^\infty \left| \int_0^1 \phi(x+iy) \left(\sum_{m=1}^{N} e(m(\alpha - x))\right) dx \right| \frac{dy}{y^\delta} < \infty. \tag{2.77}$$

To see this we evaluate the inner integral using the Fourier expansion of $\phi(z)$, (2.72). The c_0-term is then killed, and we get exactly the same expression as in (2.75), the first line. We can now apply nice majorants[4] to deduce (2.77). We also see that the formula for I obtained in the last line of (2.75) holds in the present case as well.

Our task is now to obtain a good bound on (2.77) as $N \to \infty$. This is *much* more difficult than in the cusp form case treated above.

The basic idea is to express the function $\phi(z)$ as a sum of one *bounded* part, and one very explicit "cuspidal part". This is similar to, eg., [H3, pp. 109(11.2), 154(top), 292(3.25)].

For each $k \in \{1, ..., \kappa\}$, we know from the Fourier expansion of $\phi(z)$ at η_k that

$$\phi(z) = c_0^{(k)} (\operatorname{Im} N_k(z))^{1-s} + O\left(e^{-2\pi \operatorname{Im} N_k(z)}\right) \tag{2.78}$$

whenever $\operatorname{Im} N_k(z) \geqq 1$. (Cf. (2.72); notice that $c_0^{(1)} = c_0$.)

---

[4] Specifically: $K_{s-\frac{1}{2}}(u) = O(u^{\frac{1}{2}-s})$ for $0 < u \leqq 1$ and $K_{s-\frac{1}{2}}(u) = O(e^{-u})$ for $u \geqq 1$; cf. [W, pp. 77(2), 78(6), 202(1)].

We fix some number $B > B_0$. (Hence $B > 1$.) For each $z \in \mathcal{H}$ we can find $\langle k, W\rangle \in \{1, ..., \kappa\} \times \Gamma$ such that $\text{Im } N_k W(z) = \mathcal{Y}_\Gamma(z)$, cf. Lemma 2.1.2. Notice that if $\text{Im } N_k W(z) = \mathcal{Y}_\Gamma(z) > 1$ then, for all $j \neq k$, $W' \in \Gamma$:

$$\text{Im } N_j W'(z) = \text{Im } N_j W' W^{-1} W(z) \leqq (\text{Im } N_k W(z))^{-1} < 1,$$

by Lemma 2.1.1. Hence, *if* $\mathcal{Y}_\Gamma(z) > 1$, then $k = k(z)$ is *uniquely determined* in the relation $\text{Im } N_k W(z) = \mathcal{Y}_\Gamma(z)$. We now define:

$$c(z) = \begin{cases} c_0^{(k(z))} \mathcal{Y}_\Gamma(z)^{1-s} & \text{if } \mathcal{Y}_\Gamma(z) \geqq B \\ 0 & \text{otherwise} \end{cases}. \tag{2.79}$$

Clearly, $c(z)$ is a Γ-invariant function. Next, we define $b(z)$ through

$$\phi(z) = b(z) + c(z). \tag{2.80}$$

We now have $b(z) = \phi(z)$ for all $z \in \mathcal{F}_B$, by (2.8) in Lemma 2.1.2. Also, by (2.7) and (2.78), $b(z) = O\left(e^{-2\pi \text{Im } N_k(z)}\right)$ for all $k \in \{1, ..., \kappa\}$, $z \in \mathcal{C}_{kB}$. Hence $b(z)$ is a bounded function on $\mathcal{F}$. By Γ-invariance, it is then bounded on all of $\mathcal{H}$.

Corresponding to (2.80), we split the integral I (cf. (2.73)) as

$$I = I_b + I_c, \tag{2.81}$$

in the obvious way. Since $b(z)$ is a bounded function and $b(z) = O(e^{-2\pi y})$ for $z \in \mathcal{C}_{1B} = [0, 1] \times [B, \infty)$, the computation in (2.74) carries over for I_b, and we obtain

$$|I_b| \leqq \int_0^\infty \int_0^1 \left| b(x + iy) \left(\sum_{m=1}^N e(m(\alpha - x)) \right) \right| \frac{dxdy}{y^\delta} = O(\log 2N). \tag{2.82}$$

Here, and in any "big O" estimate in the rest of this proof, the implied constant depends solely on Γ, $\phi(z)$, B and δ (all of which have now been fixed, once and for all).

We now turn to I_c. By (2.79) and (2.7) in Lemma 2.1.2 we have $c(z) = c_0^{(1)} y^{1-s}$ whenever $z \in \mathcal{C}_{1B}$. Hence

$$\int_B^\infty \int_0^1 c(x + iy) \left(\sum_{m=1}^N e(m(\alpha - x)) \right) \frac{dxdy}{y^\delta} = \int_B^\infty 0 \, dy = 0.$$

Using this, and the fact that $c(z)$ is invariant under $[S] \subset \Gamma$, we see that

$$I_c = \int_0^B \int_{\alpha - \frac{1}{2}}^{\alpha + \frac{1}{2}} c(x + iy) \left(\sum_{m=1}^N e(m(\alpha - x)) \right) \frac{dxdy}{y^\delta}$$

But by (2.79) we have for all $z \in \mathcal{H}$:

$$c(z) \leqq \left(\max_{k \in \{1, ..., \kappa\}} c_0^{(k)} \right) \lfloor \mathcal{Y}_\Gamma(z) \rfloor_B^{1-s}.$$

We also have, by a computation as in (2.74),

$$\left| \sum_{m=1}^N e(m(\alpha - x)) \right| = O\left(\min\left(N, |x - \alpha|^{-1}\right)\right) \qquad \forall x \in \left[\alpha - \tfrac{1}{2}, \alpha + \tfrac{1}{2}\right].$$

Hence:

$$I_c = O(1) \int_0^B \int_{\alpha - \frac{1}{2}}^{\alpha + \frac{1}{2}} \lfloor \mathcal{Y}_\Gamma(z) \rfloor_B^{1-s} \min\left(N, |x - \alpha|^{-1}\right) \frac{dx\,dy}{y^\delta}.$$

Using here the definition of $\mathcal{Y}_\Gamma(z)$ (cf. the computation in (2.30)) we obtain

$$I_c = O(1)\sum_{k=1}^{\kappa}\sum_{W_0\in[S]\setminus N_k\Gamma}\int_0^B\int_{\alpha-\frac12}^{\alpha+\frac12}\lfloor \operatorname{Im} W_0(z)\rfloor_B^{1-s} \times \min\left(N,|x-\alpha|^{-1}\right)\frac{dx\,dy}{y^\delta}. \tag{2.83}$$

We now consider any $k \in \{1,...,\kappa\}$ and $W_0 = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in [S]\setminus N_k\Gamma$ which give a non-zero contribution in (2.83). Then $\operatorname{Im} W_0(z) \geqq B$ for some $z = x+iy \in \left(\alpha-\frac12,\alpha+\frac12\right)\times(0,B)$. This implies $W_0 \notin [S]$, and hence $|c| \geqq 1$, by (2.4) in Lemma 2.1.1 (with $j=1$; recall that $N_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$). Also, by Remark 2.1.4, z belongs to the horoball tangent to $\mathbb{R}$ at $-\frac{d}{c}$ with Euclidean radius

$$r = \frac{1}{2c^2B} \leqq \frac{1}{2B} < \frac12.$$

In particular, $|x+\frac{d}{c}| \leqq r < \frac12$, and combining this with $|x-\alpha| < \frac12$ we see that $-\frac{d}{c} \in (\alpha-1,\alpha+1)$.

Claim 1. The contribution from each $\langle k, W_0\rangle$ in (2.83) is bounded by

$$O\left[|c|^{2s-2}\min\left(N,|\alpha'|^{-1}\right)^{\delta-s}\cdot\left(1+\log^+\left(N\,|\alpha'|\right)\right)\right], \tag{2.84}$$

where

$$\alpha' = \alpha + \frac{d}{c}. \tag{2.85}$$

Proof of Claim 1. As noticed above, the integrand in (2.83) vanishes outside the vertical strip $|x+\frac{d}{c}| \leqq r$. Using this, $\alpha' = \alpha+\frac{d}{c}$ and $\operatorname{Im} W_0(z) = y/|cz+d|^2$, we get the following upper bound on the double integral in (2.83):

$$|c|^{2s-2}\int_{-r}^{r}\left(\int_0^B\left(\frac{y}{x^2+y^2}\right)^{1-s}\frac{dy}{y^\delta}\right)\min\left(N,|x-\alpha'|^{-1}\right)dx. \tag{2.86}$$

For any $x \neq 0$, the inner integral in (2.86) is less than

$$\begin{aligned}\int_0^\infty\left(\frac{y}{x^2+y^2}\right)^{1-s}\frac{dy}{y^\delta} &\leqq \int_0^{|x|}\left(\frac{y}{x^2}\right)^{1-s}\frac{dy}{y^\delta}+\int_{|x|}^\infty\left(\frac{y}{y^2}\right)^{1-s}\frac{dy}{y^\delta}\\ &= |x|^{2s-2}\int_0^{|x|}y^{1-s-\delta}\,dy+\int_{|x|}^\infty y^{s-1-\delta}\,dy = O\left(|x|^{s-\delta}\right).\end{aligned} \tag{2.87}$$

We used the fact that $\frac12 < s < \delta < 1$, cf. (2.76).

Using (2.87) and $|x-\alpha'| \geqq ||x|-|\alpha'||$, and then noticing the symmetry $[x \leftrightarrow (-x)]$, we see that (2.86) is bounded by

$$O\left(|c|^{2s-2}\right)\int_0^\infty x^{s-\delta}\min\left(N,\left|x-|\alpha'|\right|^{-1}\right)dx. \tag{2.88}$$

We now split into two cases. First assume $|\alpha'| \leqq \frac{100}{N}$. Then,

$$\forall x \geqq \tfrac{200}{N}: \qquad \left|x-|\alpha'|\right| \geqq x - \tfrac{100}{N} \geqq x/2, \tag{2.89}$$

and our expression is bounded by

$$O\left(|c|^{2s-2}\right)\int_0^{\frac{200}{N}}x^{s-\delta}N\,dx + O\left(|c|^{2s-2}\right)\int_{\frac{200}{N}}^\infty x^{s-\delta-1}\,dx = O\left(|c|^{2s-2}N^{\delta-s}\right).$$

We again used the fact that $\frac{1}{2} < s < \delta < 1$. The bound thus obtained is the same as (2.84), since $|\alpha'| \leqq \frac{100}{N}$.

We next treat the other case, $|\alpha'| > \frac{100}{N}$. Then, by splitting the integral at $x = |\alpha'|/2$ and $x = 3|\alpha'|/2$ and using simple inequalities (similar in spirit to (2.89)), we can estimate (2.88) as follows:

$$O\left(|c|^{2s-2}\right)\left[\int_0^{|\alpha'|/2} x^{s-\delta}\left|\alpha'\right|^{-1} dx + \int_{3|\alpha'|/2}^{\infty} x^{s-\delta}x^{-1}\,dx\right.$$
$$\left. + \int_{|\alpha'|/2}^{3|\alpha'|/2} \left|\alpha'\right|^{s-\delta} \min\left(N, \left|x - |\alpha'|\right|^{-1}\right) dx\right]$$

Evaluating the first two integrals, and noticing the symmetry in the last one, we get:

$$= O\left(|c|^{2s-2}\left|\alpha'\right|^{s-\delta}\right)\left[1 + \int_0^{1/N} N\,dt + \int_{1/N}^{|\alpha'|/2} \frac{dt}{t}\right]$$
$$= O\left(|c|^{2s-2}\left|\alpha'\right|^{s-\delta}\left(1 + \log\left(N\left|\alpha'\right|\right)\right)\right).$$

This is, again, the same as (2.84). □

Claim 2. In the case $c^2 \geqq \min\left(N, |\alpha'|^{-1}\right)$, we also have the following upper bound on the contribution from $\langle k, W_0\rangle$ in (2.83):

$$O\left[|c|^{2\delta-4}\min\left(N, \left|\alpha'\right|^{-1}\right)\right]. \tag{2.90}$$

Proof of Claim 2. Assume that $c^2 \geqq \min\left(N, |\alpha'|^{-1}\right)$. We now maintain that

$$\min\left(N, |x - \alpha'|^{-1}\right) < 2\min\left(N, |\alpha'|^{-1}\right), \qquad \forall x \in [-r, r]. \tag{2.91}$$

Indeed, if $N \leqq |\alpha'|^{-1}$ then (2.91) is trivial, since the right hand side then equals $2N$. In the other case, $N > |\alpha'|^{-1}$, we prove (2.91) by first noticing that

$$(2r)^{-1} > (2Br)^{-1} = c^2 \geqq \min\left(N, |\alpha'|^{-1}\right) = |\alpha'|^{-1},$$

and thus $r < |\alpha'|/2$. We then get, for all $x \in [-r, r]$,

$$|x - \alpha'| \geqq |\alpha'| - r > |\alpha'|/2,$$

and hence $|x - \alpha'|^{-1} < 2|\alpha'|^{-1} = 2\min\left(N, |\alpha'|^{-1}\right)$, and (2.91) is proved.

By (2.91) and (2.87), we get the following upper bound on (2.86):

$$O\Big(|c|^{2s-2}\min\left(N, |\alpha'|^{-1}\right)\Big)\int_{-r}^{r} |x|^{s-\delta}\,dx$$
$$= O\left(|c|^{2s-2}\min\left(N, |\alpha'|^{-1}\right) r^{s-\delta+1}\right).$$

This is the same as (2.90), since $r = (2c^2B)^{-1}$. □

We continue onward with our proof of Proposition 2.4.2. We will add up all contributions to (2.83) for one fixed $k \in \{1, ..., \kappa\}$. As noticed on p. 165, we get a non-zero contribution from $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$ only if $|c| \geqq 1$ and $-\frac{d}{c} \in$

$(\alpha-1,\alpha+1)$. We will split the analysis into the following cases:

$$\begin{aligned}
&-\frac{d}{c}\in\left[\alpha-N^{-1},\alpha+N^{-1}\right];\\
&-\frac{d}{c}\in\left[\alpha+2^{\ell}N^{-1},\alpha+2^{\ell+1}N^{-1}\right],\quad (\ell\in\{0,1,2,...,[\log_2 N]\});\\
&-\frac{d}{c}\in\left[\alpha-2^{\ell+1}N^{-1},\alpha-2^{\ell}N^{-1}\right],\quad (\ell\in\{0,1,2,...,[\log_2 N]\}).
\end{aligned}$$

Clearly, each W_0 giving non-zero contribution to (2.83) will then be counted *at least* once.

Now for all W_0 with $-\frac{d}{c}\in[\alpha-N^{-1},\alpha+N^{-1}]$ we have $|\alpha'|\leqq N^{-1}$ and $\min\left(N,|\alpha'|^{-1}\right)=N$. Hence by Claim 1 and Claim 2, we get the following bound on the total contribution from these W_0's:

$$O(1)\sum_{\substack{W_0\\ 1\leqq|c|<\sqrt{N}}}\left(|c|^{2s-2}N^{\delta-s}\right)+O(1)\sum_{\substack{W_0\\ |c|\geqq\sqrt{N}}}\left(|c|^{2\delta-4}N\right).$$

The sums are taken over a set of representatives $W_0=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in[S]\setminus N_k\Gamma$ restricted by $-\frac{d}{c}\in[\alpha-N^{-1},\alpha+N^{-1}]$ and the stated bounds on $|c|$. Using Lemma 2.2.3 and remembering $\frac{1}{2}<s<\delta<1$, we get:

$$\begin{aligned}
&O\left[N^{\delta-s}\left(N^{-1}\sqrt{N}^{\,2s}+1\right)\right]+O\left[N\left(N^{-1}\sqrt{N}^{\,2\delta-2}+\sqrt{N}^{\,2\delta-4}\right)\right]\\
&\quad=O\left(N^{\delta-s}+N^{\delta-1}\right)=O\left(N^{\delta-s}\right).
\end{aligned}$$

Now take $\ell\in\{0,1,2,...,[\log_2 N]\}$ and consider all W_0 satisfying $-\frac{d}{c}\in\left[\alpha+2^{\ell}N^{-1},\alpha+2^{\ell+1}N^{-1}\right]$. We then have $\min\left(N,|\alpha'|^{-1}\right)=|\alpha'|^{-1}\leqq 2^{-\ell}N$, and hence by Claim 1 and Claim 2, we get the following bound on the total contribution from these W_0's:

$$\begin{aligned}
&O(1)\sum_{\substack{W_0\\ 1\leqq|c|<\sqrt{2^{-\ell}N}}}\left(|c|^{2s-2}\left(2^{-\ell}N\right)^{\delta-s}\left(1+\log\left(2^{\ell+1}\right)\right)\right)\\
&\qquad+O(1)\sum_{\substack{W_0\\ |c|\geqq\sqrt{2^{-\ell}N}}}\left(|c|^{2\delta-4}\left(2^{-\ell}N\right)\right).
\end{aligned}$$

(Of course, $2^{-\ell}N\geqq 1$, since $\ell\leqq[\log_2 N]$.) The sums are taken over a set of representatives $W_0=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in[S]\setminus N_k\Gamma$ restricted by $-\frac{d}{c}\in\left[\alpha+2^{\ell}N^{-1},\alpha+2^{\ell+1}N^{-1}\right]$ and the stated bounds on $|c|$. Using Lemma 2.2.3 we get:

$$\begin{aligned}
&O\left[\left(2^{-\ell}N\right)^{\delta-s}(\ell+1)\left(2^{\ell}N^{-1}\sqrt{2^{-\ell}N}^{\,2s}+1\right)\right]\\
&\qquad+O\left[2^{-\ell}N\left(2^{\ell}N^{-1}\sqrt{2^{-\ell}N}^{\,2\delta-2}+\sqrt{2^{-\ell}N}^{\,2\delta-4}\right)\right]\\
&\quad=O\left[(\ell+1)\left(\left(2^{-\ell}N\right)^{\delta-s}+\left(2^{-\ell}N\right)^{\delta-1}\right)\right]=O\left((\ell+1)\left(2^{-\ell}N\right)^{\delta-s}\right).
\end{aligned}$$

We get exactly the same bound for the case $-\frac{d}{c}\in\left[\alpha-2^{\ell+1}N^{-1},\alpha-2^{\ell}N^{-1}\right]$.

We now add up *all* these contributions to (2.83), for each $k\in\{1,...,\kappa\}$. Since $\sum_{\ell=0}^{\infty}(\ell+1)2^{-\ell(\delta-s)}<\infty$, we obtain:

$$I_c=O\left(N^{\delta-s}\right).$$

Because of (2.81) and (2.82), we also get $I = O\left(N^{\delta-s}\right)$, and using the formula for I in the last line of (2.75), we conclude:

$$\sum_{m=1}^{N} c_m e(m\alpha) m^{\delta-3/2} = O\left(N^{\delta-s}\right).$$

Finally, we apply partial summation in the same way as in the proof of Lemma 2.3.5, using $S(X) = \sum_{1 \leqq n \leqq X} c_n e(n\alpha) n^{\delta-3/2} = O\left(X^{\delta-s}\right)$ for $X \geqq \frac{1}{2}$. We then get:

$$\begin{aligned}\sum_{n=1}^{N} c_n e(n\alpha) &= O\left(N^{3/2-\delta} N^{\delta-s}\right) + \int_{1/2}^{N} O\left(x^{1/2-\delta} x^{\delta-s}\right) dx \\ &= O\left(N^{3/2-s}\right),\end{aligned}$$

and we are done. The proof of $\sum_{n=-N}^{-1} c_n e(n\alpha) = O\left(N^{3/2-s}\right)$ is entirely similar. □

Remark 2.4.3. Proposition 2.4.2 is complemented by the following result, which shows that the exponent $3/2 - s$ therein is the best possible.

Let $\phi(z)$ *be as in Proposition 2.4.2. Take* $k \in \{1, ..., \kappa\}$ *such that* $c_0^{(k)} \neq 0$ *in* (2.78) *(such* k *always exists, cf.* [H3, p. 282 (item 5)]*), and let* $\alpha \in \mathbb{R}$ *be any cusp equivalent to* η_k.[5] *Then at least one of*

$$\sum_{n=1}^{N} c_n e(n\alpha) = \Omega(N^{3/2-s}), \qquad \sum_{n=-N}^{-1} c_n e(n\alpha) = \Omega(N^{3/2-s})$$

must hold as $N \to \infty$.

For assume that this is not true, i.e., assume that $S_+(X) = \sum_{1 \leqq n \leqq X} c_n e(n\alpha) = o(X^{3/2-s})$ and $S_-(X) = \sum_{1 \leqq n \leqq X} c_{-n} e(-n\alpha) = o(X^{3/2-s})$ as $X \to \infty$. (Here, α has been *fixed*.) We will now study $\phi(\alpha + iy)$ as $y \to 0$.

It follows from (2.72), using partial summation (treating $n > 0$ and $n < 0$ separately), that

$$\phi(\alpha + iy) = c_0 y^{1-s} - 2\pi y^{3/2} \int_{1/2}^{\infty} K'_{s-\frac{1}{2}}(2\pi xy)\big(S_+(x) + S_-(x)\big)\, dx.$$

Convergence follows easily using our bounds on $S_\pm(X)$ and the inequality

$$\left|K'_{s-\frac{1}{2}}(2\pi u)\right| \leqq \begin{cases} Cu^{-\frac{1}{2}-s} & \text{for } 0 < u \leqq 1 \\ Ce^{-u} & \text{for } 1 \leqq u. \end{cases}$$

(Cf. [W, pp. 77(2), 78(6), 79(2), 202(1)]; the constant C depends only on s.)

Now let $\varepsilon > 0$ be given. It follows from our assumption on $S_\pm(X)$ that there is some $X_0 = X_0(\varepsilon) > 1$ such that $|S_+(X) + S_-(X)| \leqq \varepsilon X^{3/2-s}$ for all $X \geqq X_0$. Let $M = \sup_{X \in [1/2, X_0]} |S_+(X) + S_-(X)|$. We then get, for all $y < 1/X_0$:

$$\begin{aligned}|\phi(\alpha + iy)| \leqq |c_0| y^{1-s} &+ 2\pi y^{3/2} \int_{1/2}^{X_0} C(xy)^{-\frac{1}{2}-s} M\, dx \\ &+ 2\pi y^{3/2} \int_{X_0}^{1/y} C(xy)^{-\frac{1}{2}-s} \varepsilon x^{3/2-s}\, dx + 2\pi y^{3/2} \int_{1/y}^{\infty} Ce^{-xy} \varepsilon x^{3/2-s}\, dx\end{aligned}$$

---

[5] It is well-known that the set of such points α is dense in $\mathbb{R}$. A more precise statement of this nature is provided by Proposition 4.6.1 below (cf. (2.18)).

$$< \left(|c_0| + 2\pi CM \int_{1/2}^{X_0} x^{-\frac{1}{2}-s}\,dx\right) y^{1-s}$$
$$+ \varepsilon \left(\frac{2\pi C}{2-2s} + 2\pi C \int_1^{\infty} e^{-u} u^{3/2-s}\,du\right) y^{s-1}.$$

Recall that $s \in (\frac{1}{2}, 1)$; hence $s - 1 < 0 < 1 - s$. Since ε was arbitrary, and the expression inside the parentheses in front of y^{s-1} above does not depend on ε, it now follows that

$$\phi(\alpha + iy) = o(y^{s-1}) \qquad \text{as } y \to 0. \tag{2.92}$$

We will lead this to a contradiction.

We take $T \in \Gamma$ such that $\alpha = T(\eta_k)$, and write $N_k T^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We then have $N_k T^{-1}(\alpha) = N_k(\eta_k) = \infty$, and thus $c\alpha + d = 0$. Hence $c \neq 0$, and, for all $y > 0$,

$$\text{Im } N_k T^{-1}(\alpha + iy) = \frac{y}{|c(\alpha + iy) + d|^2} = \frac{y}{(cy)^2} = |c|^{-2} y^{-1}.$$

Using (2.78) and Γ-invariance, we now get:

$$\phi(\alpha + iy) = \phi(T^{-1}(\alpha + iy)) \sim c_0^{(k)} \big(\text{Im } N_k T^{-1}(\alpha + iy)\big)^{1-s} = c_0^{(k)} |c|^{2(s-1)} y^{s-1}$$

as $y \to 0$. This is a contradiction against (2.92). □

We will now use Proposition 2.4.2 to extend Proposition 2.3.6 to the case $\phi(z) \in A_1$, $\gamma \in (0, \frac{1}{2})$.

Proposition 2.4.4. *Let $\gamma \in (0, \frac{1}{2})$ and $\phi(z) \in A_1(\lambda)$, with $\lambda = s(1-s)$, $s \in (\frac{1}{2}, 1)$. We then have*

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi(x + iy)\,dx \to 0 \tag{2.93}$$

uniformly as $y \to 0$ so long as $\beta - \alpha$ remains bigger than y^γ.

Proof. Assuming $\phi(z)$ to have Fourier expansion as in (2.72), the left hand side in (2.93) equals

$$c_0 y^{1-s} + \frac{1}{\beta - \alpha} \sum_{n \neq 0} c_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y) \frac{e(n\beta) - e(n\alpha)}{2\pi i n}. \tag{2.94}$$

To treat the sum in (2.94), we split it into two parts exactly as in the proof of Proposition 2.3.6, and then apply Lemma 2.3.7, with

$$F(x) = K_{s-\frac{1}{2}}(2\pi x); \qquad a = 2\pi; \qquad h = s - \tfrac{1}{2}; \qquad b = \tfrac{3}{2} - s.$$

Using Proposition 2.4.2 and [W, pp. 77(2), 78(6), 79(2), 202(1)] one then checks that all assumptions in Lemma 2.3.7 are fulfilled.

The conclusion from Lemma 2.3.7 is that for $y \leqq 1$ and $\beta - \alpha \geqq y$, the left hand side in (2.93) (cf. (2.94)) is:

$$O\left(y^{1-s}\right) + O\left(y^{1-s}(\beta - \alpha)^{2s-2}\right) \tag{2.95}$$

Since $2s - 2 < 0$ and $\beta - \alpha \geqq y^\gamma$, we get:

$$O\left(y^{1-s}\right) + O\left(y^{(1-2\gamma)(1-s)}\right).$$

Since $\gamma < \frac{1}{2}$, the exponents $1 - s$ and $(1 - 2\gamma)(1 - s)$ are *positive.* Hence the expression tends to 0 when $y \to 0$. □

Using Proposition 2.4.4, one can immediately extend the proof of Theorem 2.3.1 on pp. 159–161 so as to also cover the case $A_1 \neq \{0\}$. Hence Theorem 2.4.1 is now proved. □ □ □

3. EXPLICIT BOUNDS ON THE ERROR TERM

The goal in this section is to give explicit information on *the rate of convergence* in the relation

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy)\,dx \to \frac{1}{\mu(\mathcal{F})}\int_{\mathcal{F}} f(z)d\mu(z), \qquad \text{when } y\to 0. \tag{3.1}$$

(Cf. (2.38) in Theorem 2.3.1.) We will show that for any given $\gamma < \frac{1}{2}$, and any function $f \in C^2(\mathcal{H}) \cap L_2(\Gamma \setminus \mathcal{H})$ satisfying $Df \in L_2(\Gamma \setminus \mathcal{H})$, the relevant difference in (3.1) goes to zero *like a small power of* y so long as $\beta-\alpha$ remains bigger than y^γ. In fact, we will prove a fairly more detailed version of this result, cf. Theorem 3.2.5 below; in particular our bound will also be uniform in f.

3.1. **Two lemmas on the K-Bessel function.**

Lemma 3.1.1. *Let $R \geqq 10$ and $y > 0$. We then have:*

$$K_{iR}(y) = O\left(e^{-\frac{\pi}{2}R}R^{-\frac{1}{3}} \cdot \min\left(1, e^{\frac{\pi}{2}R-y}\right)\right).$$

(The implied constant is absolute.)

Proof. We first prove the bound

$$K_{iR}(y) = O\left(e^{-\frac{\pi}{2}R}R^{-\frac{1}{3}}\right). \tag{3.2}$$

Notice that if restrict to any *compact* interval $R \in [10, R_0]$, then (3.2) follows directly from [W, pp. 78(6), 77(2)] for $0 < y \leqq 1$, and from the integral representation of $K_{iR}(y)$ (cf. (3.4) below) for $y \geqq 1$. In the remaining case, i.e. "R large", (3.2) follows from the asymptotic formula for $K_{iR}(y)$ given in [B1, (2), with $m = 0$][6] since the Airy function $Ai(\xi)$ occurring therein is bounded over $\xi \in \mathbb{R}$, and $Ai(\xi) = O(|\xi|^{-\frac{1}{4}})$ as $\xi \to -\infty$ (cf., e.g., [O1, §2]).

It now remains to prove, for $R \geqq 10$ and $y \geqq \frac{\pi}{2}R$

$$K_{iR}(y) = O\left(e^{-\frac{\pi}{2}R}R^{-\frac{1}{3}} \cdot e^{\frac{\pi}{2}R-y}\right). \tag{3.3}$$

This is of course again a consequence of the asymptotic formula [B1, (2)]. However, we may also derive (3.3) directly from the integral representation

$$K_{iR}(y) = \int_0^\infty e^{-y\cosh t}\cos(Rt)\,dt. \tag{3.4}$$

(Cf. [W, p. 181(5)].) Since $\cosh t \geqq 1 + t^2/2$ for all $t \geqq 0$, we get from (3.4):

$$|K_{iR}(y)| \leqq \int_0^\infty e^{-y(1+t^2/2)}\,dt = \sqrt{\frac{\pi}{2}}\frac{e^{-y}}{\sqrt{y}}. \tag{3.5}$$

Since $R \geqq 10$ and $y \geqq \frac{\pi}{2}R$, this implies (3.3). □

---

[6] Cf. also [B2, (20)] and [E, pp. 87-88]. (Note that [E, p. 88(20)] contains a misprint; there is one "π" too much.)

Lemma 3.1.2. *There exists a positive (absolute) constant c such that, for all* $R \geqq 10$,

$$\int_{R/4}^{R/2} K_{iR}(y)^2\,dy \geqq c\,e^{-\pi R}. \tag{3.6}$$

Proof. We have, uniformly for $R \geqq 10$, $\frac{R}{4} \leqq y \leqq \frac{R}{2}$,

$$K_{iR}(y)^2 = 2\pi e^{-\pi R}(R^2-y^2)^{-\frac{1}{2}}\left[\sin^2\left(R\,\theta(\tfrac{y}{R})+\tfrac{\pi}{4}\right)+O\left(R^{-1}\right)\right],$$

where

$$\theta(x) = \operatorname{arccosh}\left(\tfrac{1}{x}\right)-\sqrt{1-x^2} \qquad (0<x<1).$$

Cf. [E, p. 88(19)] and [B1, B2].

Hence, for all $R \geqq 10$:

$$\int_{R/4}^{R/2} K_{iR}(y)^2\,dy \geqq \frac{2\pi e^{-\pi R}}{R}\int_{R/4}^{R/2}\sin^2\left(R\,\theta(\tfrac{y}{R})+\tfrac{\pi}{4}\right)\,dy - O\left(e^{-\pi R}R^{-1}\right).$$

Now note that $\theta'(x) = -\sqrt{1-x^2}/x < 0$, hence $\theta(x)$ is a decreasing function for $0<x<1$. We write $\omega = R\,\theta(\frac{y}{R})+\frac{\pi}{4}$; then the integral in the last expression equals

$$\int_\beta^\alpha \sin^2(\omega)\frac{d\omega}{|\theta'(y/R)|}, \quad \text{where} \quad \alpha = R\,\theta(\tfrac{1}{4})+\tfrac{\pi}{4}, \quad \beta = R\,\theta(\tfrac{1}{2})+\tfrac{\pi}{4}.$$

But $|\theta'(x)|^{-1} = x/\sqrt{1-x^2}$ is an increasing function for $0<x<1$, hence the above integral is

$$\begin{aligned}&\geqq |\theta'\left(\tfrac{1}{4}\right)|^{-1}\int_\beta^\alpha \sin^2(\omega)\,d\omega = \frac{1}{\sqrt{15}}\left[\frac{\omega}{2}-\frac{\sin(2\omega)}{4}\right]_{\omega=\beta}^{\omega=\alpha}\\ &\geqq \frac{1}{\sqrt{15}}\left(\frac{\alpha-\beta}{2}-\frac{1}{2}\right) = \frac{R}{2\sqrt{15}}\left(\theta\left(\tfrac{1}{4}\right)-\theta\left(\tfrac{1}{2}\right)\right)-\frac{1}{2\sqrt{15}}.\end{aligned}$$

This is $\geqq c_1 R$, for some (absolute) positive constant c_1, and for all sufficiently large R. Hence for these R,

$$\int_{R/4}^{R/2} K_{iR}(y)^2\,dy \geqq 2\pi\,c_1\,e^{-\pi R} - O\left(e^{-\pi R}R^{-1}\right).$$

It follows that (3.6) holds for all sufficiently large R. By a simple continuity argument it is then possible to take c so small that (3.6) in fact holds for all $R \geqq 10$. □

3.2. **Uniform spectral bounds.** To start with, we will prove R-uniform versions of the results on the Eisenstein series from Section 2, i.e. Lemma 2.3.8 and Proposition 2.3.10.

Working through the proof of Lemma 2.3.8, the upper bound on $\sum |c_n|^2$ obtained therein is seen to depend on the following integral:

$$\mathfrak{I}_k(R,B) = \int_{\mathcal{F}_{\max(B_0,B)}} \left|E_k\left(z,\tfrac{1}{2}+iR\right)\right|^2\,d\mu(z). \tag{3.7}$$

(We use "$\mathcal{F}_{\max(B_0,B)}$" instead of just "$\mathcal{F}_B$" to make $\mathfrak{I}_k(R,B)$ well-defined for all $B>0$.)

It seems hard to give a good bound on $\mathfrak{I}_k(R,B)$ directly in terms of R and B. However, once $\mathfrak{I}_k(R,B)$ is *averaged over* R, Bessel's inequality (cf. (2.43), (2.44), and Lemma 2.3.2) can be used to obtain nice inequalities.

Lemma 3.2.1. *Let* $k \in \{1, ..., \kappa\}$ *and keep* $R \geqq 10$. *Consider the Fourier expansion of the Eisenstein series* $E_k(z, \frac{1}{2} + iR)$ *at the cusp* $\eta_1 = \infty$*:*

$$E_k\left(z, \tfrac{1}{2} + iR\right) = \delta_{k1} y^{\frac{1}{2}+iR} + \varphi_{k1}\left(\tfrac{1}{2} + iR\right) y^{\frac{1}{2}-iR} + \sum_{n\neq 0} c_n \sqrt{y} K_{iR}\left(2\pi|n|y\right) e(nx). \tag{3.8}$$

We then have, uniformly over $N \geqq 1$ *and* $R \geqq 10$*:*

$$\sum_{1 \leqq |n| \leqq N} |c_n|^2 = O\Big(e^{\pi R}(N + R)\Big)\, \mathfrak{I}_k\big(R, \max[\tfrac{100N}{R}, R]\big).$$

(The implied constant depends only on Γ*.)*

Proof. We follow the proof of Lemma 2.3.8. Defining the integral J as in that proof, and taking $Y = \frac{R}{8\pi N}$, $H = \frac{R}{4\pi}$, we obtain (cf. (2.57), (2.56) and (3.7)):

$$\begin{aligned} J &= O\left(1 + N/R\right) \mathfrak{I}_k\big(R, \max\left[\tfrac{1}{Y}, H\right]\big) \\ &= O\left(1 + N/R\right) \mathfrak{I}_k\big(R, \max\left[\tfrac{100N}{R}, R\right]\big). \end{aligned} \tag{3.9}$$

(We used the fact that $\mathfrak{I}_k(R, B)$ is an increasing function of B to clean up the expression slightly.)

On the other hand, by imitating the derivation of (2.59) we get:

$$\sum_{1 \leqq |n| \leqq N} |c_n|^2 \leqq C^{-1} J, \quad \text{where} \quad C = \int_{R/4}^{R/2} K_{iR}(y)^2 \, \frac{dy}{y}.$$

It follows from Lemma 3.1.2 that $C^{-1} = O\left(Re^{\pi R}\right)$. Using this and (3.9), we obtain the desired result. □

As in Section 2, we are going to treat the horocycle integral $\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy)\,dx$ by using the spectral expansion of $f(z)$, cf. (2.40) on p. 151. For the *Eisenstein* part, we will then apply Cauchy's inequality and (2.41). This will lead us to the integral appearing in the following lemma.

Lemma 3.2.2. *Let* $\varepsilon > 0$ *and* $k \in \{1, ..., \kappa\}$*, and keep* $y \in (0, 1)$*,* $\alpha < \beta$*. We then have:*

$$\left\{\int_{10}^{\infty} \left|\frac{1}{\beta - \alpha}\int_\alpha^\beta E_k\left(x + iy, \tfrac{1}{2} + iR\right)\, dx\right|^2 R^{-4} dR\right\}^{1/2} = O\left(\sqrt{y} + y^{\frac{1}{2}-\varepsilon}(\beta - \alpha)^{-1}\right). \tag{3.10}$$

The implied constant depends only on Γ *and* ε.

Proof. We fix k, ε, and some numbers y, α, β as in the statement of the lemma. We agree that in all "big O" estimates in the following proof, the implied constants depend solely on Γ and ε.

For any $R \geqq 10$, assuming the Fourier expansion of $E_k\left(z, \frac{1}{2} + iR\right)$ to be as in (3.8), we have

$$\frac{1}{\beta - \alpha}\int_\alpha^\beta E_k\left(x + iy, \tfrac{1}{2} + iR\right)\, dx = \delta_{k1} y^{\frac{1}{2}+iR} + \varphi_{k1}\left(\tfrac{1}{2} + iR\right) y^{\frac{1}{2}-iR} + \frac{1}{\beta - \alpha}\sum_{n\neq 0} c_n\sqrt{y} K_{iR}\left(2\pi|n|y\right) \frac{e(n\beta) - e(n\alpha)}{2\pi i n}. \tag{3.11}$$

We know that $\left|\varphi_{k1}\left(\frac{1}{2}+iR\right)\right| \leqq 1$ (cf. [H3, p. 281 def. 3.8, and p. 297(C)]), hence the first two terms in the right hand side of (3.11) are bounded by $O(\sqrt{y})$.

Using Lemma 3.1.1 and $|e(n\beta)-e(n\alpha)| \leqq 2$, we see that the second line in (3.11) is bounded by

$$O\left(e^{-\frac{\pi}{2}R}R^{-\frac{1}{3}}\frac{\sqrt{y}}{\beta-\alpha}\right)\sum_{n=1}^{\infty}\left(|c_n|+|c_{-n}|\right)\cdot a(n),$$

where $$a(n) := \begin{cases} n^{-1} & \text{if } n \leqq R/4y \\ n^{-1}e^{\frac{\pi}{2}R-2\pi ny} & \text{if } n > R/4y \end{cases}.$$

Notice that $a(n)$ is a positive, decreasing sequence (for our fixed y, R), and that $a(n) \to 0$ when $n \to \infty$. It follows that $a(n) = \sum_{m=n}^{\infty}\left(a(m)-a(m+1)\right)$, and that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(|c_n|+|c_{-n}|\right)\cdot a(n) &= \sum_{n=1}^{\infty}\left(|c_n|+|c_{-n}|\right)\cdot\sum_{m=n}^{\infty}\left(a(m)-a(m+1)\right)\\ &= \sum_{n=1}^{\infty}\left(a(n)-a(n+1)\right)\sum_{m=1}^{n}\left(|c_m|+|c_{-m}|\right).\end{aligned}$$

(These manipulations are permitted since all terms in the double sum are positive.)

It follows from Lemma 3.2.1 and Cauchy's inequality that for all $n \geqq 1$:

$$\sum_{m=1}^{n}\left(|c_m|+|c_{-m}|\right) = O\left(e^{\frac{\pi}{2}R}\sqrt{n}\sqrt{n+R}\sqrt{\mathfrak{I}_k(R,\max[\tfrac{100n}{R},R])}\right).$$

Furthermore, we have for all $n \geqq 1$:

$$a(n)-a(n+1) = O(1)\begin{cases} n^{-2} & \text{if } n+1 \leqq R/4y \\ n^{-1}ye^{\frac{\pi}{2}R-2\pi yn} & \text{if } n+1 > R/4y \end{cases}. \tag{3.12}$$

Indeed, if $n+1 \leqq R/4y$ then $a(n) = n^{-1}$ and $a(n+1) = (n+1)^{-1}$, and (3.12) follows immediately. Now assume $n+1 > R/4y$. Defining

$$f(x) = x^{-1}e^{\frac{\pi}{2}R-2\pi yx},$$

we then have $a(n) \leqq f(n)$ and $a(n+1) = f(n+1)$. Notice that

$$f'(x) = -x^{-1}\left(x^{-1}+2\pi y\right)e^{\frac{\pi}{2}R-2\pi yx}, \tag{3.13}$$

which is negative and increasing for $x > 0$. Hence:

$$a(n)-a(n+1) \leqq f(n)-f(n+1) \leqq |f'(n)|$$

and (3.12) follows from (3.13), since $n^{-1} \leqq 2(n+1)^{-1} < 8y/R < y$.

Collecting our results (and noticing that $\sqrt{n+R} = O(\sqrt{n})$ in the case $n+1 > R/4y$), we obtain:

$$\begin{aligned}\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}E_k\left(x+iy,\tfrac{1}{2}+iR\right)\,dx = O\big(\sqrt{y}\big)+O\Bigg(R^{-\frac{1}{3}}\frac{\sqrt{y}}{\beta-\alpha}\\ \times\sum_{n=1}^{\infty}\left\{\begin{array}{ll} n^{-\frac{3}{2}}\sqrt{n+R} & \text{if } n+1 \leqq R/4y \\ ye^{\frac{\pi}{2}R-2\pi yn} & \text{if } n+1 > R/4y \end{array}\right\}\sqrt{\mathfrak{I}_k(R,\max[\tfrac{100n}{R},R])}\Bigg).\end{aligned} \tag{3.14}$$

This holds uniformly for all $R \geqq 10$, $y \in (0,1)$ and $\alpha < \beta$. We write $M(n,y,R)$ for the expression within the braces above, i.e. $M(n,y,R) = n^{-\frac{3}{2}}\sqrt{n+R}$ if $n+1 \leqq R/4y$, otherwise $M(n,y,R) = ye^{\frac{\pi}{2}R-2\pi yn}$.

Using the L_2 triangle inequality[7], we get:

$$\left\{\int_{10}^{\infty}\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}E_k\left(x+iy,\tfrac{1}{2}+iR\right)\,dx\right|^2 R^{-4}dR\right\}^{1/2} = O\big(\sqrt{y}\big) \tag{3.15}$$
$$+\,O\left(\frac{\sqrt{y}}{\beta-\alpha}\right)\sum_{n=1}^{\infty}\left\{\int_{10}^{\infty}R^{-\frac{14}{3}}M(n,y,R)^2\,\mathfrak{I}_k(R,\max[\tfrac{100n}{R},R])\,dR\right\}^{1/2}.$$

Now, for each $n \geqq 1$, we have to estimate the integral

$$\int_{10}^{\infty}R^{-\frac{14}{3}}M(n,y,R)^2\,\mathfrak{I}_k(R,\max[\tfrac{100n}{R},R])\,dR. \tag{3.16}$$

We will split (3.16) into two parts, corresponding to $10 \leqq R \leqq 100n$ and $R \geqq 100n$, respectively. In the *first* part, we overestimate the integrand by replacing $\max[\frac{100n}{R},R]$ with $100n$, and then use (3.7) and Fubini's theorem. This gives:

$$\int_{10}^{100n}R^{-\frac{14}{3}}M(n,y,R)^2\,\mathfrak{I}_k(R,\max[\tfrac{100n}{R},R])\,dR \tag{3.17}$$
$$= O(1)\int_{\mathcal{F}_{\max(B_0,100n)}}\int_{10}^{100n}R^{-\frac{14}{3}}M(n,y,R)^2\big|E_k\big(w,\tfrac{1}{2}+iR\big)\big|^2\,dR\,d\mu(w).$$

Notice that for $R \leqq 100n$ we have:

$$M(n,y,R)^2 = O(1)\begin{cases} n^{-2} & \text{if } R \geqq 4y(n+1) \\ y^2\,e^{\pi R-4\pi yn} & \text{if } R < 4y(n+1)\end{cases}.$$

Case I: $10 < 4y(n+1)$. Then the inner integral in the second line of (3.17) is

$$O(1)\int_{10}^{4y(n+1)}R^{-\frac{14}{3}}y^2\,e^{\pi R-4\pi yn}\left|E_k\left(w,\tfrac{1}{2}+iR\right)\right|^2\,dR \tag{3.18}$$
$$+\,O(1)\int_{4y(n+1)}^{\infty}R^{-\frac{14}{3}}n^{-2}\left|E_k\left(w,\tfrac{1}{2}+iR\right)\right|^2\,dR.$$

One easily checks that the function $R^{-\frac{14}{3}}e^{\pi R}$ is increasing for $R \geqq 10$. Hence $R^{-\frac{14}{3}}e^{\pi R-4\pi yn} = O\left((yn)^{-\frac{14}{3}}\right)$ anytime $10 \leqq R \leqq 4y(n+1)$ (for recall that $y \in (0,1)$). Using this fact together with Lemma 2.3.2 and Bessel's inequality (cf. (2.43), (2.44)), we see that (3.18) equals

$$O\left((yn)^{-\frac{14}{3}}y^2\right)\int_{10}^{4y(n+1)}\left|E_k\left(w,\tfrac{1}{2}+iR\right)\right|^2\,dR+O\left(n^{-2}(yn)^{-\frac{8}{3}}\mathcal{Y}_\Gamma(w)\right)$$
$$= O\left(y^{-\frac{2}{3}}n^{-\frac{8}{3}}\mathcal{Y}_\Gamma(w)\right)+O\left(y^{-\frac{8}{3}}n^{-\frac{14}{3}}\mathcal{Y}_\Gamma(w)\right) = O\left(y^{-\frac{2}{3}}n^{-\frac{8}{3}}\mathcal{Y}_\Gamma(w)\right).$$

(We used the fact that $10 < 4y(n+1) \leqq 8yn$.) Hence by Lemma 2.2.4, the whole expression in (3.17) is bounded by

$$O\left(y^{-\frac{2}{3}}n^{-\frac{8}{3}}\log(2n)\right).$$

---

[7] In precise terms: $\left\{\int_{10}^{\infty}\left(\sum_{n=0}^{\infty}f_n(R)\right)^2 dR\right\}^{1/2} \leqq \sum_{n=0}^{\infty}\left\{\int_{10}^{\infty}f_n(R)^2\,dR\right\}^{1/2}$, for arbitrary measurable non-negative functions $f_0(R), f_1(R), f_2(R), \ldots$ Cf. e.g. [R5, Thm. 1.26, Thm. 3.5(2)]. Notice that we do not exclude here the possibility that the right hand side or both sides in the inequality are equal to ∞.

Case II: $10 \geqq 4y(n+1)$. Then the inner integral in the second line of (3.17) is seen to be $O(n^{-2}\mathcal{Y}_\Gamma(w))$ by Lemma 2.3.2, and hence the whole expression in (3.17) is

$$O\left(n^{-2}\log(2n)\right).$$

We now treat the second part of (3.16), i.e. that part which corresponds to $R \geqq 100n$. For these R-values, we have $\max[B_0, \frac{100n}{R}, R] < B_0 R$ (since $B_0 > 1$) and $M(n,y,R)^2 = n^{-3}(n+R) = O(Rn^{-3})$ (since $R \geqq 100n \geqq 50(n+1) > 4y(n+1)$). Using these facts, and (3.7), we get:

$$\begin{aligned}
&\int_{100n}^{\infty} R^{-\frac{14}{3}} M(n,y,R)^2\, \mathfrak{I}_k(R, \max[\tfrac{100n}{R}, R])\, dR. \\
&= O(1) \int_{100n}^{\infty} R^{-\frac{11}{3}} n^{-3} \int_{\mathcal{F}_{B_0 R}} \left|E_k\left(w, \tfrac{1}{2} + iR\right)\right|^2 d\mu(w)\, dR \\
&\qquad\qquad \left\{\begin{array}{l}\text{By (2.8) in Lemma 2.1.2 we have, for all } w \in \\ \mathcal{F}: \\ \qquad\qquad w \in \mathcal{F}_{B_0 R} \Longleftrightarrow \mathcal{Y}_\Gamma(w) < B_0 R.\end{array}\right\} \\
&= O(n^{-3}) \int_{\mathcal{F}} \int_{\max\left[100n, \frac{\mathcal{Y}_\Gamma(w)}{B_0}\right]}^{\infty} R^{-\frac{11}{3}} \left|E_k\left(w, \tfrac{1}{2} + iR\right)\right|^2 dR\, d\mu(w) \\
&\qquad\qquad\qquad \Big\{\text{Use Lemma 2.3.2.}\Big\} \\
&= O(n^{-3}) \int_{\mathcal{F}} \max\left[100n, \tfrac{\mathcal{Y}_\Gamma(w)}{B_0}\right]^{-\frac{5}{3}} \mathcal{Y}_\Gamma(w)\, d\mu(w). \\
&\qquad\qquad \Big\{\text{Use } \max[a,b]^{-\frac{5}{3}} \leqq a^{-\frac{2}{3}} b^{-1}, \text{ true for all } a, b > 0.\Big\} \\
&= O(n^{-3}) \int_{\mathcal{F}} n^{-\frac{2}{3}}\, d\mu(w) = O\left(n^{-\frac{11}{3}}\right).
\end{aligned}$$

Notice that this is $\leqq O\left(n^{-2}\log(2n)\right)$ and $\leqq O\left(y^{-\frac{2}{3}} n^{-\frac{8}{3}} \log(2n)\right)$, since we keep $y \in (0,1)$. Hence, adding up the bounds that we have obtained, we see that the integral in (3.16) is bounded by:

$$\begin{cases} O\left(n^{-2}\log(2n)\right) & \text{if } n+1 \leqq 10/4y \\ O\left(y^{-\frac{2}{3}} n^{-\frac{8}{3}} \log(2n)\right) & \text{if } n+1 > 10/4y \end{cases}.$$

Note here that $(10/4y) - 1 > 1/y$, since $y \in (0,1)$. Hence the expression in (3.15) is

$$\begin{aligned}
&O(\sqrt{y}) + O\left(\frac{\sqrt{y}}{\beta - \alpha}\right) \left\{ \sum_{1 \leqq n \leqq 10/4y} \sqrt{n^{-2}\log 2n} + \sum_{n > 1/y} \sqrt{y^{-\frac{2}{3}} n^{-\frac{8}{3}} \log 2n} \right\} \\
&= O(\sqrt{y}) + O\left(\frac{\sqrt{y}}{\beta - \alpha}\right) \left\{ \sum_{1 \leqq n \leqq 10/4y} n^{\varepsilon - 1} + y^{-\frac{1}{3}} \sum_{n > 1/y} n^{\varepsilon - \frac{4}{3}} \right\} \\
&= O\left(\sqrt{y} + y^{\frac{1}{2} - \varepsilon} (\beta - \alpha)^{-1}\right).
\end{aligned}$$

This concludes the proof. □

Next, we will treat the horocycle integral for *cusp forms* $\phi(z)$. To save time and space, we will use the same method as for $E_k(z, \frac{1}{2} + iR)$, i.e., we will imitate the proofs of Lemma 3.2.1 and Lemma 3.2.2. In particular, our basic tool will be a Rankin-Selberg type bound on the Fourier coefficients of $\phi(z)$, and *not* a bound on linear sums as in Lemma 2.3.5. With this approach, we will obtain a result that is significantly weaker than Proposition 2.3.6 in one respect; our bound on the horocycle integral of $\phi(z)$ will not imply convergence for all exponents $\gamma < 1$, but only for $\gamma < \frac{1}{2}$.

The following lemma is a slightly weakened version of [I1, Thm. 3.2, the first bound][8]. The proof is very easy to write down, given our proofs of Lemma 3.2.1 and Lemma 2.3.8.

Lemma 3.2.3. *Let* $\phi(z)$ *be a cusp form on* $\Gamma \setminus \mathcal{H}$. *We assume* $\phi(z)$ *to be* L_2-*normalized (i.e.* $\int_{\mathcal{F}} |\phi(z)|^2 \, d\mu(z) = 1$*) and to have eigenvalue* $\lambda = \frac{1}{4} + R^2$, *with* $R \geqq 10$. *Let the Fourier expansion of* $\phi(z)$ *at the cusp* $\eta_1 = \infty$ *be:*

$$\phi(z) = \sum_{n \neq 0} c_n \sqrt{y} K_{iR}\left(2\pi |n| y\right) e(nx). \tag{3.19}$$

We then have, for all $N \geqq 1$*:*

$$\sum_{1 \leqq |n| \leqq N} |c_n|^2 = O\Big(e^{\pi R}(N + R)\Big), \tag{3.20}$$

where the implied constant depends only on Γ.

Proof. We imitate the proof of Lemma 3.2.1 (cf. also the proof of Lemma 2.3.8), replacing "$E_k(z, \frac{1}{2} + iR)$" with "$\phi(z)$" throughout. We again take $Y = \frac{R}{8\pi N}$, $H = \frac{R}{4\pi}$.

Since $\int_{\mathcal{F}} |\phi(z)|^2 \, d\mu(z) = 1$, the bound (3.9) can now be replaced with $J = O(1 + N/R)$. (Cf. (2.57), with $\phi(z)$ in place of $E_k(z, \frac{1}{2} + iR)$).[9]

The rest of the proof carries over with minimal changes. The conclusion is (3.20). □

Lemma 3.2.4. *Let* $\phi(z)$ *be a cusp form on* $\Gamma \setminus \mathcal{H}$. *We assume* $\phi(z)$ *to be* L_2-*normalized (i.e.* $\int_{\mathcal{F}} |\phi(z)|^2 \, d\mu(z) = 1$*) and to have eigenvalue* $\lambda = \frac{1}{4} + R^2$, *with* $R \geqq 10$. *Keep* $y \in (0, 1)$ *and* $\alpha < \beta$. *We then have:*

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \phi\left(x + iy\right) \, dx = O\left(\frac{\sqrt{y}\, R^{-1/3}}{\beta - \alpha}\left(\sqrt{R} + \log(y^{-1})\right)\right). \tag{3.21}$$

The implied constant depends only on Γ.

Proof. By imitating the proof of Lemma 3.2.2 up to (3.14), replacing "$E_k(z, \frac{1}{2} + iR)$" with "$\phi(z)$" throughout, and using Lemma 3.2.3 in place of Lemma 3.2.1, we obtain:

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \phi\left(x + iy\right) \, dx = O\left(\frac{\sqrt{y}\, R^{-1/3}}{\beta - \alpha} \sum_{n=1}^{\infty} M(n, y, R)\right). \tag{3.22}$$

---

[8] (We remark that some minor revisions are called for in the proof given in [I1]).

[9] This time, because of $\int_{\mathcal{F}} |\phi(z)|^2 \, d\mu(z) = 1$, there is actually no need to introduce $\mathcal{F}_B$ on pp. 157–158, i.e. we may work with $\mathcal{F}$ throughout.

Here $M(n,y,R)$ is the same as in the proof of Lemma 3.2.2 (as defined just below (3.14)). We have:

$$
\begin{aligned}
\sum_{n=1}^{\infty} M(n,y,R) &= \sum_{n \leqq (R/4y)-1} n^{-\frac{3}{2}} \sqrt{n+R} + \sum_{n>(R/4y)-1} y e^{\frac{\pi}{2}R - 2\pi y n} \\
&< \sum_{n \leqq R/4y} n^{-\frac{3}{2}} (\sqrt{n} + \sqrt{R}) + y \sum_{n=-1}^{\infty} e^{-2\pi y n} \\
&= O\left(\log\left(\frac{R}{y}\right) + \sqrt{R} + \frac{y e^{2\pi y}}{1 - e^{-2\pi y}} \right) \\
&= O\Big(\log R + \log(y^{-1}) + \sqrt{R} + 1 \Big) = O\Big(\sqrt{R} + \log(y^{-1}) \Big). \qquad (3.23)
\end{aligned}
$$

Clearly, (3.21) follows from (3.22) and (3.23). □

We are now ready to prove the main result of this section. For any $f \in L_2(\Gamma\backslash\mathcal{H})$, we let $||f||$ denote the L_2-norm; $||f|| = \sqrt{\int_{\mathcal{F}} |f(z)|^2 \, d\mu(z)}$.

Theorem 3.2.5. *Let Γ be a cofinite Fuchsian group with a normalized cusp (see (2.19)), and let $\varepsilon > 0$. We then have, for all $f \in C^2(\mathcal{H}) \cap L_2(\Gamma \setminus \mathcal{H})$ such that $Df \in L_2(\Gamma \setminus \mathcal{H})$, and all $y \in (0,1)$, α, β such that $\sqrt{y} \leqq \beta - \alpha \leqq 1$:*

$$
\begin{aligned}
\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} f(x+iy)\, dx &= \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} f(z) d\mu(z) \\
&\quad + O\Big((||f|| + ||Df||)\, y^{\frac{1}{2}-\varepsilon} (\beta-\alpha)^{-1} \Big) \\
&\quad + O\Big(||f||\, y^{1-s_1-\varepsilon} (\beta-\alpha)^{s_1-1} \Big) \\
&\quad + O\Big(||f||\, y^{1-s_1'} (\beta-\alpha)^{2(s_1'-1)} \Big).
\end{aligned} \tag{3.24}
$$

(In each "big O", the implied constant depends solely on Γ and ε.) Here $s_1 \in (\frac{1}{2}, 1)$ is the largest number such that there is a cusp form on $\Gamma \setminus \mathcal{H}$ of eigenvalue $\lambda = s_1(1-s_1)$, and $s_1' \in (\frac{1}{2}, 1)$ is the largest number such that there is an A_1-eigenfunction on $\Gamma \setminus \mathcal{H}$ of eigenvalue $\lambda = s_1'(1-s_1')$. If there are no such cusp forms or A_1-functions, it is understood that the corresponding error term in (3.24) *is omitted.*

Proof. Recall the spectral expansion (2.40). Because of the uniform convergence on compactas in this expansion, we have

$$
\begin{aligned}
\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} f(x+iy)\, dx &= \sum_{m \geqq 0} d_m \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \phi_m(x+iy)\, dx \\
&\quad + \sum_{k=1}^{\kappa} \int_0^{\infty} g_k(R) \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} E_k\left(x+iy, \tfrac{1}{2}+iR\right) dx\, dR.
\end{aligned} \tag{3.25}
$$

We will split the right hand side of (3.25) into $4 + 2\kappa$ parts, and treat each of these separately. Specifically, for f, y, α, β satisfying the assumptions stated in the theorem, we will prove the following:

$$
d_0 \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \phi_0(x+iy)\, dx = \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} f(z) d\mu(z); \tag{3.26}
$$

$$\sum_{0<\lambda_m<\frac{1}{4}} d_m \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi_m(x+iy)\,dx = O\Big(||f||\,\Big[y^{1-s_1-\varepsilon}(\beta-\alpha)^{s_1-1}+y^{1-s_1'}(\beta-\alpha)^{2(s_1'-1)}\Big]\Big); \tag{3.27}$$

$$\sum_{0\leqq r_m<10} d_m \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi_m(x+iy)\,dx = O\Big(||f||\,y^{\frac{1}{2}-\varepsilon}(\beta-\alpha)^{-\frac{1}{2}}\Big); \tag{3.28}$$

$$\sum_{10\leqq r_m} d_m \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi_m(x+iy)\,dx = O\Big((||f||+||Df||)\,y^{\frac{1}{2}-\varepsilon}(\beta-\alpha)^{-1}\Big); \tag{3.29}$$

$$\int_0^{10} g_k(R)\frac{1}{\beta-\alpha}\int_\alpha^\beta E_k\left(x+iy,\tfrac{1}{2}+iR\right)\,dx\,dR = O\Big(||f||\,y^{\frac{1}{2}-\varepsilon}(\beta-\alpha)^{-1}\Big); \tag{3.30}$$

$$\int_{10}^{\infty} g_k(R)\frac{1}{\beta-\alpha}\int_\alpha^\beta E_k\left(x+iy,\tfrac{1}{2}+iR\right)\,dx\,dR = O\Big((||f||+||Df||)\,y^{\frac{1}{2}-\varepsilon}(\beta-\alpha)^{-1}\Big). \tag{3.31}$$

(In (3.27), the "s_1-term" is to be omitted if there is no cusp form of eigenvalue $\lambda \in (0,\frac{1}{4})$, and similarly for the "s_1'-term". Also, (3.30) and (3.31) hold *for each* $k \in \{1,...,\kappa\}$.)

Clearly, the proof of Theorem 3.2.5 will be complete once we manage to prove (3.26) – (3.31).

Proof of (3.26)**.** This is a direct consequence of $\phi_0(z) \equiv 1/\sqrt{\mu(\mathcal{F})}$ and $d_0 = \langle f,\phi_0\rangle$. □

Proof of (3.27)**.** Consider any fixed eigenfunction $\phi_m(z)$ with $0<\lambda_m<\frac{1}{4}$, and write $\lambda_m = s(1-s)$, $\frac{1}{2}<s<1$. Reviewing the proofs of Proposition 2.3.6 and Proposition 2.4.4 (cf. in particular (2.52) and (2.95)), we see that

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi_m(x+iy)\,dx = \begin{cases} O\Big(y^{1-s-\varepsilon}(\beta-\alpha)^{s-1}\Big) & \text{if } \phi_m \in A_0 \\ O\Big(y^{1-s}(\beta-\alpha)^{2(s-1)}\Big) & \text{if } \phi_m \in A_1 \end{cases}.$$

whenever $y\in(0,1)$ and $\sqrt{y}\leqq\beta-\alpha\leqq 1$.

Notice that $|d_m| = |\langle f,\phi_m\rangle| \leqq ||f||\cdot||\phi_m|| = ||f||$, for all $m \geqq 0$. Now (3.27) follows from the fact that that there are only finitely many eigenfunctions ϕ_m with $0<\lambda_m<\frac{1}{4}$. (To see which m-values give the "worst" contributions, we simply note that both $y^{1-s-\varepsilon}(\beta-\alpha)^{s-1}$ and $y^{1-s}(\beta-\alpha)^{2(s-1)}$ are nondecreasing functions of $s\in(\frac{1}{2},1)$ for any fixed y,α,β satisfying $\sqrt{y}\leqq\beta-\alpha\leqq 1$.) □

Proof of (3.28)**.** Consider any fixed eigenfunction $\phi_m(z)$ with $0\leqq r_m<10$. Reviewing the proof of Proposition 2.3.6 (cf. in particular (2.52)), we see that

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi_m(x+iy)\,dx = O\Big(y^{\frac{1}{2}-\varepsilon}(\beta-\alpha)^{-\frac{1}{2}}\Big),$$

whenever $y \in (0,1)$ and $\sqrt{y} \leqq \beta - \alpha \leqq 1$.[10]

Now (3.28) follows from the inequality $|d_m| \leqq ||f||$ and the fact that there are only finitely many eigenfunctions ϕ_m with $0 \leqq r_m < 10$. □

Proof of (3.29)**.** By Lemma 3.2.4, the left hand side in (3.29) is

$$O\left(\frac{\sqrt{y}}{\beta-\alpha}\right) \sum_{10 \leqq r_m} |d_m| r_m^{-1/3} \left(\sqrt{r_m} + \log(y^{-1})\right).$$

Using Cauchy's inequality, we see that this is

$$O\left(\frac{\sqrt{y}}{\beta-\alpha}\right) \sqrt{\sum_{10 \leqq r_m} |d_m|^2 r_m^4} \times \sqrt{\sum_{10 \leqq r_m} r_m^{-\frac{14}{3}} \left(\sqrt{r_m} + \log(y^{-1})\right)^2}.$$

But by (2.41) (with $a = 2$, say) we have

$$\sqrt{\sum_{10 \leqq r_m} |d_m|^2 r_m^4} = O\big(||f|| + ||Df||\big).$$

Also, Lemma 2.3.2 clearly implies that for $y \in (0,1)$,

$$\sqrt{\sum_{10 \leqq r_m} r_m^{-\frac{14}{3}} \left(\sqrt{r_m} + \log(y^{-1})\right)^2} = O\left(1 + \log(y^{-1})\right) = O\left(y^{-\varepsilon}\right).$$

Combining these facts, we obtain (3.29). □

Proof of (3.30)**.** By reviewing the proof of Lemma 2.3.10 (cf. in particular (2.62)), wherein we take $T = 10$ and $\varepsilon' = \min(\frac{1}{10}, \frac{\varepsilon}{2})$, we see that

$$\begin{aligned} &\frac{1}{\beta-\alpha} \int_\alpha^\beta E_k\left(x+iy, \tfrac{1}{2}+iR\right) dx \\ &= O\left(y^{\frac{1}{2}}\right) + O\left(y^{\frac{1}{2}-2\varepsilon'} (\beta-\alpha)^{\varepsilon'-1}\right) = O\left(y^{\frac{1}{2}-\varepsilon} (\beta-\alpha)^{-1}\right), \end{aligned}$$

uniformly over all $R \in [0,10]$, $y \in (0,1)$ and all $\alpha, \beta \in \mathbb{R}$ such that $\sqrt{y} \leqq \beta-\alpha \leqq 1$.

We also have, by Cauchy's inequality and [H3, pp. 91(9.36), 242(Prop. 2.3), 291(3.23), 317(line 4)],

$$\int_0^{10} |g_k(R)|\, dR \leqq \sqrt{10 \int_0^{10} |g_k(R)|^2\, dR} = O\big(||f||\big).$$

Combining these two facts, we obtain (3.30). □

---

[10] (In fact, we need only require $y \leqq 1$, $y \leqq \beta - \alpha$ here.)

Proof of (3.31)**.** By Cauchy's inequality and Lemma 3.2.2, we have

$$\int_{10}^{\infty} |g_k(R)| \left| \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} E_k\left(x+iy, \tfrac{1}{2}+iR\right)\, dx \right| dR$$
$$\leqq \left\{ \int_{10}^{\infty} |g_k(R)|^2 R^4 \, dR \right\}^{1/2}$$
$$\times \left\{ \int_{10}^{\infty} \left| \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} E_k\left(x+iy, \tfrac{1}{2}+iR\right)\, dx \right|^2 R^{-4} dR \right\}^{1/2}$$
$$\leqq O(1) \left\{ \int_{10}^{\infty} |g_k(R)|^2 R^4 \, dR \right\}^{1/2} \left(\sqrt{y} + y^{\frac{1}{2}-\varepsilon} (\beta-\alpha)^{-1} \right)$$

Here, the $\sqrt{y}$–term can be removed, since $y \in (0,1)$ and $\beta - \alpha \leqq 1$. Notice that by (2.41) (with $a = 2$, say),

$$\left\{ \int_{10}^{\infty} |g_k(R)|^2 R^4 \, dR \right\}^{1/2} = O\big(||f|| + ||Df|| \big).$$

Clearly, (3.31) follows from these facts. □

This concludes the proof of Theorem 3.2.5. □ □ □

4. GENERIC EQUIDISTRIBUTION FOR $\beta - \alpha \geqq y^{1-\varepsilon}$

It is easily seen that in Theorem 2.3.1 and Theorem 2.4.1, the restriction $\gamma < \frac{1}{2}$ is essentially the weakest possible (cf. [H6, p. 3(top)], or the discussion below).

However, Proposition 2.3.6 seems to indicate that *some* type of asymptotic equidistribution result should hold all the way up to $\gamma < 1$. In this section we will prove a theorem stating exactly this; see Theorem 4.5.1 below (and Definition 4.2.1, Proposition 4.2.2).

4.1. **A genericity assumption.**

In this subsection we give a discussion (partly informal) concerning the precise formulation of the new equidistribution result to be proved.

As always, we let Γ be a cofinite Fuchsian group having a normalized cusp, cf. (2.19). For simplicity, we will fix some $\gamma \in (\frac{1}{2}, 1)$, and then restrict attention to the horocycle segments

$$[\alpha, \alpha + y^{\gamma}] + iy, \qquad \text{with} \quad 0 < y < 1. \tag{4.1}$$

Any attempt to formulate an equidistribution result for these horocycle segments will now have to deal with the following obstacle.

Take $\eta \in \mathbb{R}$ to be a cusp ($\eta \sim \eta_k$, say) and fix any number $B > 2B_0$. It follows from (2.13) and (2.16) that, as $y \longrightarrow 0$, the whole segment (4.1) may be kept *inside* $\mathfrak{B}_{\eta}^{(B)}$. The pullback of this segment to $\mathcal{F}$ will then be confined to the cuspidal region $\mathcal{C}_{kB}$, and hence not even close to being equidistributed.

A very similar problem exists for arbitrarily small B. To see this, keep η fixed as above, fix *any* two numbers $0 < B < B'$, and consider the region $R = \mathfrak{B}_{\eta}^{(B)} - \mathfrak{B}_{\eta}^{(B')}$. Taking B'' to be any number larger than $\max(B_0, 1/B, B')$, we have, by the last statement in Lemma 2.1.3 (p. 142), $\mathfrak{B}_{\eta}^{(B)} \cap \mathfrak{B}_{\eta'}^{(B'')} = \emptyset$ for each cusp $\eta' \neq \eta$. Hence, by (2.15) applied with B'', the pullback of the whole region R to $\mathcal{F}$ is actually confined to $\mathcal{F}_{B''} = \mathcal{F} - \cup_{j=1}^{\kappa} \mathcal{C}_{jB''}$! Notice that it follows from (2.13) and (2.16) that, as $y \to 0$, the whole segment (4.1) may be kept *inside* R. Then, again, the horocycle segment does not become equidistributed over $\Gamma \setminus \mathcal{H}$ as $y \to 0$.

Notice that we cannot extend the validity of Theorem 2.3.1 by simply shrinking the class of test functions f in any reasonable way. In fact, choosing the segments (4.1) as described above, we see that for $\gamma \in (\frac{1}{2}, 1)$, the conclusion (2.38) of Theorem 2.3.1 is in general false even for C^∞-functions f with compact support on $\Gamma\backslash\mathcal{H}$.

Following a suggestion by Professor Hejhal, we will try to circumvent the obstacles just mentioned by imposing a certain *genericity assumption* on α.[11] Specifically, we will try to find, for each y, an explicit set $\mathcal{G}(y) \subseteq \mathbb{R}$ of "permissible" α, such that:

$$\text{(4.2a)} \qquad \begin{cases} \mathcal{G}(y) \text{ contains asymptotically } 100\% \text{ of all points in any fixed} \\ \text{interval, i.e. } \lim\limits_{y \to 0} m([\mu, \nu] \cap \mathcal{G}(y)) = \nu - \mu \text{ for all } \mu < \nu. \\ \text{(Here } m \text{ signifies ordinary Lebesgue measure.)} \end{cases}$$

and

$$\text{(4.2b)} \qquad \begin{cases} \text{If we keep } \alpha \in \mathcal{G}(y), \text{ then the horocycle segment } [\alpha, \alpha + y^\gamma] + iy \\ \text{necessarily becomes equidistributed over } \Gamma \setminus \mathcal{H}, \text{ as } y \to 0. \end{cases}$$

The question, now, is how to define the set $\mathcal{G}(y)$. In the following discussion, η is understood to denote any cusp $\in \mathbb{R}$, and B any number > 0. In order to get round the obstructions described above, it seems natural to require, for all choices of η, B, that $\alpha \in \mathcal{G}(y)$ *should imply* $\alpha + iy \notin \mathfrak{B}_\eta^{(B)}$ whenever y is sufficiently small — i.e. whenever $y \leqq y_0 = y_0(\eta, B)$, say. Our tentative definition then reads:

$$\text{(4.3)} \qquad \mathcal{G}(y) \overset{?}{:=} \left\{ \alpha \in \mathbb{R} \;\middle|\; \forall \eta, B : \, y \leqq y_0(\eta, B) \Longrightarrow \alpha + iy \notin \mathfrak{B}_\eta^{(B)} \right\}.$$

We must try to choose the function $y_0(\eta, B)$ in a way such that (4.2a) and (4.2b) are both fulfilled.

For each cusp $\eta \in \mathbb{R}$, let us define $c = c(\eta) > 0$ through the relations

$$\text{(4.4)} \qquad \eta = U(\eta_k), \quad \begin{pmatrix} * & * \\ c & * \end{pmatrix} = N_k U^{-1}, \quad \text{where } U \in \Gamma, \; k \in \{1, ..., \kappa\}.$$

Then by (2.13) and Remark 2.1.4, the Euclidean diameter of the horoball $\mathfrak{B}_\eta^{(B)}$ is $1/c^2B$. In particular, c is uniquely determined by η and Γ (of course, this is also trivial to verify directly). Using (2.4) with $j = 1$, $N_j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (cf. (2.19)), we see that we actually have $c \geqq 1$. We will let $y_0(\eta, B)$ depend on η only in terms of $c = c(\eta)$, i.e. *we will let* y_0 *be a function of* c *and* B.

We will now find several constraints that are reasonable to impose on this function y_0.

First, we *fix* some arbitrary number $B > 0$. Let $\eta \in \mathbb{R}$ be any cusp, and write $B' = \max(B, c^{-2} y_0^{-1})$ (where $c = c(\eta)$ and $y_0 = y_0(\eta, B)$). Then for each $y \in (0, 1)$, our proposed definition of $\mathcal{G}(y)$ forces

$$\{\alpha + iy \mid \alpha \in \mathcal{G}(y)\} \, \cap \, \mathfrak{B}_\eta^{(B')} \, = \, \emptyset.$$

(For notice that $\mathfrak{B}_\eta^{(B')} \subset \mathfrak{B}_\eta^{(B)}$ and $\mathfrak{B}_\eta^{(B')} \subset \mathbb{R} \times (0, y_0]$, cf. (2.16).) Now, *if* $c^{-2} y_0^{-1}$ would remain *bounded* as η varies through all cusps $\in \mathbb{R}$, then by Lemma 2.1.3 the pullback to $\mathcal{F}$ of each set $\{\alpha + iy \mid \alpha \in \mathcal{G}(y)\}$ $(0 < y < 1)$ would be disjoint from the following cuspidal region:

---

[11] Or on β; cf. Remark 4.5.4 below.

$$\bigcup_k \mathcal{C}_{kB''}, \qquad \text{with } B'' = \max(B_0, B, \sup_\eta c^{-2}y_0^{-1}).$$

This would make it impossible for (4.2a) to hold for $\mu = 0$, $\nu = 1$; cf., e.g., [H5, Cor. 2].

Hence, we have to make sure that $c^{-2}y_0^{-1}$ does not remain bounded as η varies. This makes it natural to require that, for each fixed $B > 0$,

$$y_0 = o(c^{-2}) \qquad \text{as } c \to \infty. \tag{4.5}$$

Next, assuming that our fixed number B is $> 2B_0$, say, we must make sure that no sequence of horocycle segments (4.1) with $y \to 0$, $\alpha \in \mathcal{G}(y)$ can have all their $\mathcal{F}$-pullbacks confined to the cuspidal region $\cup_k \mathcal{C}_{kB}$; otherwise (4.2b) would not hold. In other words (cf. Lemma 2.1.3), it must not be possible for any such sequence of horocycle segments to lie entirely inside $\cup_\eta \mathfrak{B}_\eta^{(B)}$. Following the implicit intention of our proposed definition of $\mathcal{G}(y)$, this leads us to require that for some (small) $y_1 > 0$, and for *all* η, y satisfying $y_0(\eta, B) < y \leqq y_1$, the segment (4.1) should be too long to fit inside the horoball $\mathfrak{B}_\eta^{(B)}$. Taking "$y = y_0(\eta, B) + 0$" and using (2.16) (and (4.5)), this forces

$$2\sqrt{\frac{y_0}{c^2B} - y_0^2} < y_0^\gamma,$$

for c large. This implies (using (4.5) again) that $\sqrt{y_0}/c = O(y_0^\gamma)$, i.e.,

$$y_0 \geqq [\text{const}] \cdot c^{-2/(2\gamma-1)} \qquad \text{as } c \to \infty. \tag{4.6}$$

This should (at least) hold for each fixed number $B > 2B_0$; the constant in the inequality may depend on B.

Finally, let us now take η to be fixed, and vary the number $B > 0$. Notice that by (2.13), the horoball $\mathfrak{B}_\eta^{(B)}$ grows larger and larger as $B \to 0$, and $\cup_{B>0} \mathfrak{B}_\eta^{(B)} = \mathcal{H}$. Hence we must have

$$y_0 \to 0 \qquad \text{as} \quad B \to 0, \tag{4.7}$$

for otherwise (4.3) would give $\mathcal{G}(y) = \emptyset$ for each sufficiently small y, and (4.2a) would not hold. Of course, (4.7) is to hold for each *fixed* cusp $\eta \in \mathbb{R}$.

A tentative, simple choice of the function y_0, fulfilling the above three constraints (4.5), (4.6) and (4.7), is as follows:

$$y_0 = \min(c^{-\delta_1}, B^{\delta_2}),$$

where δ_1 and δ_2 are some fixed exponents satisfying $2 < \delta_1 < 2/(2\gamma - 1)$ and $0 < \delta_2$.

With this choice, the proposed set $\mathcal{G}(y)$ (cf. (4.3)) can be given in a very explicit form. Namely, by (2.16), $\alpha \in \mathcal{G}(y)$ will now hold if and only if:

$$\forall \eta, B : \quad y \leqq \min(c^{-\delta_1}, B^{\delta_2}) \Longrightarrow \frac{y}{c^2B} - y^2 < |\alpha - \eta|^2,$$

i.e., if and only if

$$\forall \eta : \quad y \leqq c^{-\delta_1} \Longrightarrow \frac{y}{c^2 y^{1/\delta_2}} - y^2 < |\alpha - \eta|^2. \tag{4.8}$$

But $y \leqq c^{-\delta_1}$ implies

$$\frac{y}{c^2 y^{1/\delta_2}} \geqq y^{1+2/\delta_1 - 1/\delta_2},$$

and here the exponent $1 + 2/\delta_1 - 1/\delta_2$ is certainly < 2. Hence we do not change much by disregarding the negative y^2-term in (4.8). We write $\gamma_1 = \delta_1^{-1}$ and $\gamma_2 = \frac{1}{2}(1 - \delta_2^{-1})$ (then $\gamma - \frac{1}{2} < \gamma_1 < \frac{1}{2}$ and $\gamma_2 < \frac{1}{2}$), and recall that in (4.4), η runs through all cusps (with repetitions) as $\langle U, k\rangle$ runs through $\Gamma \times \{1, ..., \kappa\}$, and $\eta \in \mathbb{R}$ if and only if $c \neq 0$. We then get:

$$\mathcal{G}(y) = \Big\{\alpha \in \mathbb{R} \;\Big|\; \forall\, k \in \{1, ..., \kappa\},\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in N_k\Gamma : \quad 0 < |c| \leqq y^{-\gamma_1} \Longrightarrow \left|\alpha + \frac{d}{c}\right| > \frac{y^{\gamma_2}}{|c|}\Big\}.$$

With this definition of $\mathcal{G}(y)$, where γ_1, γ_2 are any numbers satisfying[12] $\gamma - \frac{1}{2} < \gamma_1 < \gamma_2 < \frac{1}{2}$, it turns out that *both (4.2a) and (4.2b) are indeed true!* The proof of this will occupy Sections 4.2–4.5. (We will actually obtain slightly stronger statements than (4.2a), (4.2b).)

4.2. **Some fundamental estimates.** In the discussion in last section, we arrived at the following definition (for completeness, we here include the case $\gamma = \frac{1}{2}$).

Definition 4.2.1. *Let Γ be a cofinite Fuchsian group with a normalized cusp (see (2.19)). Let a number $\gamma \in [\frac{1}{2}, 1)$ be given, and fix, once and for all, two numbers γ_1, γ_2 satisfying*

$$\gamma - \frac{1}{2} < \gamma_1 < \gamma_2 < \frac{1}{2}. \tag{4.9}$$

We then define, for all $y > 0$:

$$\mathcal{G}(y) = \Big\{\alpha \in \mathbb{R} \;\Big|\; \forall\, k \in \{1, ..., \kappa\},\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in N_k\Gamma : \quad 0 < |c| \leqq y^{-\gamma_1} \Longrightarrow \left|\alpha + \frac{d}{c}\right| > \frac{y^{\gamma_2}}{|c|}\Big\}. \tag{4.10}$$

It is now fairly easy to prove that (4.2a) holds:

Proposition 4.2.2. *We have, for each pair of real numbers $\mu < \nu$:*

$$\lim_{y \to 0} m([\mu, \nu] \cap \mathcal{G}(y)) = \nu - \mu.$$

Proof. Since $\frac{d}{c}$ and $|c|$ remain invariant when $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is multiplied from the left by any power of $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$\mathcal{G}(y) = \mathbb{R} - \bigcup_{k=1}^{\kappa} \bigcup_{W_0}^{(k)} \left[-\frac{d}{c} - \frac{y^{\gamma_2}}{|c|}, -\frac{d}{c} + \frac{y^{\gamma_2}}{|c|}\right],$$

where the inner union ("$\bigcup^{(k)}$") is taken over a set of representatives $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$ restricted by $0 < |c| \leqq y^{-\gamma_1}$. Hence:

$$[\mu, \nu] \cap \mathcal{G}(y) = [\mu, \nu] - \bigcup_{k=1}^{\kappa} \bigcup_{W_0}^{(k)} \left[-\frac{d}{c} - \frac{y^{\gamma_2}}{|c|}, -\frac{d}{c} + \frac{y^{\gamma_2}}{|c|}\right]. \tag{4.11}$$

---

[12] The "extra" restriction $\gamma_1 < \gamma_2$ is needed for the proof of (4.2a) to work. Cf. the proof of Proposition 4.2.2 below.

Let us now keep $y \in (0,1)$. Recall that for all W_0 occurring above we have $|c| \geqq 1$ (by (2.4), with $j = 1$, $N_j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$), and hence $y^{\gamma_2}/|c| < 1$. Hence we may restrict the inner union in (4.11) further by requiring $-\frac{d}{c} \in [\mu - 1, \nu + 1]$. It follows that

$$(4.12) \qquad (\nu - \mu) - \sum_{k=1}^{\kappa} \sum_{W_0}^{(k)} \frac{2y^{\gamma_2}}{|c|} \leqq m\big([\mu,\nu] \cap \mathcal{G}(y)\big) \leqq \nu - \mu,$$

where the inner sum ("$\sum^{(k)}$") is taken over a set of representatives $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$ restricted by $1 \leqq |c| \leqq y^{-\gamma_1}$ and $-\frac{d}{c} \in [\mu - 1, \nu + 1]$. Hence by Lemma 2.2.3,

$$m\big([\mu,\nu] \cap \mathcal{G}(y)\big) \geqq (\nu - \mu) - \kappa\, 2y^{\gamma_2}\big(2(\nu - \mu + 2)y^{-\gamma_1} + 1\big).$$

Here the right hand side tends to $\nu - \mu$ as $y \to 0$, since $0 < \gamma_1 < \gamma_2$ (cf. (4.9)). The proposition now follows. □

As a first step towards the proof of the equidistribution result (4.2b), we will now show that Proposition 2.2.5 can be extended to all $\gamma \in [\frac{1}{2}, 1)$, as long as we keep $\alpha \in \mathcal{G}(y)$.

Proposition 4.2.3. *Let* $\gamma \in [\frac{1}{2}, 1)$ *and let* γ_1, γ_2, $\mathcal{G}(y)$ *be given as in Definition 4.2.1. We then have*

$$(4.13) \qquad \frac{1}{\beta - \alpha} \int_\alpha^\beta \sqrt{\mathcal{Y}_\Gamma(x + iy)}\, dx = O(1),$$

for all $0 < y < 1$ *and all* $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$. *The implied constant depends only on* Γ *and* γ, γ_1, γ_2. *Furthermore, given any* $M \geqq 10$ *there is some* $y_0 = y_0(M, \gamma, \gamma_1, \gamma_2) > 0$ *such that*

$$(4.14) \qquad \frac{1}{\beta - \alpha} \int_\alpha^\beta \sqrt{\lfloor \mathcal{Y}_\Gamma(x + iy) \rfloor_M}\, dx \leqq \frac{40\kappa}{\sqrt{M}},$$

for all $0 < y < y_0$ *and all* $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$.

We will use the first part of the technical Lemma 2.2.7 in the proof, and to this we add the following complement (which will also be useful later):

Lemma 4.2.4. *Let* $\gamma, \gamma_1, \gamma_2, \mathcal{G}(y)$ *be given as above, and let* $M > 0$ *and* $k \in \{1, ..., \kappa\}$. *Then there is some* $y_1 = y_1(M, \gamma_2) \in (0, 1]$ *for which the following holds: For all* $y \in (0, y_1)$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$ *and* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in N_k\Gamma$ *satisfying*

$$(4.15) \qquad c \neq 0 \quad \textit{and} \quad -\frac{d}{c} \in \left[\alpha - \frac{\sqrt{y}}{\sqrt{M}|c|}, \beta + \frac{\sqrt{y}}{\sqrt{M}|c|}\right],$$

we have

$$(4.16) \qquad \frac{y^{\max(\gamma_2, \gamma - \gamma_1)}}{2(\beta - \alpha)} \leqq |c|,$$

and, in particular,

$$(4.17) \qquad \frac{\sqrt{y}}{\sqrt{M}|c|} \leqq \frac{2(\beta - \alpha)}{\sqrt{M}} y^{\frac{1}{2} - \max(\gamma_2, \gamma - \gamma_1)} < \frac{2(\beta - \alpha)}{\sqrt{M}}.$$

If $M \geqq 4$, *then we can actually take* $y_1 = 1$.

Proof. We choose $y_1 = y_1(M, \gamma_2)$ to be any number in $(0,1]$ such that

$$y^{\gamma_2} - \sqrt{y/M} \geqq y^{\gamma_2}/2 \qquad \text{for all } \ y \in (0, y_1). \tag{4.18}$$

This is possible since $\gamma_2 < \frac{1}{2}$. We note that in the case $M \geqq 4$ we can actually take $y_1 = 1$.

Now take any $y, \alpha, \beta, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as in the statement of the lemma. To prove (4.16) it suffices to prove *one* of $|c| \geqq \frac{y^{\gamma_2}}{2(\beta-\alpha)}$, $|c| \geqq \frac{y^{\gamma-\gamma_1}}{2(\beta-\alpha)}$, since $0 < y < y_1 \leqq 1$.

If $|c| \geqq y^{-\gamma_1}$, then (4.16) follows directly from $\beta - \alpha \geqq y^{\gamma}$:

$$|c| \geqq y^{-\gamma_1} \geqq \frac{y^{\gamma-\gamma_1}}{\beta - \alpha}.$$

Now assume $|c| < y^{-\gamma_1}$. In this case, it follows from $\alpha \in \mathcal{G}(y)$ that

$$\left|\alpha + \frac{d}{c}\right| > \frac{y^{\gamma_2}}{|c|}.$$

On the other hand, (4.15) implies

$$\left|\alpha + \frac{d}{c}\right| \leqq \frac{\sqrt{y}}{\sqrt{M}|c|} + (\beta - \alpha).$$

Combining these two facts, we get:

$$\begin{aligned} \frac{y^{\gamma_2}}{|c|} < \frac{\sqrt{y}}{\sqrt{M}|c|} + (\beta - \alpha) &\Longrightarrow \frac{y^{\gamma_2} - \sqrt{y/M}}{|c|} < \beta - \alpha \\ \{\text{use } (4.18)\} \Longrightarrow \frac{y^{\gamma_2}}{2|c|} < \beta - \alpha &\Longrightarrow |c| > \frac{y^{\gamma_2}}{2(\beta-\alpha)}, \end{aligned}$$

i.e. (4.16) holds in the present case as well.

Finally, using $y < y_1 \leqq 1$ and $\frac{1}{2} - \max(\gamma_2, \gamma - \gamma_1) > 0$ (cf. (4.9)), we immediately deduce (4.17) from (4.16). □

Proof of Proposition 4.2.3. The proof of Proposition 2.2.5 on pp. 149–150 goes over almost word by word up to and including (2.35), except that we add the extra restriction $\alpha \in \mathcal{G}(y)$, and replace the application of Lemma 2.2.7 by an application of *the first part* of Lemma 2.2.7 together with Lemma 4.2.4[13]; the inequalities (2.32) and (2.33) can then be replaced by

$$A \leqq |c| \leqq \frac{1}{\sqrt{My}}, \qquad \text{where } \ A = \max\left(1, \frac{y^{\max(\gamma_2, \gamma-\gamma_1)}}{2(\beta - \alpha)}\right), \tag{4.19}$$

and

$$-\frac{d}{c} \in [\mu, \nu], \qquad \text{where } \ \mu = \alpha - \frac{2(\beta-\alpha)}{\sqrt{M}}, \ \nu = \beta + \frac{2(\beta - \alpha)}{\sqrt{M}}. \tag{4.20}$$

Now if $A > 1/\sqrt{My}$ then the right hand side in (2.35) is 0 by (4.19); otherwise we obtain the following bound, by Lemma 2.2.3:

$$\leqq \frac{2\sqrt{y}}{\beta - \alpha}\left[\frac{2(\nu-\mu)}{\sqrt{My}} + A^{-1} + \frac{4(\nu - \mu)}{\sqrt{My}} + \frac{2}{A}\log\left(\frac{1}{A\sqrt{My}}\right)\right].$$

---

[13] Notice that we can take $y_1 = 1$ in Lemma 4.2.4, since $M \geqq 10 \geqq 4$.

From (4.20) and $M \geqq 10$ it follows that $\nu - \mu \leqq 3(\beta - \alpha)$. Using $\beta - \alpha \geqq y^\gamma$ and $A^{-1} \leqq 1,\ A^{-1} \leqq 2(\beta-\alpha)y^{-\max(\gamma_2,\gamma-\gamma_1)}$ (cf. (4.19)), we conclude:

$$\begin{aligned}(4.21)\qquad & \frac{1}{\beta-\alpha}\int_\alpha^\beta \sqrt{\lfloor \mathcal{Y}_\Gamma(x+iy)\rfloor_M}\, dx \\ & \leqq \frac{36\kappa}{\sqrt{M}} + 4\kappa y^{\frac{1}{2}-\max(\gamma_2,\gamma-\gamma_1)}\left[1 + 2\log^+\left(\frac{1}{\sqrt{My}}\right)\right].\end{aligned}$$

This holds for all $0 < y < 1$ and all $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$. Notice here that $\frac{1}{2} - \max(\gamma_2, \gamma - \gamma_1) > 0$. The proof can now be completed just as the proof of Proposition 2.2.5. □

4.3. **The incomplete Eisenstein series.** Up to now, we have been stuck at $\gamma < \frac{1}{2}$ when treating the horocycle integral of the Eisenstein series, cf. Proposition 2.3.10 (and Lemma 3.2.2). The same barrier also exists for the A_1-functions, cf. Proposition 2.4.4. We must now try to extend these results all the way up to $\gamma < 1$, for generic horocycle segments. As a tool to deal with this task, we will use the *incomplete Eisenstein series.*

In this section we recollect some basic facts about this series, and then prove an approximation result, Proposition 4.3.4, which will play a fundamental role in later arguments.

The incomplete Eisenstein series is defined as follows:

$$(4.22)\qquad \theta_{\psi,k}(z) = \sum_{U_0\in[T_k]\backslash\Gamma} \psi\left(\text{Im } N_k U_0 z\right).$$

(Cf. [H3, p. 282 (3.10)].) Here $z \in \mathcal{H}$, $k \in \{1, ..., \kappa\}$, and $\psi(y)$ is a function on $0 < y < \infty$.

Concerning convergence, we have the following lemma.

Lemma 4.3.1. *Let constants $C > 0$, $\delta_1 > 1$ and $\delta_2 \in [0, \delta_1]$ be given. Let $\psi : (0,\infty) \to \mathbb{C}$ be a piecewise continuous function satisfying the inequality*

$$(4.23)\qquad |\psi(y)| \leqq C \min\bigl(y^{\delta_1}, y^{\delta_2}\bigr) \qquad \text{for all } y > 0.$$

We then have uniform and absolute convergence over $z \in \mathcal{H}$-compacta in (4.22), and $\theta_{\psi,k}(z)$ is Γ-invariant. If $\psi(y)$ is a continuous function, then so is $\theta_{\psi,k}(z)$. We also have:

$$(4.24)\qquad \theta_{\psi,k}(z) = O\left(\mathcal{Y}_\Gamma(z)^{\delta_2}\right), \qquad \forall z \in \mathcal{H}.$$

(The implied constant depends only on $C, \delta_1, \delta_2, \Gamma$.)

Proof. We use the majorant $|\psi(y)| \leqq Cy^{\delta_1}$ and [H3, pp. 56(Prop. 8.1(b)), 271(line 10)] (with $n = 0$) to show the uniform absolute convergence on compacta. The statements about Γ-invariance and continuity are now trivial.

Finally, applying [H3, pp. 56(Prop. 8.1(e)), 275], again using the majorant $|\psi(y)| \leqq Cy^{\delta_1}$, we see that for each $j \in \{1, ..., \kappa\}$,

$$(4.25)\qquad \sideset{}{^*}\sum_{U_0\in[T_k]\backslash\Gamma} |\psi(\text{Im } N_k U_0 z)| \to 0 \qquad \text{uniformly as } \text{Im } N_j(z) \to \infty.$$

Here the asterisk denotes that *when $j = k$, the sum is restricted by $U_0 \notin [T_k]$.* Combining (4.25) with $|\psi(\text{Im } N_k z)| \leqq C(\text{Im } N_k z)^{\delta_2}$, we obtain:

$$(4.26)\qquad \theta_{\psi,k}(z) = O\left((\text{Im } N_j(z))^{\delta_2}\right)$$

uniformly as $\text{Im } N_j(z) \to \infty$. But for any $B > \mathcal{Y}_\Gamma^{(0)}$, the function $\theta_{\psi,k}(z)$ is uniformly bounded over the compact set $z \in N_j^{-1}\big([0,1] \times [\mathcal{Y}_\Gamma^{(0)}, B]\big)$, and hence, by $[T_j]$-invariance, over the whole set $z \in N_j^{-1}\big(\mathbb{R} \times [\mathcal{Y}_\Gamma^{(0)}, B]\big)$. It follows from this that (4.26) holds anytime $\text{Im } N_j(z) \geqq \mathcal{Y}_\Gamma^{(0)}$ (possibly with a larger implied constant). Hence by Lemma 2.1.2 ("(2.9)⇒(2.10)"), we obtain (4.24). □

Definition 4.3.2. *We define $\mathcal{S}$ to be the joint set of all poles in $\{Re\ s \geqq \frac{1}{2}\}$ of the functions $E_k(z,s)$, $k = 1, ..., \kappa$.*

Recall that by [H3, pp. 78(Claim 9.6), 284(item 12), 297(E)], $\mathcal{S}$ is a finite set, $\mathcal{S} \subset (\frac{1}{2}, 1]$, and $\xi \in \mathcal{S}$ holds if and only if $A_1(\xi(1-\xi)) \neq 0$. In particular, $1 \in \mathcal{S}$. Also, for each $\xi \in \mathcal{S}$, it is known (cf. [H3, p. 288(3.19)]) that the functions $f_k(z) = \text{Res}_{s=\xi} E_k(z,s)$, $k = 1, ..., \kappa$, span the space $A_1(\xi(1-\xi))$.

Lemma 4.3.3. *Let $G(s)$ be an analytic function in the strip $|Re\ s| \leqq 2$. Assume that $|G(s)| \leqq Ce^{-50|Im\ s|}$ holds throughout this strip, for some constant $C > 0$. Let $\Delta \in [-2,2]$, and define:*

$$\psi(y) = \frac{1}{2\pi i} \int_{(\Delta)} G(-s) y^s \, ds \qquad \text{for } y > 0. \tag{4.27}$$

(I.e., ψ is the inverse Mellin transform of G.) Then $\psi(y)$ is continuous, and

$$\psi(y) = O(1) \cdot \min(y^2, y^{-2}) \qquad \text{for all } y > 0. \tag{4.28}$$

We have:

$$\theta_{\psi,k}(z) = \frac{1}{2\pi i} \int_{(\Delta)} G(-s) E_k(z,s) \, ds \qquad \text{for any } 1 < \Delta \leqq 2. \tag{4.29}$$

$$\theta_{\psi,k}(z) = \sum_{\xi \in \mathcal{S}} G(-\xi) Res_{s=\xi} E_k(z,s) + \frac{1}{2\pi} \int_{-\infty}^{\infty} G\left(-\tfrac{1}{2} - it\right) E_k\left(z, \tfrac{1}{2} + it\right) dt. \tag{4.30}$$

The integrals in (4.29) and (4.30) converge uniformly and absolutely over $z \in \mathcal{H}$-compacta. Furthermore, in (4.30),

$$\int_{-\infty}^{\infty} \left|G\left(-\tfrac{1}{2} - it\right)\right| \cdot \left|E_k\left(z, \tfrac{1}{2} + it\right)\right| dt \leqq O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right), \quad \forall z \in \mathcal{H}. \tag{4.31}$$

(The implied constant depends only on C, Γ.) Finally, for any discrete eigenfunction $\phi \in A(\xi(1-\xi))$ (where $Re\ \xi \geqq \frac{1}{2}$) we have:

$$\langle \theta_{\psi,k}, \phi \rangle = G(-\xi) \cdot \big\langle Res_{s=\xi} E_k(\cdot, s), \phi \big\rangle. \tag{4.32}$$

(Of course, this is 0 anytime $\xi \notin \mathcal{S}$.)

Proof. It is clear from $|G(s)| \leqq Ce^{-50|\text{Im } s|}$ that the integral in (4.27) converges absolutely and uniformly over $y \in \mathbb{R}^+$-compacta, and that the integral does not depend on the choice of $\Delta \in [-2,2]$. Hence $\psi(y)$ is continuous. Taking $\Delta = 2$ and $\Delta = -2$ in (4.27) we see that $\psi(y)$ satisfies (4.28). In particular, by Lemma 4.3.1 (with $\delta_1 = 2$, $\delta_2 = 0$), $\theta_{\psi,k}(z)$ is well defined, continuous, and uniformly bounded on all of $\mathcal{H}$.

Using (4.27) we obtain, for any $1 < \Delta \leqq 2$:

$$\begin{aligned}\theta_{\psi,k}(z) &= \sum_{U_0 \in [T_k]\backslash\Gamma} \psi(\operatorname{Im} N_k U_0 z) \\ &= \frac{1}{2\pi i} \sum_{U_0 \in [T_k]\backslash\Gamma} \int_{(\Delta)} G(-s)\,(\operatorname{Im} N_k U_0 z)^s \, ds \\ &= \frac{1}{2\pi i} \int_{(\Delta)} G(-s) \sum_{U_0 \in [T_k]\backslash\Gamma} (\operatorname{Im} N_k U_0 z)^s \, ds = \frac{1}{2\pi i} \int_{(\Delta)} G(-s) E_k(z,s)\, ds.\end{aligned}$$

The change of order of summation is justified by using the obvious majorants;

$$\begin{aligned}\int_{-\infty}^{\infty} |G(-\Delta - it)| \sum_{U_0 \in [T_k]\backslash\Gamma} \left|(\operatorname{Im} N_k U_0 z)^{\Delta+it}\right| dt \\ \leqq \int_{-\infty}^{\infty} C e^{-50|t|} E_k(z,\Delta)\, dt < \infty, \qquad \text{for each } z \in \mathcal{H}.\end{aligned}$$

In this way we also obtain the uniform absolute convergence on compacta. Now (4.29) is proved.

To prove (4.30), we have to move the contour in (4.29) to $\Delta = 1/2$. Recall Definition 4.3.2. We write $s = \sigma + it$ and let $\mathcal{R}$ be the vertical strip $\frac{1}{2} \leqq \sigma \leqq \frac{3}{2}$ minus a tiny neighborhood of each point $\xi \in \mathcal{S}$. We then have the following uniform bound, for $s = \sigma + it \in \mathcal{R}$ and $z \in \mathcal{H}$:

$$E_k(z,s) = O\left(\mathcal{Y}_\Gamma(z)^\sigma + \sqrt{\omega(t)} e^{3|t|}\right). \tag{4.33}$$

This follows from [H3, p. 301(a)(d)] and Lemma 2.1.2 ("(2.9)⇒(2.10)"). Here $\omega(t)$ is an even function satisfying $\omega(t) \geqq 1$ everywhere, and

$$\int_{-T}^{T} \omega(t)\, dt = O(T^2) \qquad \text{as } T \to \infty. \tag{4.34}$$

Cf. [H3, pp. 161, 299(line 14), 315(ii)].

Using (4.33) with $\sigma = \frac{1}{2}$ we get

$$\begin{aligned}&\int_{-\infty}^{\infty} \left|G\left(-\tfrac{1}{2} - it\right)\right| \cdot \left|E_k\left(z, \tfrac{1}{2} + it\right)\right| dt \\ &\leqq \int_{-\infty}^{\infty} C e^{-50|t|} \cdot O\left(\sqrt{\mathcal{Y}_\Gamma(z)} + \sqrt{\omega(t)} e^{3|t|}\right) dt \qquad \left\{\text{Use } \omega(t) \geqq 1.\right\} \\ &= O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right) + O(1) \int_0^{\infty} e^{-47t} \omega(t)\, dt \qquad \left\{\begin{matrix}\text{Integrate by parts,} \\ \text{and use (4.34).}\end{matrix}\right\} \\ &= O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right) + O(1) = O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right).\end{aligned}$$

This proves (4.31). Inspecting the computation we also see that the integral converges uniformly on compacta.

It follows from (4.34) that there exists a sequence of points $1 \leqq T_1 < T_2 < \ldots \to \infty$ satisfying $\omega(T_j) \leqq T_j^2$. For $\Delta = 3/2$ and fixed z in (4.29) we now apply the Cauchy residue theorem over the box $\left[\frac{1}{2}, \frac{3}{2}\right] \times [-T_j, T_j]$, and then let $j \to \infty$. This yields (4.30), because of (4.33) and $|G(s)| \leqq C e^{-50|\operatorname{Im} s|}$.

Finally, to prove (4.32) we first remark that the integral in (4.30) also converges *in* $L_2(\Gamma \setminus \mathcal{H})$*–norm*; this follows from $E_k(z,\overline{s}) \equiv \overline{E_k(z,s)}$ and [H3, pp. 85(Claim 9.8), 288(item 16)], since $\int_{T_1}^{T_2} |G(-\frac{1}{2} - it)|^2\, dt \to 0$ uniformly as $T_1 < T_2$, $T_1 \to \infty$ (or $T_2 \to -\infty$). Clearly, the $L_2(\Gamma \setminus \mathcal{H})$–limit of the integral in (4.30) has to agree with the (locally uniform) pointwise limit on every compact set $\subset \mathcal{F}$, and hence on all of $\mathcal{H}$.

Hence by [H3, pp. 90 (9.30–31), 291 (item 17)], the "$\int_0^\infty$-part" of the integral in (4.30) is *orthogonal* to any $\phi \in A(\xi(1-\xi))$. Using $E_k(z,\overline{s}) \equiv \overline{E_k(z,s)}$ and $\overline{\phi} \in A(\xi(1-\xi))$, we see that the same is true for the "$\int_{-\infty}^0$-part". Now (4.32) follows from (4.30), by use of

$$\text{Res}_{s=\xi'} E(z,s) \in A_1(\xi'(1-\xi')), \qquad \forall \xi' \in \mathcal{S}, \tag{4.35}$$

and [H3, pp. 71(Claim 9.3), 282(item 4)]. □

The next proposition is the main result of this section. Recall that we have defined $A_1 = \oplus_{\lambda>0} A_1(\lambda)$; hence $\mathbb{C} \oplus A_1 = \oplus_{\lambda \geqq 0} A_1(\lambda)$.

Proposition 4.3.4. *Let* $a(z) \in \mathbb{C} \oplus A_1$ *and* $g_1(t), ..., g_\kappa(t) \in L_2(0,\infty)$. *Then for any* $T, \varepsilon > 0$ *there exist functions* $G_1(s), ..., G_\kappa(s)$ *satisfying the conditions in Lemma 4.3.3, such that, if* $\psi_1(y), ..., \psi_\kappa(y)$ *denotes the inverse Mellin transforms (cf. (4.27)) of* $G_1(s), ..., G_\kappa(s)$*:*

$$\left| a(z) + \sum_{k=1}^{\kappa} \int_0^T g_k(t) E_k(z, \tfrac{1}{2} + it)\, dt - \sum_{k=1}^{\kappa} \theta_{\psi_k,k}(z) \right| \leqq \varepsilon \sqrt{\mathcal{Y}_\Gamma(z)}$$

for all $z \in \mathcal{H}$*, and, for each discrete eigenfunction* $\phi \in A(\lambda)$ $(\lambda \geqq 0)$*:*

$$\langle \sum_{k=1}^{\kappa} \theta_{\psi_k,k}\,, \phi \rangle = \langle a, \phi \rangle.$$

Our proof of Proposition 4.3.4 is similar to [H3, pp. 238–241]. We need the following auxiliary result.

Lemma 4.3.5. *Let* $g(t)$ *be a continuous function on* $t \in \mathbb{R}$ *such that* $g(t) = o(t^{-2})$ *as* $|t| \to \infty$. *Then for each* $\eta > 0$ *there exists a function* $G(s)$ *satisfying the following conditions:*

(4.36) $G(s)$ *is analytic in the strip* $|Re\ s| \leqq 2$;

(4.37) $G(s) = O(e^{-50|Im\ s|})$ *when* $|Re\ s| \leqq 2$;

(4.38) $G(-\xi) = 0$ *for each* $\xi \in \mathcal{S}$ *(cf. Def. 4.3.2);*

$$\left| g(t) - \frac{1}{2\pi} G(-\tfrac{1}{2} - it) \right| \leqq \frac{\eta}{1+t^2} \ \ \textit{for all}\ \ t \in \mathbb{R}. \tag{4.39}$$

Proof. Let

$$h(t) = \left(t^2 + 16\right) \cdot g(t) \cdot \prod_{\xi \in \mathcal{S}} \frac{t + 4i}{t + (\xi - 1/2)i} \qquad \text{for } t \in \mathbb{R}.$$

This is a continuous function, and $h(t) \to 0$ as $|t| \to \infty$. We define

$$H(u) = \begin{cases} h(4\tan u) & \text{for } -\frac{\pi}{2} < u < \frac{\pi}{2} \\ 0 & \text{for } u = \pm\frac{\pi}{2} \end{cases}.$$

(The factor 4 will later ensure that the *inverse* of $4\tan(u)$ — i.e., $\arctan(t/4)$ — has an analytic continuation to a sufficiently large horizontal strip in the complex plane.) Then $H(u)$ is a continuous function in the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and hence by Weierstrass approximation theorem there is a polynomial P such that

$$|H(u) - P(u)| \leqq \frac{\eta}{3} \qquad \text{for all } u \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

It follows that, for all $t \in \mathbb{R}$,

$$\left| \left(t^2+16\right) \cdot g(t) \cdot \prod_{\xi \in \mathcal{S}} \frac{t+4i}{t+(\xi-1/2)i} - P\left(\arctan(t/4)\right) \right| \leqq \frac{\eta}{3}.$$

Defining

$$g_0(t) = \frac{1}{t^2+16} \cdot P\left(\arctan(t/4)\right) \cdot \prod_{\xi \in \mathcal{S}} \frac{t+(\xi-1/2)i}{t+4i}, \tag{4.40}$$

we then have, for all $t \in \mathbb{R}$,

$$|g(t) - g_0(t)| \leqq \frac{\eta/3}{t^2+16} \cdot \prod_{\xi \in \mathcal{S}} \frac{|t+(\xi-1/2)i|}{|t+4i|} \leqq \frac{\eta/3}{t^2+16}.$$

We take $T > 0$ so large that

$$|g(t)| \leqq \frac{\eta/3}{t^2+16} \qquad \text{anytime } |t| \geqq T.$$

This is possible since $g(t) = o(t^{-2})$ as $|t| \to \infty$. We also take $\varepsilon > 0$ so small that

$$\left| g_0(t) - e^{-\varepsilon t^2} g_0(t) \right| \leqq \frac{\eta/3}{t^2+16} \qquad \text{for all } t \in [-T, T].$$

We now maintain that

$$\left| g(t) - e^{-\varepsilon t^2} g_0(t) \right| \leqq \frac{\eta}{t^2+16} \qquad \text{for } \textit{all } t \in \mathbb{R}. \tag{4.41}$$

Indeed, if $|t| \leqq T$ then

$$\left| g(t) - e^{-\varepsilon t^2} g_0(t) \right| \leqq |g(t) - g_0(t)| + \left| g_0(t) - e^{-\varepsilon t^2} g_0(t) \right| \leqq \frac{2\eta/3}{t^2+16},$$

i.e. (4.41) holds. Otherwise, if $|t| > T$, (4.41) follows from

$$\left| g(t) - e^{-\varepsilon t^2} g_0(t) \right| \leqq |g(t)| + |g_0(t)| \leqq 2|g(t)| + |g(t) - g_0(t)| \leqq \frac{\eta}{t^2+16}.$$

Note that in (4.40), $g_0(t)$ is in fact defined as an analytic function in the strip $|\text{Im } t| < 4$ (using the standard branch for $\arctan(t/4) = \frac{1}{2i} \log(\frac{1+it/4}{1-it/4})$). We let

$$G(s) = 2\pi \cdot \exp\left(\varepsilon(s+\tfrac{1}{2})^2\right) \cdot g_0\left(i(s+\tfrac{1}{2})\right).$$

Using (4.40), (4.41), this function $G(s)$ is easily verified to satisfy all the conditions (4.36), (4.37), (4.38) and (4.39). □

Proof of Proposition 4.3.4. We fix $a(z)$, $g_1(t), ..., g_\kappa(t)$ and T, ε as in the statement of the proposition. As noted in the proof of Lemma 2.3.8 (just below (2.57); cf. also (4.33)), there is some $C = C(T, \Gamma) > 0$ such that

$$\left| E_k(z, \tfrac{1}{2}+it) \right| \leqq C\sqrt{\mathcal{Y}_\Gamma(z)}, \qquad \forall k \in \{1, ..., \kappa\},\ z \in \mathcal{H},\ t \in [0, T]. \tag{4.42}$$

Also, $g_k(t) \in L_1(0, T)$, since $g_k(t) \in L_2(0, \infty)$. Now for each $k \in \{1, ..., \kappa\}$, we let $g_k^{(0)}(t)$ be a continuous function on $\mathbb{R}$ with support contained in $(0, T)$ such that $||g_k - g_k^{(0)}||_{L_1(0,T)} < \varepsilon/2\kappa C$ (cf. [R5, Thm. 3.14]); then

$$\left| \int_0^T \left(g_k(t) - g_k^{(0)}(t) \right) E_k(z, \tfrac{1}{2}+it)\, dt \right| \leqq \frac{\varepsilon}{2\kappa} \sqrt{\mathcal{Y}_\Gamma(z)} \tag{4.43}$$

for all $z \in \mathcal{H}$.

Using $E_k(z,\overline{s}) = \overline{E_k(z,s)}$ and (4.42) with T replaced by 1, we obtain, uniformly over all $z \in \mathcal{H}$,

$$\int_{-\infty}^{\infty} \frac{\left|E_k(z,\frac{1}{2}+it)\right|}{1+t^2}\,dt \leqq O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right) + 2\int_1^{\infty} t^{-2}\left|E_k(z,\tfrac{1}{2}+it)\right|\,dt$$
$$\leqq O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right) + 2\sqrt{\int_1^{\infty} t^{-4/3}\,dt}\sqrt{\int_1^{\infty} t^{-8/3}\left|E_k(z,\tfrac{1}{2}+it)\right|^2\,dt}$$
$$\leqq O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right),$$

by Cauchy's inequality and Lemma 2.3.2. Hence we can fix $\eta > 0$ so small that

$$\eta\int_{-\infty}^{\infty} \frac{\left|E_k\left(z,\frac{1}{2}+it\right)\right|}{1+t^2}\,dt \leqq \frac{\varepsilon}{2\kappa}\sqrt{\mathcal{Y}_\Gamma(z)} \qquad \forall k \in \{1,\ldots,\kappa\}, z \in \mathcal{H}. \tag{4.44}$$

Recall that $a(z) \in \mathbb{C} \oplus A_1 = \oplus_{\lambda \geqq 0} A_1(\lambda)$. By [H3, p. 288(3.19)], there are complex numbers $d_{\xi,k}$ such that

$$a(z) \equiv \sum_{\xi\in\mathcal{S}}\sum_{k=1}^{\kappa} d_{\xi,k}\,\mathrm{Res}_{s=\xi} E_k(z,s). \tag{4.45}$$

For each $\xi \in \mathcal{S}$ we fix P_ξ to be a polynomial satisfying $P_\xi(-\xi') = \delta_{\xi'\xi}$ for all $\xi' \in \mathcal{S}$. We then define:

$$g_k^{(1)}(t) = g_k^{(0)}(t) - \frac{1}{2\pi}\sum_{\xi\in\mathcal{S}} d_{\xi,k} P_\xi\left(-\tfrac{1}{2}-it\right)\cdot\exp\left[\left(-\tfrac{1}{2}-it+\xi\right)^2\right].$$

Next, for each $k \in \{1,\ldots,\kappa\}$ we can find, by Lemma 4.3.5, a function $G_k^{(1)}(s)$ which satisfies the conditions (4.36)–(4.39) with respect to $g_k^{(1)}$ and η. We let

$$G_k(s) = G_k^{(1)}(s) + \sum_{\xi\in\mathcal{S}} d_{\xi,k}P_\xi(s)\cdot\exp\left[(s+\xi)^2\right].$$

These functions $G_1(s),\ldots,G_\kappa(s)$ are analytic in the strip $|\mathrm{Re}\, s| \leqq 2$, and satisfy the bound $G_k(s) = O(e^{-50|\mathrm{Im}\, s|})$ in this strip. We also have $G_k(-\xi) = d_{\xi,k}$ for all $\xi \in \mathcal{S}$, and

$$\left|g_k^{(0)}(t) - \frac{1}{2\pi}G_k(-\tfrac{1}{2}-it)\right| \leqq \frac{\eta}{1+t^2} \qquad \text{for all } t \in \mathbb{R}. \tag{4.46}$$

We let $\psi_1(y),\ldots,\psi_\kappa(y)$ be the inverse Mellin transforms (cf. (4.27)) of $G_1(s),\ldots,G_\kappa(s)$. We then have, by (4.30) in Lemma 4.3.3,

$$\theta_{\psi_k,k}(z) = \sum_{\xi\in\mathcal{S}} d_{\xi,k}\mathrm{Res}_{s=\xi}E_k(z,s) + \frac{1}{2\pi}\int_{-\infty}^{\infty} G_k(-\tfrac{1}{2}-it)E_k(z,\tfrac{1}{2}+it)\,dt. \tag{4.47}$$

Hence, using (4.45) and (4.43) and recalling that the support of $g_k^{(0)}(t)$ is contained in $(0,T)$, we get

$$\begin{aligned}
&\left| a(z) + \sum_{k=1}^{\kappa} \int_0^T g_k(t) E_k(z, \tfrac{1}{2} + it)\, dt - \sum_{k=1}^{\kappa} \theta_{\psi_k,k}(z) \right| \\
&= \left| \sum_{k=1}^{\kappa} \int_0^T g_k(t) E_k(z, \tfrac{1}{2} + it)\, dt - \sum_{k=1}^{\kappa} \frac{1}{2\pi} \int_{-\infty}^{\infty} G_k(-\tfrac{1}{2} - it) E_k(z, \tfrac{1}{2} + it)\, dt \right| \\
&\leqq \left| \sum_{k=1}^{\kappa} \int_{-\infty}^{\infty} \left(g_k^{(0)}(t) - \frac{1}{2\pi} G_k(-\tfrac{1}{2} - it) \right) E_k(z, \tfrac{1}{2} + it)\, dt \right| + \frac{\varepsilon}{2} \sqrt{\mathcal{Y}_\Gamma(z)}.
\end{aligned}$$

By (4.46) and (4.44), this is seen to be

$$\leqq \varepsilon \sqrt{\mathcal{Y}_\Gamma(z)},$$

and the first part of the proposition is proved.

The second part of the proposition follows by evaluating $\langle a, \phi \rangle$ using (4.45), and then applying (4.32) in Lemma 4.3.3. □

4.4. **The horocycle integral of $\theta_{\psi,k}$.** We will now study the following integral, for functions $\psi(y)$ as in Lemma 4.3.1:

$$\begin{aligned}
&\frac{1}{\beta - \alpha} \int_\alpha^\beta \theta_{\psi,k}(x + iy)\, dx \\
&= \frac{1}{\beta - \alpha} \sum_{W_0 \in [S] \setminus N_k \Gamma} \int_\alpha^\beta \psi\left(\operatorname{Im} W_0(x + iy)\right)\, dx. \qquad (4.48)
\end{aligned}$$

(The identity follows directly from (4.22), by noticing that if U_0 runs through a set of representatives $[T_k] \setminus \Gamma$, then $W_0 = N_k U_0$ runs through $[S] \setminus N_k \Gamma$. The change of order of summation and integration is justified by uniform absolute convergence on the compact set $[\alpha, \beta] + iy$. Cf. Lemma 4.3.1.)

Our goal in this section is to prove that the sum in (4.48) can be sufficiently well approximated by another sum, in which the terms have very nice invariance properties. In §4.5, the replacement of (4.48) by this other sum will be one of the central steps in the proof of our main theorem on generic asymptotic equidistribution.

The basic idea is very simple. To give an informal outline, it is easiest to consider the case $\gamma < \frac{1}{2}$, and to assume that $\psi(y)$ vanishes identically on $(0, h]$, for some (small) number $h > 0$. Let us keep y *very* small, and $(\beta - \alpha)/\sqrt{y}$ *very* large. We consider any $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k \Gamma$ (with $c \neq 0$) which occurs in the sum in (4.48). One knows (cf. (2.27) on p. 148, and recall $|c| \geqq 1$) that $\operatorname{Im} W_0(x+iy) > h$ can hold only if the distance between x and $-\frac{d}{c}$ is less than or equal to $\varepsilon = \sqrt{y/h}$. By our assumptions on y, α, β, this distance ε is *much* smaller than $\beta - \alpha$.

It now follows that the whole $\int_\alpha^\beta$-integral will *vanish* if $-\frac{d}{c} \notin [\alpha - \varepsilon, \beta + \varepsilon]$, and that we may, whenever $-\frac{d}{c} \in [\alpha + \varepsilon, \beta - \varepsilon]$, replace "$\int_\alpha^\beta$" by "$\int_{-\infty}^\infty$" without changing the value of the integral!

Hence (pretending "$\varepsilon = 0$"), it seems reasonable to expect that the sum in (4.48) is well approximated by

$$\frac{1}{\beta - \alpha} \sum_{W_0}^{[\alpha,\beta];k} \int_{-\infty}^{\infty} \psi(\operatorname{Im} W_0(x + iy))\, dx, \qquad (4.49)$$

where the summation symbol is defined as follows.

Definition 4.4.1. *For fixed* $k \in \{1, ..., \kappa\}$, *and* $\alpha, \beta \in \mathbb{R}$ *satisfying* $\alpha < \beta$, *we let*

$$\sum_{W_0}^{[\alpha,\beta];k}$$

denote summation over a set of representatives $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$ *restricted by* $c \neq 0$ *and* $-\frac{d}{c} \in [\alpha, \beta]$.

Theorem 4.4.2. *Let* $\gamma \in [\frac{1}{2}, 1)$, *and let* γ_1, γ_2, $\mathcal{G}(y)$ *be given as in Definition 4.2.1. Take* $\psi : (0, \infty) \to \mathbb{C}$ *to be any piecewise continuous function satisfying* $|\psi(y)| \leqq C \min(y^2, \sqrt{y})$ *for all* $y > 0$, *where* C *is some positive constant. Let* $k \in \{1, ..., \kappa\}$. *We then have*

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta \theta_{\psi,k}(x + iy)\, dx - \frac{1}{\beta - \alpha} \sum_{W_0}^{[\alpha,\beta];k} \int_{-\infty}^{\infty} \psi(Im\ W_0(x + iy))\, dx \right| \to 0 \tag{4.50}$$

uniformly as $y \to 0$ *so long as* $\alpha \in \mathcal{G}(y)$ *and* $\beta - \alpha \geqq y^\gamma$. *(Here* $\Gamma, \gamma, \gamma_1, \gamma_2, \psi$ *are kept fixed.)*

Proof. The key step in the following proof (i.e., the detailed carrying out of the above discussion) does not occur until (4.61) on p. 195.

First, to avoid any mysteries, let us note that the sum occurring to the right in (4.50) is absolutely convergent; in fact, we have

$$\sum_{W_0}^{[\alpha,\beta];k} \int_{-\infty}^{\infty} \Big| \psi(\text{Im}\ W_0(x + iy)) \Big|\, dx < \infty \qquad \text{for } \textit{all}\ \ y > 0,\ \alpha < \beta. \tag{4.51}$$

To see this, notice that for any $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \neq 0$, Lemma 2.2.6 (together with $|\psi(1/w)| \leqq Cw^{-2}$) implies that:

$$\int_{-\infty}^{\infty} \Big| \psi(\text{Im}\ W_0(x + iy)) \Big|\, dx \leqq C \frac{\sqrt{y}}{|c|} \int_{c^2 y}^{\infty} \frac{dw}{w^2 \sqrt{w - c^2 y}}$$

$$= C \frac{\sqrt{y}}{|c|} (c^2 y)^{-3/2} \int_1^\infty \frac{du}{u^2 \sqrt{u - 1}} = \frac{\pi}{2} C y^{-1} |c|^{-4}. \tag{4.52}$$

By Lemma 2.1.1 (with $j = 1, N_j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$) we have $|c| \geqq 1$ for all W_0 occurring in $\sum_{W_0}^{[\alpha,\beta];k}$. Now (4.51) follows from Lemma 2.2.3 (with $A = 1$, $B = \infty$, $\delta = 4$).

In the argument below, we will use the following majorant function:

$$\Psi(y) = \begin{cases} y^{3/2} & \text{for } 0 < y \leqq 1 \\ 0 & \text{for } y > 1. \end{cases}$$

Notice that by Lemma 4.3.1 (with $\delta_1 = \frac{3}{2}$, $\delta_2 = 0$) there is some constant $C_1 = C_1(\Gamma) > 0$ such that

$$|\theta_{\Psi,k}(z)| \leqq C_1, \qquad \forall z \in \mathcal{H}. \tag{4.53}$$

We let $\varepsilon > 0$ be given, and then fix a number $h \in (0, 1)$ so small that

$$C\sqrt{h} \leqq \min\big(\varepsilon / C_1, \varepsilon / 21\big). \tag{4.54}$$

We will split the given function $\psi(y)$ as a sum $\psi \equiv \psi_1 + \psi_2$, where

$$\psi_1(y) = \begin{cases} \psi(y) & \text{for } y \leqq h \\ 0 & \text{for } y > h \end{cases}; \qquad \psi_2(y) = \begin{cases} 0 & \text{for } y \leqq h \\ \psi(y) & \text{for } y > h \end{cases}. \tag{4.55}$$

Clearly then, $\theta_{\psi,k} \equiv \theta_{\psi_1,k} + \theta_{\psi_2,k}$.

To start with, we will study *the contribution from* ψ_1 to each side in the difference in (4.50).

Notice that for each $y \leqq h$ we have

$$|\psi_1(y)| = |\psi(y)| \leqq Cy^2 \leqq C\sqrt{h}y^{3/2} \leqq (\varepsilon/C_1)y^{3/2}. \qquad \{\text{Cf. (4.54).}\}$$

Hence $|\psi_1(y)| \leqq (\varepsilon/C_1)\Psi(y)$ for *all* $y > 0$, since $h < 1$, and since $\psi_1(y) = 0$ when $y > h$. We thus conclude, by (4.53),

$$|\theta_{\psi_1,k}(z)| \leqq \varepsilon \qquad \text{for all } z \in \mathcal{H}.$$

In particular, we have

$$\frac{1}{\beta - \alpha}\int_\alpha^\beta \left|\theta_{\psi_1,k}(x+iy)\right| dx \leqq \varepsilon \tag{4.56}$$

for all $y > 0$ and all $\alpha < \beta$.

Next, using Lemma 2.2.6 and the inequality

$$\left|\psi_1\left(\frac{1}{w}\right)\right| \leqq \begin{cases} Cw^{-2} & \text{for } w \geqq h^{-1} \\ 0 & \text{for } w < h^{-1} \end{cases}$$

we obtain, for any $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying $c \neq 0$ and $c^2 y < h^{-1}$:

$$\begin{aligned} \int_{-\infty}^{\infty} \left|\psi_1(\operatorname{Im} W_0(x+iy))\right| dx &\leqq C\frac{\sqrt{y}}{|c|}\int_{h^{-1}}^{\infty} \frac{dw}{w^2\sqrt{w - c^2y}} \\ &< C\frac{\sqrt{y}}{|c|}\left(\int_{h^{-1}}^{h^{-1}+1} \frac{h^2\,dw}{\sqrt{w-h^{-1}}} + \int_{h^{-1}+1}^{\infty}\frac{dw}{w^2}\right) \\ &= C\frac{\sqrt{y}}{|c|}\left(2h^2 + \frac{1}{h^{-1}+1}\right) < 3Ch\sqrt{y}|c|^{-1}, \end{aligned}$$

since $h < 1$. For $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c^2 y \geqq h^{-1}$ we can instead repeat the computation in (4.52), with ψ replaced by ψ_1. Collecting these inequalities, we obtain (since $\pi/2 < 3$):

$$\begin{aligned} &\frac{1}{\beta-\alpha}\sum_{W_0}^{[\alpha,\beta];k}\int_{-\infty}^{\infty}\left|\psi_1(\operatorname{Im} W_0(x+iy))\right| dx \\ &\leqq \frac{3C}{\beta-\alpha}\sum_{W_0}^{[\alpha,\beta];k}\begin{cases} h\sqrt{y}|c|^{-1} & \text{if } |c| < 1/\sqrt{hy} \\ y^{-1}|c|^{-4} & \text{if } |c| \geqq 1/\sqrt{hy} \end{cases}. \end{aligned} \tag{4.57}$$

We now let $y_1 = y_1(h,\gamma_2) \in (0,1]$ be as in Lemma 4.2.4 on p. 184 (with $M := h$), and temporarily fix some numbers $y \in (0, y_1)$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$. Then, for each $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ occurring in $\sum_{W_0}^{[\alpha,\beta];k}$ we have $|c| \geqq 1$ (as noted above) and

$$-\frac{d}{c} \in [\alpha,\beta] \subset \left[\alpha - \frac{\sqrt{y}}{\sqrt{h}|c|}, \beta + \frac{\sqrt{y}}{\sqrt{h}|c|}\right],$$

and hence by Lemma 4.2.4 on p. 184:

$$A \leqq |c|, \quad \text{where } A = \max\left(1, \frac{y^{\max(\gamma_2, \gamma-\gamma_1)}}{2(\beta-\alpha)}\right). \tag{4.58}$$

We can now apply Lemma 2.2.3 to get an upper bound on the right hand side in (4.57). If $A < 1/\sqrt{hy}$ we get the following upper bound:

$$\begin{aligned}
&\frac{3Ch\sqrt{y}}{\beta-\alpha}\left(2(\beta-\alpha)\frac{1}{\sqrt{hy}} + A^{-1}\right) + \frac{3Cy^{-1}}{\beta-\alpha}\left(2(\beta-\alpha)(hy) + (hy)^2\right) \\
&\leqq 6C\sqrt{h} + 6Chy^{\frac{1}{2}-\max(\gamma_2,\gamma-\gamma_1)} + 6Ch + 3Ch^2\frac{y}{\beta-\alpha} \\
&\qquad \left\{\begin{matrix}\text{Recall that } h<1,\, y<y_1\leqq 1,\, \beta-\alpha \geqq y^\gamma > y, \\ \text{and } \tfrac{1}{2} - \max(\gamma_2, \gamma-\gamma_1) > 0.\end{matrix}\right\} \\
&\leqq (6C + 6C + 6C + 3C)\sqrt{h} = 21C\sqrt{h} \leqq \varepsilon.
\end{aligned} \tag{4.59}$$

(Cf. (4.54).)

In the other case, $A \geqq 1/\sqrt{hy}$, we obtain:

$$\begin{aligned}
&\frac{3Cy^{-1}}{\beta-\alpha}\left(2(\beta-\alpha)A^{-2} + A^{-4}\right) \leqq \frac{3Cy^{-1}}{\beta-\alpha}\left(2(\beta-\alpha)(hy) + (hy)^2\right) \\
&= 6Ch + 3Ch^2\frac{y}{\beta-\alpha} \leqq 9C\sqrt{h} \leqq \varepsilon.
\end{aligned}$$

We have thus proved:

$$\frac{1}{\beta-\alpha}\sum_{W_0}^{[\alpha,\beta];k}\int_{-\infty}^{\infty}\left|\psi_1(\operatorname{Im} W_0(x+iy))\right| dx \leqq \varepsilon \tag{4.60}$$

for all $y \in (0, y_1)$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$.

Next, we will study *the contribution from* ψ_2 in (4.50).

We let $y_1 = y_1(h, \gamma_2)$ be as above, and fix attention to some choice of numbers $0 < y < \min(h, y_1)$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$. Then notice that *if* $c = 0$ in $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$, then $W_0 \in [S]$ by Lemma 2.1.1 (with $j = 1$, $N_j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$), and hence $\psi_2(\operatorname{Im} W_0(x+iy)) = \psi_2(y) = 0$ for all $x \in \mathbb{R}$, because of $y < h$ and (4.55). Using (4.48) (for ψ_2), we now see that

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\int_\alpha^\beta \theta_{\psi_2,k}(x+iy)\,dx - \frac{1}{\beta-\alpha}\sum_{W_0}^{[\alpha,\beta];k}\int_{-\infty}^{\infty}\psi_2(\operatorname{Im} W_0(x+iy))\,dx \\
&= \frac{1}{\beta-\alpha}\sum_{\substack{W_0\in[S]\setminus N_k\Gamma \\ c\neq 0}}\int_{-\infty}^{\infty}\Big(I\big[x\in[\alpha,\beta]\big] - I\big[-\tfrac{d}{c}\in[\alpha,\beta]\big]\Big) \\
&\qquad\qquad\times \psi_2(\operatorname{Im} W_0(x+iy))\,dx.
\end{aligned} \tag{4.61}$$

(Recall here that $I[\cdot]$ denotes the indicator function, cf. (2.55) on p. 157.)

We now consider any $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S]\setminus N_k\Gamma$ which gives a non-zero contribution to the right hand side in (4.61). Then there is some $x \in \mathbb{R}$ such that $\operatorname{Im} W_0(x + iy) > h$ and

$$I\big[x \in [\alpha,\beta]\big] \quad \neq \quad I\big[-\tfrac{d}{c} \in [\alpha,\beta]\big]. \tag{4.62}$$

Reviewing the first half of the *proof* of Lemma 2.2.7 (with $M := h$), we then get:

$$1 \leqq |c| \leqq \frac{1}{\sqrt{hy}} \tag{4.63}$$

and

$$\left| x + \frac{d}{c} \right| \leqq \frac{\sqrt{y}}{\sqrt{h}|c|}. \tag{4.64}$$

But notice that by (4.62), one of the two points α, β must lie *between* x *and* $-\frac{d}{c}$ on the real line; this forces either $|\alpha + \frac{d}{c}| \leqq |x + \frac{d}{c}|$ or $|\beta + \frac{d}{c}| \leqq |x + \frac{d}{c}|$ to hold. Combining this with (4.64), we obtain:

$$-\frac{d}{c} \in \left[\alpha - \frac{\sqrt{y}}{\sqrt{h}|c|}, \alpha + \frac{\sqrt{y}}{\sqrt{h}|c|}\right] \cup \left[\beta - \frac{\sqrt{y}}{\sqrt{h}|c|}, \beta + \frac{\sqrt{y}}{\sqrt{h}|c|}\right]. \tag{4.65}$$

Finally, applying Lemma 4.2.4, we conclude:

$$A \leqq |c| \leqq \frac{1}{\sqrt{hy}}, \qquad -\frac{d}{c} \in \big[\alpha - D, \alpha + D\big] \cup \big[\beta - D, \beta + D\big], \tag{4.66}$$

where A is as in (4.58), and where

$$D = \frac{2(\beta - \alpha)}{\sqrt{h}} y^{\frac{1}{2} - \max(\gamma_2, \gamma - \gamma_1)}.$$

(Concerning (4.66), we remark that $A < 1/\sqrt{hy}$ is *always* true for our choice of h, y, α, β, i.e., regardless of whether there exists any W_0 such as we are considering here or not; indeed, $A < 1/\sqrt{hy}$ is a direct consequence of $y < h < 1$, $\beta - \alpha \geqq y^{\gamma}$ and $\max(\gamma_2, \gamma - \gamma_1) \geqq \gamma - \gamma_1 > \gamma - \frac{1}{2}$.)

Furthermore, using $|\psi(u)| \leqq C\sqrt{u}$ and (4.55), (2.21), we see that $|\psi_2(u)| \leqq C\sqrt{\lfloor u \rfloor_h}$ holds for all $u > 0$, and hence by the inequality (2.34) (with $M := h$)[14]:

$$\begin{aligned} &\int_{-\infty}^{\infty} \Big| I\big[x \in [\alpha, \beta]\big] - I\big[-\tfrac{d}{c} \in [\alpha, \beta]\big] \Big| \cdot \Big| \psi_2(\text{Im } W_0(x + iy)) \Big| \, dx \\ &\leqq C \int_{-\infty}^{\infty} \sqrt{\lfloor \text{Im } W_0(x + iy) \rfloor_h} \, dx \\ &\leqq C \frac{2\sqrt{y}}{|c|} \left(1 + 2 \log\left(\frac{1}{\sqrt{hy}|c|}\right)\right). \end{aligned} \tag{4.67}$$

We now get the following upper bound on the modulus of (4.61), by Lemma 2.2.3 and (4.66), (4.67):

$$\begin{aligned} &\frac{4C\sqrt{y}}{\beta - \alpha} \left(\frac{4D}{\sqrt{hy}} + A^{-1}\right) + \frac{8C\sqrt{y}}{\beta - \alpha} \left(\frac{4D}{\sqrt{hy}} + A^{-1} \log\left(\frac{1}{A\sqrt{hy}}\right)\right) \\ &\leqq \frac{48C}{\beta - \alpha} \frac{D}{\sqrt{h}} + \frac{4C\sqrt{y}}{\beta - \alpha} A^{-1} \left(1 + 2 \log\left(\frac{1}{\sqrt{hy}}\right)\right) \\ &\leqq \frac{96C}{h} y^{\frac{1}{2} - \max(\gamma_2, \gamma - \gamma_1)} + 8C y^{\frac{1}{2} - \max(\gamma_2, \gamma - \gamma_1)} \left(1 + 2 \log\left(\frac{1}{\sqrt{hy}}\right)\right). \end{aligned} \tag{4.68}$$

This tends to 0 as $y \to 0$, since $\frac{1}{2} - \max(\gamma_2, \gamma - \gamma_1) > 0$ (cf. (4.9)). Hence there is some $y_2 = y_2(\varepsilon, C, h, \gamma, \gamma_1, \gamma_2) > 0$ such that the above expression is $\leqq \varepsilon$ whenever $y \in (0, y_2)$.

---

[14] Notice that the deduction of (2.34) is valid anytime $0 < |c| \leqq 1/\sqrt{My}$ (which holds is the present case; cf. (4.66)).

We have thus proved:

$$\left| \frac{1}{\beta-\alpha} \int_\alpha^\beta \theta_{\psi_2,k}(x+iy)\,dx - \frac{1}{\beta-\alpha} \sum_{W_0}^{[\alpha,\beta];k} \int_{-\infty}^{\infty} \psi_2(\operatorname{Im} W_0(x+iy))\,dx \right| \leqq \varepsilon$$

for all $y \in \big(0, \min(h, y_1, y_2)\big)$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$.

Together with (4.60) and (4.56), this shows that the difference in (4.50) is $\leqq 3\varepsilon$ anytime $y \in \big(0, \min(h, y_1, y_2)\big)$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$. This concludes the proof. $\square$

Remark 4.4.3. For exponents $\gamma \in (0, \frac{1}{2})$, we have a non-generic version of Theorem 4.4.2. In precise terms:

Let $\gamma \in (0, \frac{1}{2})$, and let $\psi : (0,\infty) \to \mathbb{C}$ be any piecewise continuous function satisfying $|\psi(y)| \leqq C \min(y^2, \sqrt{y})$ for all $y > 0$, where C is some positive constant. Let $k \in \{1, ..., \kappa\}$. We then have

$$\left| \frac{1}{\beta-\alpha} \int_\alpha^\beta \theta_{\psi,k}(x+iy)\,dx - \frac{1}{\beta-\alpha} \sum_{W_0}^{[\alpha,\beta];k} \int_{-\infty}^{\infty} \psi(Im\ W_0(x+iy))\,dx \right| \to 0$$

uniformly as $y \to 0$ so long as $\beta - \alpha$ remains bigger than y^γ. (Here Γ, γ, ψ are kept fixed.)

The proof is very similar to the proof of Theorem 4.4.2. The necessary modifications are as follows.

Throughout the proof we replace "y_1" by 1, and "$\mathcal{G}(y)$" by $\mathbb{R}$.

In (4.58), we simply take $A = 1$. (Of course, (4.58) is now immediate from Lemma 2.1.1; we do not use Lemma 4.2.4.) Then $A < 1/\sqrt{hy}$ always holds on p. 195, and as upper bound on the right hand side in (4.57) we obtain the first line in (4.59), unchanged. This expression is again seen to be $\leqq \varepsilon$, by use of $A = 1$, $\beta - \alpha \geqq y^\gamma$, $\gamma < \frac{1}{2}$ and (4.54).

In (4.66) we take $D = \sqrt{y/h}$ (and $A = 1$). Notice that (4.66) now follows directly from (4.63) and (4.65); we do not use Lemma 4.2.4.

The rest of the proof then goes through with very minor changes, except that the last line in (4.68) is replaced by

$$\frac{48C}{h} \frac{\sqrt{y}}{\beta-\alpha} + \frac{4C\sqrt{y}}{\beta-\alpha} \left(1 + 2\log\left(\frac{1}{\sqrt{hy}} \right) \right).$$

This tends to 0 as $y \to 0$ because of $\beta - \alpha \geqq y^\gamma$ and $\gamma < \frac{1}{2}$. $\square$

4.5. **Proof of the main theorem.** At last, we are now ready to prove our main theorem on generic equidistribution. What we will prove is a strengthened version of the goal we set in Section 4.1; (4.2b).

Theorem 4.5.1. *Let* Γ *be a cofinite Fuchsian group with a normalized cusp (see (2.19)). Let* $\gamma \in [\frac{1}{2}, 1)$*, and let* γ_1*,* γ_2*,* $\mathcal{G}(y)$ *be given as in Definition 4.2.1. Take* $f : \mathcal{H} \to \mathbb{C}$ *to be any continuous,* Γ*-invariant function satisfying the growth condition*

$$|f(z)| \leqq C\sqrt{\mathcal{Y}_\Gamma(z)} \qquad \forall z \in \mathcal{H}, \tag{4.69}$$

for some positive constant C*. Then:*

$$\frac{1}{\beta - \alpha}\int_\alpha^\beta f(x+iy)\,dx \to \frac{1}{\mu(\mathcal{F})}\int_{\mathcal{F}} f(z)d\mu(z)$$

uniformly as $y \to 0$ *so long as* $\alpha \in \mathcal{G}(y)$ *and* $\beta - \alpha \geqq y^\gamma$*. (Here* $\Gamma, \gamma, \gamma_1, \gamma_2, f$ *are kept fixed.)*

The central new result that we need for the proof of Theorem 4.5.1 is given in the following proposition.

Proposition 4.5.2. *Let* $G(s)$ *be any given analytic function satisfying the conditions in Lemma 4.3.3, and let* $\psi(y)$ *be its inverse Mellin transform (cf. (4.27), (4.28) on p. 187). Let* $k \in \{1, ..., \kappa\}$*. We then have:*

$$\frac{1}{\beta - \alpha}\int_\alpha^\beta \theta_{\psi,k}(x+iy)\,dx \to \frac{1}{\mu(\mathcal{F})}\int_{\mathcal{F}} \theta_{\psi,k}(z)\,d\mu(z)$$

uniformly as $y \to 0$ *so long as* $\alpha \in \mathcal{G}(y)$ *and* $\beta - \alpha \geqq y^\gamma$*. (Here* $\Gamma, \gamma, \gamma_1, \gamma_2, G$ *are kept fixed.)*

Proof. The main idea in the proof is to study what happens in Lemma 4.3.3 and Theorem 4.4.2 when the function $\psi(y)$ is replaced by "$\Lambda^{-1}\psi(\Lambda y)$", for some large constant Λ. (Cf. $\widetilde{\psi}(y)$ below.)

In the following, we will need to apply the results from earlier sections using slightly "sharper" exponents than $\gamma, \gamma_1, \gamma_2$. Specifically: We choose some $\gamma' \in (\gamma, \gamma_1 + \frac{1}{2})$, and then take γ_1', γ_2' such that

$$\left(\gamma - \frac{1}{2} <\right) \gamma' - \frac{1}{2} < \gamma_1' < \gamma_1 < \gamma_2 < \gamma_2' < \frac{1}{2}. \tag{4.70}$$

This is clearly possible, by (4.9) on p. 183. Note in particular that the new exponents $\gamma', \gamma_1', \gamma_2'$ again satisfy the condition (4.9), *i.e.,* $\gamma' - \frac{1}{2} < \gamma_1' < \gamma_2' < \frac{1}{2}$. We now define $\mathcal{G}'(y)$ as in (4.10), but with γ_1', γ_2' in place of γ_1, γ_2.

Let $\varepsilon > 0$ be given. By Theorem 4.4.2 (applied to ψ, γ', $\mathcal{G}'(y)$), there is some $y_1 = y_1(\Gamma, \gamma', \gamma_1', \gamma_2', \psi, \varepsilon) > 0$ such that

$$\left| \frac{1}{\beta - \alpha}\int_\alpha^\beta \theta_{\psi,k}(x+iy)\,dx - \frac{1}{\beta - \alpha}\sum_{W_0}^{[\alpha,\beta];k}\int_{-\infty}^{\infty} \psi(\operatorname{Im} W_0(x+iy))\,dx \right| \leqq \varepsilon \tag{4.71}$$

anytime $y \in (0, y_1)$, $\alpha \in \mathcal{G}'(y)$, $\beta \geqq \alpha + y^{\gamma'}$.

We let $C_0 = C_0(\Gamma, \gamma', \gamma_1', \gamma_2')$ be the implied geometric constant in (4.13) in Proposition 4.2.3 (on p. 184), *applied to* γ', $\mathcal{G}'(y)$.

Recall Definition 4.3.2. For each $\xi \in \mathcal{S}$ we have $\operatorname{Res}_{s=\xi} E_k(z,s) = O(\mathcal{Y}_\Gamma(z)^{1-\xi}) = O\big(\sqrt{\mathcal{Y}_\Gamma(z)}\big)$ uniformly over $z \in \mathcal{H}$, by (4.35), [H3, pp. 271(Def. 3.4), 282(3.9)] and

Lemma 2.1.2 ("(2.9)⇒(2.10)"). Using this fact together with (4.31) in Lemma 4.3.3, we can now *fix* $\Lambda > 1$ *so large that,* for all $z \in \mathcal{H}$:

$$\sum_{\substack{\xi \in \mathcal{S} \\ (\xi \neq 1)}} \Lambda^{\xi - 1} \big| G(-\xi) \cdot \mathrm{Res}_{s=\xi} E_k(z,s) \big|$$

$$+ \frac{\Lambda^{-\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} \left| G\left(-\tfrac{1}{2} - it\right) E_k\left(z, \tfrac{1}{2} + it\right) \right| dt \leqq C_0^{-1} \varepsilon \sqrt{\mathcal{Y}_\Gamma(z)}. \tag{4.72}$$

We define

$$\widetilde{G}(s) = \Lambda^{-1-s} G(s).$$

Clearly, this function again satisfies the conditions in Lemma 4.3.3, and the corresponding incomplete Eisenstein series is $\theta_{\widetilde{\psi},k}(z)$, where

$$\widetilde{\psi}(y) = \frac{1}{2\pi i} \int_{(0)} \widetilde{G}(-s) y^s \, ds = \frac{1}{2\pi i} \int_{(0)} \Lambda^{-1+s} G(-s) y^s \, ds = \Lambda^{-1} \psi(\Lambda y).$$

It now follows from (4.30) in Lemma 4.3.3 (applied to $\widetilde{G}$) and (4.72) that

$$\left| \theta_{\widetilde{\psi},k}(z) - \widetilde{G}(-1) \mathrm{Res}_{s=1} E_k(z,s) \right| \leqq C_0^{-1} \varepsilon \sqrt{\mathcal{Y}_\Gamma(z)}, \qquad \forall z \in \mathcal{H}. \tag{4.73}$$

But $\widetilde{G}(-1) = G(-1)$ and $\mathrm{Res}_{s=1} E_k(z,s) \equiv \mu(\mathcal{F})^{-1}$ (cf. [I1, Prop. 6.13], and also [H3, pp. 224 (note 85), 286 (3.14)]), and hence, using (4.32) in Lemma 4.3.3, we have:

$$\begin{aligned} \widetilde{G}(-1) \mathrm{Res}_{s=1} E_k(z,s) \equiv \frac{G(-1)}{\mu(\mathcal{F})} &= \frac{1}{\mu(\mathcal{F})} \langle \theta_{\psi,k}, 1 \rangle \\ &= \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} \theta_{\psi,k}(w) \, d\mu(w). \end{aligned} \tag{4.74}$$

Using (4.73), (4.74) and our choice of C_0, we see that

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta \theta_{\widetilde{\psi},k}(x+iy) \, dx - \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} \theta_{\psi,k}(w) \, d\mu(w) \right| \leqq \varepsilon \tag{4.75}$$

for all $y \in (0,1)$, $\alpha \in \mathcal{G}'(y)$, $\beta \geqq \alpha + y^{\gamma'}$.

But, by Theorem 4.4.2 (applied to $\widetilde{\psi}$, γ', $\mathcal{G}'(y)$) there is some $y_2 = y_2(\Gamma, \gamma', \gamma_1', \gamma_2', \widetilde{\psi}, \varepsilon) \in (0,1)$ such that

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta \theta_{\widetilde{\psi},k}(x+iy) \, dx - \frac{1}{\beta - \alpha} \sum_{W_0}^{[\alpha,\beta];k} \int_{-\infty}^{\infty} \widetilde{\psi}(\mathrm{Im}\, W_0(x+iy)) \, dx \right| \leqq \varepsilon \tag{4.76}$$

anytime $y \in (0, y_2)$, $\alpha \in \mathcal{G}'(y)$, $\beta \geqq \alpha + y^{\gamma'}$.

We can now evaluate, for $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, $c \neq 0$:

$$\begin{aligned} &\int_{-\infty}^{\infty} \widetilde{\psi}(\mathrm{Im}\, W_0(x+iy)) \, dx = \Lambda^{-1} \int_{-\infty}^{\infty} \psi\left(\frac{\Lambda y}{|c(x+iy)+d|^2} \right) dx \\ &= \Lambda^{-1} \int_{-\infty}^{\infty} \psi\left(\frac{y/\Lambda}{|c(x+iy)/\Lambda + d/\Lambda|^2} \right) dx \qquad \left\{ \text{Let } x = \Lambda x_1 + \tfrac{d}{c}(\Lambda - 1). \right\} \\ &= \int_{-\infty}^{\infty} \psi\left(\frac{y/\Lambda}{|c(x_1 + iy/\Lambda) + d|^2} \right) dx_1 \\ &= \int_{-\infty}^{\infty} \psi\left(\mathrm{Im}\, W_0\left(x + iy\Lambda^{-1} \right) \right) dx. \end{aligned}$$

Combining this with (4.75) and (4.76), we obtain:
(4.77)

$$\left| \frac{1}{\beta-\alpha} \sum_{W_0}^{[\alpha,\beta];k} \int_{-\infty}^{\infty} \psi\left(\text{Im } W_0\left(x+iy\Lambda^{-1}\right)\right) dx - \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} \theta_{\psi,k}(z)\, d\mu(z) \right| \leqq 2\varepsilon.$$

This holds for all $y \in (0, y_2)$, $\alpha \in \mathcal{G}'(y)$, $\beta \geqq \alpha + y^{\gamma'}$.

In (4.77) we replace y by Λy. Combining the resulting inequality (which holds for all $y \in (0, \Lambda^{-1} y_2)$, $\alpha \in \mathcal{G}'(\Lambda y)$, $\beta \geqq \alpha + (\Lambda y)^{\gamma'}$) with (4.71), we get:

$$\left| \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \theta_{\psi,k}(x+iy)\, dx - \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} \theta_{\psi,k}(z)\, d\mu(z) \right| \leqq 3\varepsilon \tag{4.78}$$

anytime $y \in \big(0, \min(y_1, \Lambda^{-1} y_2)\big)$, $\alpha \in \mathcal{G}'(y) \cap \mathcal{G}'(\Lambda y)$, $\beta \geqq \alpha + (\Lambda y)^{\gamma'}$. (For notice that $(\Lambda y)^{\gamma'} > y^{\gamma'}$.)

Finally, for all sufficiently small y we have, because of our choice of γ' and (4.70):

$$(\Lambda y)^{\gamma'} < y^{\gamma}, \quad \text{and} \quad (\Lambda y)^{-\gamma_1'} < y^{-\gamma_1'} < y^{-\gamma_1}, \quad y^{\gamma_2'} < (\Lambda y)^{\gamma_2'} < y^{\gamma_2}.$$

For these y, one immediately verifies (using (4.10)) that

$$\mathcal{G}(y) \subset \mathcal{G}'(y) \cap \mathcal{G}'(\Lambda y).$$

Hence, for each sufficiently small y, (4.78) holds for all $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^{\gamma}$. Since ε was arbitrary, this concludes the proof. □

Proof of Theorem 4.5.1. Recall the proof of Theorem 2.3.1, pp. 159–161. Notice that this proof carries over almost entirely to the present situation with $\gamma \in [\frac{1}{2}, 1)$, if we keep α restricted by $\alpha \in \mathcal{G}(y)$ and always use Proposition 4.2.3 instead of Proposition 2.2.5.

The *only* part of this proof which cannot easily be extended is the argument leading from (2.63) (which now holds for all $y \in (0,1)$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^{\gamma}$) to (2.64) (which we want to prove for all sufficiently small y, and all $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^{\gamma}$).

More precisely, since Proposition 2.3.6 holds for arbitrary $\gamma \in (0,1)$, the only thing that remains to show is that the A_1–part of the λ_n–sum in (2.63), together with the whole Eisenstein part, tends to 0 uniformly as $y \to 0$ so long as $\alpha \in \mathcal{G}(y)$, $\beta - \alpha \geqq y^{\gamma}$. In other words, Theorem 4.5.1 would be proved in general if we could only prove that for any function $f_0(z)$ of the form

$$f_0(z) = a(z) + \sum_{k=1}^{\kappa} \int_0^T g_k(t) E_k(z, \tfrac{1}{2} + it)\, dt, \tag{4.79}$$

where $T > 0$, $a(z) \in A_1$, and $g_1(t), ..., g_\kappa(t)$ are functions in $L_2(0,\infty)$, we have

$$\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} f_0(x+iy)\, dx \to 0$$

uniformly as $y \to 0$ so long as $\alpha \in \mathcal{G}(y)$ and $\beta - \alpha \geqq y^{\gamma}$.

We let $f_0(z)$ be any function as in (4.79), and let $\varepsilon > 0$ be given. We let $C_0 = C_0(\Gamma, \gamma, \gamma_1, \gamma_2)$ be the implied constant in (4.13) in Proposition 4.2.3. By Proposition 4.3.4 there are analytic functions $G_1(s), ..., G_\kappa(s)$ satisfying the conditions in Lemma 4.3.3, such that, if $\psi_1(y), ..., \psi_\kappa(y)$ denotes the corresponding inverse Mellin transforms:

$$\left| f_0(z) - \sum_{k=1}^{\kappa} \theta_{\psi_k,k}(z) \right| \leqq C_0^{-1} \varepsilon \sqrt{\mathcal{Y}_\Gamma(z)}, \qquad \text{for all } \ z \in \mathcal{H}.$$

It then follows from our choice of C_0 that

$$\left| \frac{1}{\beta-\alpha} \int_\alpha^\beta f_0(x+iy)\,dx - \sum_{k=1}^{\kappa} \frac{1}{\beta-\alpha} \int_\alpha^\beta \theta_{\psi_k,k}(x+iy)\,dx \right| \leqq \varepsilon \tag{4.80}$$

for all $y \in (0,1)$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$.

By the second part of Proposition 4.3.4 (with $\phi \equiv 1 \in A(0)$), we have

$$\sum_{k=1}^{\kappa} \int_{\mathcal{F}} \theta_{\psi_k,k}(z)\,d\mu(z) = \langle a, 1 \rangle = 0,$$

since $A_1 \perp \mathbb{C}$. Hence by Proposition 4.5.2 there is some $y_1 \in (0,1)$ such that

$$\left| \sum_{k=1}^{\kappa} \frac{1}{\beta-\alpha} \int_\alpha^\beta \theta_{\psi_k,k}(x+iy)\,dx \right| \leqq \varepsilon$$

for all $y \in (0,y_1)$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$. Combining this with (4.80) we get:

$$\left| \frac{1}{\beta-\alpha} \int_\alpha^\beta f_0(x+iy)\,dx \right| \leqq 2\varepsilon, \qquad \forall y \in (0,y_1),\ \alpha \in \mathcal{G}(y),\ \beta \geqq \alpha + y^\gamma.$$

Since ε was arbitrary, this concludes the proof. □ □ □

Remark 4.5.3. The arguments given on pp. 198–201 also apply, with very small changes[15], to prove a non-generic version Theorem 4.5.1 for exponents $\gamma \in (0, \frac{1}{2})$. In other words, we have here a new proof of Theorem 2.4.1, independent of Lemma 2.3.8, Proposition 2.3.10 and Proposition 2.4.2.

Remark 4.5.4. *Theorem 4.5.1 remains true if the constraint "$\alpha \in \mathcal{G}(y)$" is replaced by "$\beta \in \mathcal{G}(y)$".* Indeed, one can check that the whole proof, from p. 184 to p. 201, remains valid, modulo a small number of very simple changes, if "$\alpha \in \mathcal{G}(y)$" is replaced by "$\beta \in \mathcal{G}(y)$" throughout.

Remark 4.5.5. There is an alternative equidistribution result for generic horocycle segments, which can be obtained as a simple corollary to Theorem 4.5.1:

Given Γ as above, and given $\gamma \in [\frac{1}{2}, 1)$, there is a set $S \subset \mathbb{R}$ of measure zero such that the following holds: For each given continuous and Γ-invariant function $f : \mathcal{H} \to \mathbb{C}$ satisfying the (usual) growth condition (4.69)*, and for each* fixed *$\alpha_0 \in \mathbb{R} - S$, we have*

$$\frac{1}{\beta-\alpha_0} \int_{\alpha_0}^\beta f(x+iy)\,dx \to \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} f(z)d\mu(z) \tag{4.81}$$

uniformly as $y \to 0$ so long as $\beta \geqq \alpha_0 + y^\gamma$.

---

[15] Among the modifications are that on p. 198 we now simply take any $\gamma' \in (\gamma, \frac{1}{2})$ as a "sharper" exponent than γ; no "γ_1, γ_2" take part in the proof. Also, throughout the proof, we use Prop. 2.2.5 instead of Prop. 4.2.3, and Remark 4.4.3 instead of Thm. 4.4.2.

The proof of this is based on the observation that Proposition 4.2.2 (on p. 183) can be strengthened to read, for any two fixed numbers $\mu < \nu$,

$$\lim_{y\to 0} m\Big([\mu,\nu] \cap \bigcap_{v \leqq y} \mathcal{G}(v)\Big) \;=\; \nu - \mu. \tag{4.82}$$

This is proved by modifying the original proof of Proposition 4.2.2. We first notice that by (4.10), a given point $\alpha \in [\mu,\nu]$ belongs to $\bigcap_{v\leqq y} \mathcal{G}(v)$ if and only if it lies outside the intervals $\big[-\frac{d}{c} - \frac{v^{\gamma_2}}{|c|}, -\frac{d}{c} + \frac{v^{\gamma_2}}{|c|}\big]$ for all $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$ ($c \neq 0$, and k arbitrary) and for all positive numbers v satisfying $v \leqq y$ and $v \leqq |c|^{-1/\gamma_1}$. For each W_0, we need here only care about the largest possible choice of v. This yields the following modified version of (4.11), for each $y < 1$:

$$\begin{aligned}
[\mu,\nu] \cap \bigcap_{v\leqq y} \mathcal{G}(v) = [\mu,\nu] &- \bigcup_{k=1}^{\kappa} \; \underset{1\leqq |c| \leqq y^{-\gamma_1}}{\bigcup_{W_0}^{(k)}} \left[-\frac{d}{c} - \frac{y^{\gamma_2}}{|c|}, -\frac{d}{c} + \frac{y^{\gamma_2}}{|c|}\right] \\
&- \bigcup_{k=1}^{\kappa} \; \underset{y^{-\gamma_1} < |c|}{\bigcup_{W_0}^{(k)}} \left[-\frac{d}{c} - \frac{(|c|^{-1/\gamma_1})^{\gamma_2}}{|c|}, -\frac{d}{c} + \frac{(|c|^{-1/\gamma_1})^{\gamma_2}}{|c|}\right].
\end{aligned}$$

(Notice that the *first* double union here is exactly the same as in (4.11).)

Using this new relation in place of (4.11), we get the following extra term in the left hand side of (4.12):

$$-\sum_{k=1}^{\kappa} \sum_{W_0}^{(k)} 2|c|^{-1-\gamma_2/\gamma_1},$$

the inner sum ("$\sum^{(k)}$") here being taken over a set of representatives $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$ restricted by $y^{-\gamma_1} < |c|$ and $-\frac{d}{c} \in [\mu - 1, \nu + 1]$. But by Lemma 2.2.3 on p. 145, this sum tends to 0 as $y \to 0$, since $-1 - \gamma_2/\gamma_1 < -2$ by (4.9) (p. 183). Now (4.82) follows as in the proof of Proposition 4.2.2.

We can now prove our assertion regarding (4.81) as follows. We define

$$S = \mathbb{R} - \bigcup_{y>0} \bigcap_{v\leqq y} \mathcal{G}(v).$$

For any fixed numbers $\mu < \nu$, and any fixed $y > 0$, we then have $[\mu,\nu] \cap S \subset [\mu,\nu] - \bigcap_{v\leqq y} \mathcal{G}(v)$. Hence, by (4.82), $m([\mu,\nu] \cap S) = 0$. Since μ, ν were arbitrary, this gives $m(S) = 0$.

Furthermore, for any fixed $\alpha_0 \in \mathbb{R} - S$, there exists some $y > 0$ such that $\alpha_0 \in \cap_{v\leqq y} \mathcal{G}(v)$; hence, (4.81) follows immediately from Theorem 4.5.1. □

4.6. **A result concerning the distribution of cusps.** A key role in the above proof of Theorem 4.5.1 was played by Theorem 4.4.2. We will now see that this proposition entails yet another interesting consequence. Specifically, by use of Theorem 4.4.2 in conjunction with Lemma 2.2.6 and our main theorems on equidistribution of horocycle segments, we can compute the limit of

$$\frac{\sqrt{y}}{\beta - \alpha} \sum_{W_0}^{[\alpha,\beta];k} \frac{1}{|c|} \int_{c^2 y}^{\infty} \psi\left(\frac{1}{w}\right) \frac{dw}{\sqrt{w - c^2 y}} \qquad \text{as } y \to 0,$$

for any "nice" function ψ. Exploiting here the arbitrariness of ψ, we will obtain asymptotic information about the cusp distribution function $\mathfrak{C}^k_{\alpha\beta}(X)$ (cf. (2.17), (2.18)) as $X \to \infty$.

Proposition 4.6.1. *Let* Γ *be a cofinite Fuchsian group with a normalized cusp (see (2.19)). Let* $k \in \{1, ..., \kappa\}$ *and* $\gamma \in (0, \frac{1}{2})$*. We then have*

$$\frac{1}{X^2(\beta-\alpha)}\, \mathfrak{C}^k_{\alpha\beta}(X) \longrightarrow \frac{1}{\pi\, \mu(\mathcal{F})}$$

uniformly as $X \to \infty$ *so long as* $\beta - \alpha \geqq X^{-2\gamma}$*.*

Proposition 4.6.2. *Let* Γ *be a cofinite Fuchsian group with a normalized cusp (see (2.19)). Let* $k \in \{1, ..., \kappa\}$ *and* $\gamma \in [\frac{1}{2}, 1)$*, and let* γ_1*,* γ_2*,* $\mathcal{G}(y)$ *be given as in Definition 4.2.1. We then have*

$$\frac{1}{X^2(\beta-\alpha)}\, \mathfrak{C}^k_{\alpha\beta}(X) \longrightarrow \frac{1}{\pi\, \mu(\mathcal{F})}$$

uniformly as $X \to \infty$ *so long as* $\alpha \in \mathcal{G}(X^{-2})$ *and* $\beta - \alpha \geqq X^{-2\gamma}$*.*

Proof. We first prove Proposition 4.6.2. Let $\varepsilon \in (0,1)$ be given. We fix $F(x)$ to be a function in $C^\infty(\mathbb{R}^+)$ such that

$$\begin{cases} F(x) = x & \text{for } 0 < x \leqq 1; \\ 0 \leqq F(x) \leqq x & \text{for } 1 \leqq x \leqq 1+\varepsilon; \\ F(x) = 0 & \text{for } 1+\varepsilon \leqq x. \end{cases} \tag{4.83}$$

To this function F we will now apply the inverse of the integral transform $\psi \mapsto \int_{x^2}^\infty \psi(\frac{1}{w}) \frac{dw}{\sqrt{w-x^2}}$. The relevant facts[16] about this inverse transform will be verified explicitly below. To start with, we define the function ψ on $\mathbb{R}^+$ through

$$\psi\left(\frac{1}{w}\right) := -\frac{1}{2\pi}\int_w^\infty \frac{F'\big(\sqrt{u}\big)}{\sqrt{u}\sqrt{u-w}}\, du = -\frac{1}{\pi}\int_0^\infty \frac{F'\big(\sqrt{w+v^2}\big)}{\sqrt{w+v^2}}\, dv. \tag{4.84}$$

It is clear from the second form of the integral that $\psi \in C^\infty(\mathbb{R}^+)$. It follows from (4.84) and (4.83) that $\psi(1/w) = 0$ for all $w \geqq 4$. Also, substituting $u = wr$ in (4.84) and using the fact that $F'(x)$ is bounded on $[0,2]$ and 0 on $[2,\infty)$, we get

$$\forall\, w \in (0,1): \qquad \left|\psi\left(\frac{1}{w}\right)\right| = O(1)\int_1^{4/w} \frac{dr}{\sqrt{r}\sqrt{r-1}} = O\left(\log\left(\frac{4}{w}\right)\right).$$

(Of course, the implied constant depends here on F, ε.) In particular, there is a constant $C = C(F,\varepsilon) > 0$ such that $|\psi(y)| \leqq C \min(y^2, \sqrt{y})$ for all $y > 0$.

Furthermore, for each $x > 0$ we have

$$\begin{aligned}
\int_{x^2}^\infty \psi\left(\frac{1}{w}\right) \frac{dw}{\sqrt{w-x^2}} &= -\frac{1}{2\pi}\int_{x^2}^\infty \left[\int_w^\infty \frac{F'\big(\sqrt{u}\big)}{\sqrt{u}\sqrt{u-w}}\, du\right] \frac{dw}{\sqrt{w-x^2}} \\
&= -\frac{1}{2\pi}\int_{x^2}^\infty \frac{F'\left(\sqrt{u}\right)}{\sqrt{u}} \left[\int_{x^2}^u \frac{dw}{\sqrt{u-w}\sqrt{w-x^2}}\right] du \qquad \left\{\begin{matrix}\text{Substitute} \\ w = x^2 + (u-x^2)v.\end{matrix}\right\} \\
&= -\frac{1}{2\pi}\int_{x^2}^\infty \frac{F'\left(\sqrt{u}\right)}{\sqrt{u}} \left[\int_0^1 v^{-\frac{1}{2}}(1-v)^{-\frac{1}{2}}\, dv\right] du \qquad \{\text{Cf. [AS, p. 258(§6.2)].}\} \\
&= -\frac{1}{2}\int_{x^2}^\infty \frac{F'(\sqrt{u})}{\sqrt{u}}\, du = -\int_x^\infty F'(r)\, dr = F(x).
\end{aligned}$$

(Since $F(x) = 0$ for all $x \geqq 2$, the above manipulations are trivially justified by absolute convergence in the double integral.) Combining this with Lemma 2.2.6

---

[16] Of course, these facts are well-known; cf., e.g., [H2, pp. 15(i), 16(d)].

we obtain, for each $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \neq 0$,

$$\int_{-\infty}^{\infty} \psi(\mathrm{Im}\, W_0(x+iy))\, dx = \frac{\sqrt{y}}{|c|} F\left(|c|\sqrt{y}\right). \tag{4.85}$$

With these preliminaries in place, the proof can now be concluded in a prompt manner. By Lemma 4.3.1, the incomplete Eisenstein series $\theta_{\psi,k}(z)$ is continuous, and uniformly bounded by $O\left(\sqrt{\mathcal{Y}_\Gamma(z)}\right)$ for all $z \in \mathcal{H}$. Hence by Theorem 4.5.1 we have

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \theta_{\psi,k}(x+iy)\, dx \to \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} \theta_{\psi,k}(z)\, d\mu(z)$$

uniformly as $y \to 0$ so long as $\alpha \in \mathcal{G}(y)$ and $\beta - \alpha \geqq y^\gamma$. Together with Theorem 4.4.2 and (4.85), this implies that

$$\frac{1}{\beta - \alpha} \sum_{W_0}^{[\alpha,\beta];k} \frac{\sqrt{y}}{|c|} F\left(|c|\sqrt{y}\right) \to \frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} \theta_{\psi,k}(z)\, d\mu(z) \tag{4.86}$$

uniformly as $y \to 0$ so long as $\alpha \in \mathcal{G}(y)$ and $\beta - \alpha \geqq y^\gamma$.

Concerning the left hand side in (4.86), recall Definition 4.4.1 and (2.17). Since $F(x) = x$ when $x \leqq 1$ and $F(x) \geqq 0$ when $x \geqq 1$, we clearly obtain, for all $y > 0$ and all $\alpha < \beta$,

$$\frac{1}{\beta - \alpha} \sum_{W_0}^{[\alpha,\beta];k} \frac{\sqrt{y}}{|c|} F\left(|c|\sqrt{y}\right) \geqq \frac{y}{\beta - \alpha}\, \mathfrak{C}^k_{\alpha\beta}\left(y^{-\frac{1}{2}}\right). \tag{4.87}$$

We now turn to the right hand side in (4.86). We have

$$\int_{\mathcal{F}} \theta_{\psi,k}(z)\, d\mu(z) = \int_0^\infty \psi(y)\, \frac{dy}{y^2}.$$

This is proved by "unfolding", as in [H3, pp.73(with $\varphi_n \equiv$ const), 283(3.11)]. The necessary manipulations are justified by nicely majorized convergence using $\psi(y) \leqq C \min(y^2, \sqrt{y})$. Further evaluation using (4.84) gives

$$\begin{aligned}
&\int_{\mathcal{F}} \theta_{\psi,k}(z)\, d\mu(z) = \int_0^\infty \psi(y)\, \frac{dy}{y^2} = \int_0^\infty \psi\left(\frac{1}{w}\right) dw \\
&= -\frac{1}{2\pi} \int_0^\infty \int_w^\infty \frac{F'(\sqrt{u})}{\sqrt{u}\sqrt{u-w}}\, du\, dw \\
&= -\frac{1}{2\pi} \int_0^\infty \frac{F'(\sqrt{u})}{\sqrt{u}} \left[\int_0^u \frac{dw}{\sqrt{u-w}}\right] du \\
&= -\frac{1}{\pi} \int_0^\infty F'(\sqrt{u})\, du = -\frac{2}{\pi} \int_0^\infty F'(x) x\, dx = \frac{2}{\pi} \int_0^\infty F(x)\, dx.
\end{aligned}$$

(We integrated by parts in the last step.) Notice here that, by (4.83),

$$\int_0^\infty F(x)\, dx \leqq \int_0^{1+\varepsilon} x\, dx = \tfrac{1}{2}(1+\varepsilon)^2 < \tfrac{1}{2} + 2\varepsilon$$

(since $\varepsilon < 1$). Hence,

$$\frac{1}{\mu(\mathcal{F})} \int_{\mathcal{F}} \theta_{\psi,k}(z)\, d\mu(z) < \frac{1}{\pi\mu(\mathcal{F})}\, (1 + 4\varepsilon). \tag{4.88}$$

Together, (4.86), (4.87) and (4.88) imply that whenever y is sufficiently small and $\alpha \in \mathcal{G}(y)$ and $\beta - \alpha \geqq y^{\gamma}$, we have

$$\frac{y}{\beta-\alpha}\,\mathfrak{C}^{k}_{\alpha\beta}\left(y^{-\frac{1}{2}}\right) < \frac{1}{\pi\mu(\mathcal{F})}\,(1+5\varepsilon). \tag{4.89}$$

Next, we repeat the whole argument, but with a function $F(x)$ in $C^{\infty}(\mathbb{R}^{+})$ satisfying

$$\begin{cases} F(x) = x & \text{for } 0 < x \leqq 1-\varepsilon; \\ 0 \leqq F(x) \leqq x & \text{for } 1-\varepsilon \leqq x \leqq 1; \\ F(x) = 0 & \text{for } 1 \leqq x. \end{cases}$$

This time, (4.87) holds with "$\geqq$" replaced by "$\leqq$", and in place of (4.88) we have

$$\frac{1}{\mu(\mathcal{F})}\int_{\mathcal{F}} \theta_{\psi,k}(z)\,d\mu(z) > \frac{1}{\pi\mu(\mathcal{F})}\,(1-2\varepsilon),$$

since now $\int_0^{\infty} F(x)\,dx \geqq \int_0^{1-\varepsilon} x\,dx = \frac{1}{2}(1-\varepsilon)^2 > \frac{1}{2}-\varepsilon$. The conclusion (analogous to (4.89)) is that whenever y is sufficiently small and $\alpha \in \mathcal{G}(y)$ and $\beta - \alpha \geqq y^{\gamma}$, we have

$$\frac{y}{\beta-\alpha}\,\mathfrak{C}^{k}_{\alpha\beta}\left(y^{-\frac{1}{2}}\right) > \frac{1}{\pi\mu(\mathcal{F})}\,(1-3\varepsilon). \tag{4.90}$$

Since ε was arbitrary, Proposition 4.6.2 now follows from (4.89) and (4.90) upon taking $y = X^{-2}$.

Of course, almost exactly the same proof works for Proposition 4.6.1, if we use Theorem 2.4.1 and Remark 4.4.3 in place of Theorem 4.5.1 and Theorem 4.4.2, respectively. ☐

Remark 4.6.3. The constraint "$\alpha \in \mathcal{G}(X^{-2})$" can be replaced by "$\beta \in \mathcal{G}(X^{-2})$" in Proposition 4.6.2. Cf. Remark 4.5.4.

To conclude this section, we will make an explicit study of the case $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ in connection with Proposition 4.6.1 and Proposition 4.6.2. This will be seen to lead to a result concerning the asymptotic distribution *Farey points.*

We recall that for $N \in \mathbb{Z}^{+}$, the Farey series $\mathfrak{F}_N$ of order N is defined as the ascending series of all reduced fractions between 0 and 1 whose denominators do not exceed N. Thus, as a set,

$$\mathfrak{F}_N = \left\{ \frac{h}{k} \;\middle|\; 1 \leqq k \leqq N,\ 0 \leqq h \leqq k,\ (h,k) = 1 \right\}. \tag{4.91}$$

(In this formula and below, h and k are implicitly understood to be *integers.*)

Corollary 4.6.4. *Let a number* $\delta \in (0,2)$ *be given. If* $\delta < 1$ *we have*

$$\frac{1}{N^2(\beta-\alpha)} \cdot \Big|\mathfrak{F}_N \cap [\alpha,\beta]\Big| \to \frac{3}{\pi^2} \tag{4.92}$$

uniformly as $N \to \infty$ *so long as* $\alpha, \beta \in [0,1]$ *and* $\beta - \alpha \geqq N^{-\delta}$.

If $1 \leqq \delta < 2$, *we let* δ_1, δ_2 *be numbers satisfying* $\delta - 1 < \delta_1 < \delta_2 < 1$, *and define, for all* $N \in \mathbb{Z}^+$,

$$\mathbb{G}(N) := \big[0,1\big] - \bigcup_{\mathfrak{F}_{[N^{\delta_1}]}} \left[\frac{h - N^{-\delta_2}}{k}, \frac{h + N^{-\delta_2}}{k}\right], \tag{4.93}$$

the union being taken over all fractions $\frac{h}{k}$ *in the Farey series of order* $[N^{\delta_1}]$. *Then,* (4.92) *holds uniformly as* $N \to \infty$ *so long as* $\alpha \in \mathbb{G}(N)$, $\beta \in [0,1]$ *and* $\beta - \alpha \geqq N^{-\delta}$ *(or* $\alpha \in [0,1]$, $\beta \in \mathbb{G}(N)$ *and* $\beta - \alpha \geqq N^{-\delta}$*).*

Note that the sum of the lengths of all intervals appearing in the union to the right in (4.93) is

$$\sum_{\mathfrak{F}_{[N^{\delta_1}]}} \frac{2N^{-\delta_2}}{k} \leqq \sum_{k=1}^{[N^{\delta_1}]} \sum_{h=0}^{k} \frac{2N^{-\delta_2}}{k} \leqq [N^{\delta_1}] \cdot 4N^{-\delta_2}, \tag{4.94}$$

and this tends to 0 as $N \to \infty$. Hence $m\big(\mathbb{G}(N)\big) \to 1$ as $N \to \infty$, i.e. $\mathbb{G}(N)$ contains asymptotically 100% of all points in $[0,1]$. (The same fact also follows from Proposition 4.2.2 and the formula (4.97) below.)

Proof of Corollary 4.6.4. The whole corollary is nothing but a restatement of Proposition 4.6.1 and Proposition 4.6.2, Remark 4.6.3 (with $\gamma = \delta/2$, $\gamma_1 = \delta_1/2$, $\gamma_2 = \delta_2/2$) in the special case $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, $X = N \in \mathbb{Z}^+$ and $\alpha, \beta \in [-1,0]$.

To see this, we first recall some well-known facts about the modular group, $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$.

The usual choice of fundamental region for Γ is

$$\mathcal{F} = \left\{ z \in \mathcal{H} \quad \Big| \quad |z| \geqq 1,\ |\mathrm{Re}\, z| \leqq \tfrac{1}{2} \right\}.$$

It follows that $\mu(\mathcal{F}) = \frac{\pi}{3}$.

We can take $\eta_1 = \infty$, $N_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (i.e., (2.19) is fulfilled), and this is the *only* cusp up to equivalence (i.e., $\kappa = 1$).

It is also known (and easy to show) that Γ can be decomposed as a disjoint union

$$\Gamma \ = \ [S] \ \sqcup \ \bigsqcup_{c,d} [S]\begin{pmatrix} * & * \\ c & d \end{pmatrix}, \tag{4.95}$$

where $\langle c, d \rangle$ runs through all pairs of integers such that $c > 0$ and $(c,d) = 1$, and $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ denotes *any* matrix in Γ whose bottom row is c, d. (Cf. [M4, Lemma 4.1.5].)

It follows from (4.95), (4.91) and (2.17) that, for all $\alpha, \beta \in [0,1]$ with $\alpha < \beta$,

$$\Big|\mathfrak{F}_N \cap [\alpha,\beta]\Big| = \left|\left\{ \frac{h}{k} \quad \Big| \quad 0 < k \leqq N,\ \frac{h}{k} \in [\alpha,\beta],\ (h,k) = 1 \right\}\right| = \mathfrak{C}^1_{-\beta,-\alpha}(N). \tag{4.96}$$

Also, when $\delta \in [1,2)$ and $N \geqq 2$, notice that in (4.93), we may as well extend the union to all $h \in \mathbb{Z}$ and all $k \in \{1, ..., [N^{\delta_1}]\}$ such that $(h,k) = 1$, for if $h \leqq -1$ or $h \geqq k+1$ then the interval $\big[(h - N^{-\delta_2})/k, (h + N^{-\delta_2})/k\big]$ is trivially disjoint from $[0,1]$. Hence, using (4.95) and Definition 4.2.1 (with $\gamma_1 = \delta_1/2, \gamma_2 = \delta_2/2$), we get (for $N \geqq 2$)

$$\begin{aligned} \mathbb{G}(N) &= \Big\{ \alpha \in [0,1] \ \Big| \ \forall k,h \in \mathbb{Z}: \quad 0 < k \leqq N^{\delta_1} \wedge (h,k) = 1 \\ &\qquad\qquad\qquad\qquad\qquad \Longrightarrow \left| \alpha - \frac{h}{k} \right| > \frac{N^{-\delta_2}}{k} \Big\} \\ &= \big[0,1\big] \cap \big(-\mathcal{G}(N^{-2})\big). \end{aligned} \tag{4.97}$$

In view of (4.96) and (4.97), it is now clear that the corollary indeed follows directly from Proposition 4.6.1 and Proposition 4.6.2. □

We remark that it is possible to give an alternative proof of the above result, using only more elementary, non-spectral methods. For details on this, cf. Appendix 2 below.

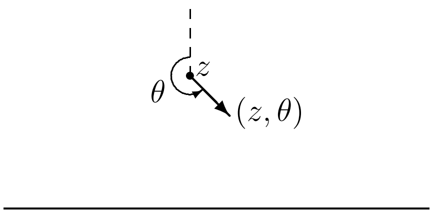
5. Alternative approach, using invariant measures

In this section, we will give an alternative proof of Theorem 2.4.1 and Theorem 4.5.1. This proof involves no spectral theory, but instead makes use of the classification of invariant (finite) measures of the horocycle flow on $\Gamma \setminus \mathrm{SL}(2,\mathbb{R})$, obtained by S. G. Dani in [D1, D2]. (Notice that [D1, D2] actually treats the more general case of a (maximal) horospherical flow on any quotient space of the form G/Γ, where G is a reductive Lie group and Γ a lattice. Dani's result has later been vastly generalized by M. Ratner in [R2].)

The proof given below will in fact lead to a more general—phase space— version of Theorem 2.4.1 and Theorem 4.5.1. However, the methods which we will use are of a more indirect nature than those employed in earlier sections. In particular, it does not seem possible to obtain estimates of error terms (akin to those given in Theorem 3.2.5) along the lines of argument used below.

5.1. **Preliminaries.** We let $T_1\mathcal{H}$ denote the *unit tangent bundle* of the hyperbolic plane $\mathcal{H}$. To start with, we will recall some basic facts about the horocycle flow on $T_1\mathcal{H}$. Our presentation closely follows [S1, §1], [M1, §3.1] and [M2, §2.3].

$T_1\mathcal{H}$ can be parametrized by $(z,\theta) \in \mathcal{H} \times (\mathbb{R}/2\pi\mathbb{Z})$, where θ is an angle which describes the orientation of an arrow attached to z. The standard choice is to let θ measure the angle counterclockwise against the upward vertical.



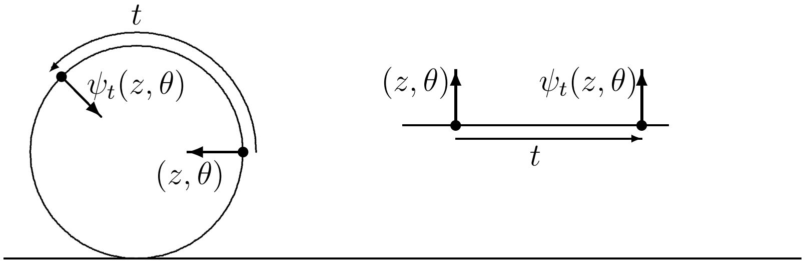
With this choice, a quick computation shows that the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathcal{H}$ by isometries extends to an action on $T_1\mathcal{H}$ through the formula:

$$T(z,\theta) := \big(T(z), \theta - 2\vartheta_T(z)\big), \tag{5.1}$$

where we have defined

$$\vartheta_T(z) = \arg(cz+d), \qquad \text{for} \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To each $(z,\theta) \in T_1\mathcal{H}$ we associate the unique horocycle $\mathfrak{h}$ which goes through the *point* $z \in \mathcal{H}$ and is orthogonal to (z,θ), such that the geodesic ray determined by (z,θ) approaches the point where $\mathfrak{h}$ touches $\mathbb{R} \cup \{\infty\}$. For $t \in \mathbb{R}$ we then define $\psi_t(z,\theta) \in T_1\mathcal{H}$ as the element obtained by moving (z,θ) around $\mathfrak{h}$ in the counterclockwise direction for a (hyperbolic) arclength t (keeping the orthogonality to $\mathfrak{h}$). This is illustrated in the following picture, in two cases:



It is clear that $\psi_0 \equiv Id$ and $\psi_t \circ \psi_s \equiv \psi_{s+t}$, i.e., ψ_t is a one-parameter flow on $T_1\mathcal{H}$. We call ψ_t the *horocycle flow.*

Notice that, since $\mathrm{PSL}(2,\mathbb{R})$ acts by isometries on $\mathcal{H}$ we have, for all $T \in \mathrm{PSL}(2,\mathbb{R})$,

$$\psi_t \circ T \equiv T \circ \psi_t : \quad T_1\mathcal{H} \to T_1\mathcal{H}. \tag{5.2}$$

The horocycle flow can be very neatly expressed using the standard identification between the unit tangent bundle $T_1\mathcal{H}$ and $\mathrm{PSL}(2,\mathbb{R})$. This identification is given by

$$\mathrm{PSL}(2,\mathbb{R}) \to T_1\mathcal{H}, \qquad U \quad \longmapsto \quad U(i,0) = \big(U(i), -2\vartheta_U(i)\big). \tag{5.3}$$ [17]

(Clearly then, the action of $T \in \mathrm{PSL}(2,\mathbb{R})$ on $T_1\mathcal{H}$ simply corresponds to $U \to TU$ on $\mathrm{PSL}(2,\mathbb{R})$.) Under the identification (5.3), the horocycle flow is given by

$$\psi_t : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R}), \qquad U \longmapsto U \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \tag{5.4}$$

This can be proved by a quick computation using (5.2), (5.3), and the fact that $\psi_t(i,0) = (i+t,0) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)(i,0)$.

Using (5.1), one easily verifies that the inverse of the identification map (5.3) is given by

$$T_1\mathcal{H} \to \mathrm{PSL}(2,\mathbb{R}), \qquad (x+iy,\theta) \quad \longmapsto \quad S^x \boldsymbol{a}(y)\boldsymbol{k}(-\theta/2), \tag{5.5}$$

where we use the following notation:

$$S^x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; \quad \boldsymbol{a}(y) := \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}; \quad \boldsymbol{k}(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}.$$ [18]

The formula (5.5) is of course closely related to the Iwasawa decomposition $\mathrm{PSL}(2,\mathbb{R}) = NAK$, according to which each $U \in \mathrm{PSL}(2,\mathbb{R})$ has a unique factorization

$$U = S^x \boldsymbol{a}(y)\boldsymbol{k}(\vartheta), \qquad x \in \mathbb{R}, \quad y > 0, \quad \vartheta \in \mathbb{R}/\pi\mathbb{Z}. \tag{5.6}$$

---

[17] In this connection, notice that there is a misprint in [S1, pp.721-722]: If "SH" and $\mathrm{SL}(2,\mathbb{R})$ are identified as stated in [S1, p.721(bot)], the horocycle flow does *not* correspond to $\psi_t(g) = g\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ on $\mathrm{SL}(2,\mathbb{R})$ as claimed on p.722(top).

[18] Thus $\boldsymbol{k}(\vartheta)$ depends on $\vartheta \in \mathbb{R}$ mod $2\pi\mathbb{Z}$ as a matrix in $\mathrm{SL}(2,\mathbb{R})$, but only on $\vartheta \in \mathbb{R}$ mod $\pi\mathbb{Z}$ as an element in $\mathrm{PSL}(2,\mathbb{R})$.

We define a volume measure ν on $T_1\mathcal{H}$ through

$$d\nu(z,\theta) := d\mu(z)\,d\theta = \frac{dx\,dy\,d\theta}{y^2}. \tag{5.7}$$

Under the identification with $\mathrm{PSL}(2,\mathbb{R})$, this measure equals

$$dU = 2\,\frac{dx\,dy\,d\vartheta}{y^2}, \tag{5.8}$$

where x, y, ϑ are the coordinates given by (5.6). This measure dU is in fact the left- and right-invariant Haar measure on $\mathrm{PSL}(2,\mathbb{R})$, in a somewhat non-standard normalization. It follows from the bi-invariance of the Haar measure that the volume measure ν is invariant under the action of $\mathrm{PSL}(2,\mathbb{R})$, and under the horocycle flow ψ_t.

We now let Γ be a cofinite Fuchsian group $\subset \mathrm{PSL}(2,\mathbb{R})$. Then Γ acts on $T_1\mathcal{H}$ through (5.1), and we obtain a quotient space $\boldsymbol{M} = \Gamma \setminus T_1\mathcal{H}$. Clearly, a (closed) fundamental domain is given by

$$\widetilde{\mathcal{F}} = \mathcal{F} \times (\mathbb{R}/2\pi\mathbb{Z}) \quad \subset \quad T_1\mathcal{H}.$$

It follows from this that $\nu(\boldsymbol{M}) = 2\pi\,\mu(\mathcal{F})$; cf. (5.7). (As usual, we write ν also for the measure induced by ν on $\Gamma \setminus T_1\mathcal{H}$, just as for μ on $\mathcal{H}$ and $\Gamma \setminus \mathcal{H}$.) If Γ does not contain any elliptic elements, then $\boldsymbol{M} = \Gamma \setminus T_1\mathcal{H}$ can be naturally identified with $T_1(\Gamma \setminus \mathcal{H})$, the unit tangent bundle of the hyperbolic surface $\Gamma \setminus \mathcal{H}$. (In the general case, a similar identification can be made after removing the finite set of elliptic fixpoints from $\Gamma \setminus \mathcal{H}$.)

Notice that in view of (5.2), ψ_t induces a well-defined flow *on* $\Gamma \setminus T_1\mathcal{H}$. This is the *horocycle flow on* $\boldsymbol{M}$, which we again denote by ψ_t.

It is well known (cf., e.g., [S1, §1]) that a point $p \in T_1\mathcal{H}$ belongs to a *closed orbit* of the horocycle flow on $\boldsymbol{M}$ if and only if it determines a closed horocycle $\mathfrak{h}$ encircling one of the κ cusps, i.e., if and only if

$$p \in \Gamma N_k^{-1}(z,0) \qquad \text{for some } k \in \{1, ..., \kappa\},\ z \in \mathcal{H}.$$

We let $\mathfrak{S}$ be the union of all closed orbits on $\boldsymbol{M}$, i.e.

$$\mathfrak{S} = \bigcup_{k=1}^{\kappa} \pi\big(N_k^{-1}(\mathcal{H} \times \{0\})\big) \quad \subset \boldsymbol{M}, \tag{5.9}$$

where π is the projection map $T_1\mathcal{H} \to \Gamma \setminus T_1\mathcal{H} = \boldsymbol{M}$. We will call $\mathfrak{S}$ the "singular set".

To conclude this section, we will now state the result on invariant measures — a special case of Dani's results in [D1, D2] — on which we will build our proof in §5.2. A convenient, "down to earth" reference for our particular case with $G = \mathrm{SL}(2,\mathbb{R})$ is Ratner's paper [R3].

Proposition 5.1.1. *Let ω be a Borel probability measure on $\boldsymbol{M}$ invariant under the horocycle flow (i.e., $\omega(\psi_t^{-1}(A)) = \omega(A)$ holds for each Borel set $A \subset \boldsymbol{M}$ and all $t \in \mathbb{R}$). Assume that $\omega(\mathfrak{S}) = 0$. Then ω is equal to $(2\pi\mu(\mathcal{F}))^{-1}\nu$, the unit normalized volume measure on $\boldsymbol{M}$.*

Proof. Clearly, under the identification (5.3), our space $\boldsymbol{M}$ corresponds to $\Gamma \setminus \mathrm{PSL}(2,\mathbb{R})$. Recall that ψ_t is then described through (5.4).

We let $G = \mathrm{SL}(2,\mathbb{R})$, $U = \{S^t \mid t \in \mathbb{R}\} \subset G$, and let $\overline{\Gamma}$ be the inverse image of Γ under the double covering $G \to \mathrm{PSL}(2,\mathbb{R})$. (In particular, $\pm \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \in \overline{\Gamma}$.)

Then, through the obvious identification $\Gamma \setminus \mathrm{PSL}(2,\mathbb{R}) = \overline{\Gamma} \setminus G$, we can view ω as a U-invariant Borel probability measure on $\overline{\Gamma} \setminus G$ (where U is acting from the right). By our definition of $\mathfrak{S}$, we know that ω vanishes on the union of all periodic orbits of U.

We now apply ergodic decomposition to ω (cf., e.g., [R4, §§0.5, 2.5, 2.6]). By [R3, Theorem 2] (or by [D1]), ω can have only one ergodic component, the (right)-G-invariant Borel probability measure on $\overline{\Gamma} \setminus G$, induced by the Haar measure on G. The proposition follows from this. □

5.2. **The theorem.**

Theorem 5.2.1. *Let Γ be a cofinite Fuchsian group with a normalized cusp (see (2.19)). Let $\gamma \in (0,1)$, and let $f : T_1\mathcal{H} \to \mathbb{C}$ be any continuous, Γ-invariant function satisfying the growth condition*

$$|f(z,\theta)| \leqq C\sqrt{\mathcal{Y}_\Gamma(z)} \qquad \forall (z,\theta) \in T_1\mathcal{H}, \tag{5.10}$$

for some positive constant C. Then, if $\gamma < \frac{1}{2}$,

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy,0)\,dx \to \frac{1}{2\pi\mu(\mathcal{F})}\int_{\mathcal{F}}\int_0^{2\pi} f(z,\theta)\,d\theta\,d\mu(z) \tag{5.11}$$

uniformly as $y \to 0$ so long as $\beta - \alpha$ remains bigger than y^γ.

If $\frac{1}{2} \leqq \gamma < 1$, we let $\gamma_1, \gamma_2, \mathcal{G}(y)$ be given as in Definition 4.2.1. Then (5.11) *holds uniformly as $y \to 0$ so long as $\alpha \in \mathcal{G}(y)$ and $\beta - \alpha \geqq y^\gamma$.*

Remark 5.2.2. Clearly, Theorem 5.2.1 is a generalization of both Theorem 2.4.1 and Theorem 4.5.1.

The proof of Theorem 5.2.1 will be a fairly standard application of Proposition 5.1.1 once we succeed in proving the following: If s is a horocycle segment $[\alpha + iy, \beta + iy] \times \{0\}$ as in Theorem 5.2.1, then it is not possible, as $y \to 0$, for a positive portion of s to vanish into some cusp. Nor is it possible for a positive portion of s to get concentrated along the singular set $\mathfrak{S}$.

These two facts are proved below, formulated in a precise way suitable for our application, in Lemma 5.2.3 and Lemma 5.2.4. Notice that the proofs of these two lemmas are perfectly "elementary", in that they do not rely on spectral results or ergodic theory.

In the following, to allow for statements which simultaneously cover the two cases $0 < \gamma < \frac{1}{2}$ and $\frac{1}{2} \leqq \gamma < 1$, we will agree that *if $0 < \gamma < \frac{1}{2}$, then $\mathcal{G}(y) := \mathbb{R}$ for all $y > 0$.*

Lemma 5.2.3. *Let $\Gamma, \gamma, \mathcal{G}(y)$ be as above. Then, for any $\varepsilon > 0$, there exist a number $y_0 > 0$ and a continuous, Γ-invariant function $f : T_1\mathcal{H} \to [0,1]$ which has* compact support *on $\boldsymbol{M} = \Gamma \setminus T_1\mathcal{H}$, such that*

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x+iy,0)\,dx \geqq 1-\varepsilon$$

for all $0 < y < y_0$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$.

Proof. This lemma can be viewed as a weaker version of Proposition 2.2.5 and Proposition 4.2.3 (on pp. 147, 184).

We fix a number $B \geqq \max(B_0, 10)$ so large that $40\kappa/\sqrt{B} \leqq \varepsilon$, and take $F : \mathbb{R}^+ \to [0,1]$ to be any continuous function such that $F(y) = 1$ on $(0, B]$ and $F(y) = 0$ on $[B+1, \infty)$. We then define $f(z,\theta) := F(\mathcal{Y}_\Gamma(z))$.

By Lemma 2.1.2, $f(z,\theta)$ is continuous and Γ-invariant. Furthermore, by (2.8) in the same lemma, we have $f(z,\theta) = 0$ for all $(z,\theta) \in \widetilde{\mathcal{F}} - \widetilde{\mathcal{F}}_{B+1}$, where $\widetilde{\mathcal{F}}_{B+1}$ denotes the bounded region $\mathcal{F}_{B+1} \times (\mathbb{R}/2\pi\mathbb{Z})$. Hence $f(z,\theta)$ has compact support on $\boldsymbol{M} = \Gamma \setminus T_1\mathcal{H}$.

Since $B \geqq 10 \geqq 1$, we clearly have

$$f(z,\theta) \geqq 1 - \sqrt{\lfloor \mathcal{Y}_\Gamma(z) \rfloor_B} \qquad \text{for all } (z,\theta) \in T_1\mathcal{H}.$$

Using this (and the fact that $40\kappa/\sqrt{B} \leqq \varepsilon$), the desired conclusion follows at once from (2.25) in Proposition 2.2.5 (if $\gamma < \frac{1}{2}$) or (4.14) in Proposition 4.2.3 (if $\gamma \geqq \frac{1}{2}$). □

In the next lemma, we will use the following notation for subsets of the singular set $\mathfrak{S}$ (cf. (5.9)):

$$\mathfrak{S}_{h,B} = \bigcup_{k=1}^{\kappa} \pi\big(N_k^{-1}(\mathbb{R} \times [h,B] \times \{0\})\big) \qquad \subset \mathfrak{S}.$$

Here h, B are any numbers such that $0 < h < B$.

Lemma 5.2.4. *Let $\Gamma, \gamma, \mathcal{G}(y)$ be as above. Then, for any numbers $0 < h < B$ and $\varepsilon > 0$, there exist a number $y_0 > 0$ and a continuous, Γ-invariant function $f : T_1\mathcal{H} \to [0,\infty)$ which has compact support on $\boldsymbol{M} = \Gamma \setminus T_1\mathcal{H}$, such that*

$$f(p) \geqq 1 \qquad \text{for all } p \in \mathfrak{S}_{h,B} \tag{5.12}$$

and

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta f(x+iy, 0)\, dx \leqq \varepsilon \tag{5.13}$$

for all $0 < y < y_0$, $\alpha \in \mathcal{G}(y)$, $\beta \geqq \alpha + y^\gamma$.

Proof. We fix a number $\eta \in (0, \frac{1}{10})$ so small that

$$\frac{16\eta}{h}\Big(1 + 8/\sqrt{h}\Big) < \frac{\varepsilon}{2\kappa}. \tag{5.14}$$

We then define, for $(z,\theta) \in T_1\mathcal{H}$,

$$\Psi(z,\theta) = t_1(\operatorname{Im} z) \cdot t_2(\theta), \tag{5.15}$$

where t_1, t_2 are some arbitrarily fixed continuous functions $t_1 : \mathbb{R}^+ \to [0,1]$ and $t_2 : \mathbb{R}/2\pi\mathbb{Z} \to [0,1]$ satisfying the following conditions:

$$\begin{cases} t_1(y) = 1 & \text{for } y \in [h,B]; \\ t_1(y) = 0 & \text{for } y \in (0, h/4] \cup [B+1, \infty); \end{cases} \tag{5.16}$$

$$\begin{cases} t_2(0) = 1; \\ t_2(\theta) = 0 & \text{for } \theta \notin [-\eta, \eta] + 2\pi\mathbb{Z}. \end{cases} \tag{5.17}$$

Now define, for each $k \in \{1, ..., \kappa\}$:

$$\Theta_k(p) = \sum_{U_0 \in [T_k] \setminus \Gamma} \Psi(N_k U_0(p)) \qquad (\text{where } p \in T_1\mathcal{H}). \tag{5.18}$$

Notice that this sum does not depend on the choice of the set of representatives $[T_k] \setminus \Gamma$, since $N_k T_k^n U_0 = S^{-n} N_k U_0$ for all $n \in \mathbb{Z}$, and

$$\Psi(S^{-n}(x+iy,\theta)) \equiv \Psi(x-n+iy,\theta) \equiv \Psi(x+iy,\theta).$$

Using $0 \leqq \Psi(z,\theta) \leqq t_1(\text{Im } z)$ and Lemma 4.3.1 (p. 186), we see that we have uniform (absolute) convergence over $p \in T_1\mathcal{H}$-compacta in (5.18). It now follows that $\Theta_k(p)$ is continuous. Also, $\Theta_k(p)$ is Γ-invariant, for if $T \in \Gamma$, then $U_0 T$ runs through a set of representatives $[T_k] \setminus \Gamma$ whenever U_0 does so.

We maintain that $\Theta_k(p)$ has compact support on $\boldsymbol{M} = \Gamma \setminus T_1\mathcal{H}$. More specifically, we claim that $\Theta_k(p) = 0$ for all $p \in \widetilde{\mathcal{F}} - \widetilde{\mathcal{F}}_{B'}$, where $B' = \max(B_0, 4/h, B+1)$ and $\widetilde{\mathcal{F}}_{B'} = \mathcal{F}_{B'} \times (\mathbb{R}/2\pi\mathbb{Z})$. To prove this, notice that if $(z,\theta) \in \widetilde{\mathcal{F}} - \widetilde{\mathcal{F}}_{B'}$, then there is some $j \in \{1, ..., \kappa\}$ such that $z \in \mathcal{C}_{jB'}$, i.e. $\text{Im } N_j(z) \geqq B'$. By (2.5) in Lemma 2.1.1, this implies, for each $U_0 \in \Gamma$,

$$\begin{aligned}
&\text{Im } N_k U_0(z) \leqq \big(\text{Im } N_j(z)\big)^{-1} \leqq B'^{-1} \leqq h/4 \\
&\textit{or } \; k = j, \; U_0 \in [T_k], \text{ and } \; \text{Im } N_k U_0(z) = \text{Im } N_k(z) \geqq B' \geqq B+1.
\end{aligned}$$

This forces $\Psi(N_k U_0(z,\theta)) = 0$ for all $U_0 \in \Gamma$, because of (5.15), (5.16). Hence, by (5.18), $\Theta_k(z,\theta) = 0$. This proves our claim.

We now let

$$f(p) \equiv \sum_{k=1}^{\kappa} \Theta_k(p). \tag{5.19}$$

Clearly, this function is continuous, Γ-invariant, and of compact support on $\boldsymbol{M} = \Gamma \setminus T_1\mathcal{H}$. Also, $f(p)$ trivially satisfies (5.12); for if $z \in \mathbb{R} \times [h,B] \subset \mathcal{H}$, then by considering only the term corresponding to $U_0 \in [T_k]$ in (5.18), we get:

$$f(N_k^{-1}(z,0)) \geqq \Theta_k(N_k^{-1}(z,0)) \geqq \Psi(z,0) = t_1(\text{Im } z) \cdot t_2(0) = 1.$$

It remains to prove (5.13). We have:

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\int_\alpha^\beta \Theta_k(x+iy,0)\,dx \\
&= \frac{1}{\beta-\alpha} \sum_{W_0 \in [S]\setminus N_k\Gamma} \int_\alpha^\beta \Psi\left(W_0(x+iy,0)\right)\,dx.
\end{aligned} \tag{5.20}$$

(Cf. (4.48) on p. 192 and its justification.) We now consider some arbitrary, temporarily fixed numbers y, α, β, restricted by $0 < y < \min(1, h/4)$ and $\alpha \in \mathcal{G}(y)$, $\beta - \alpha \geqq y^\gamma$. If $\gamma \geqq \frac{1}{2}$, we also impose the restriction that $y < y_1$, where $y_1 = y_1(h/4, \gamma_2) \in (0,1]$ is as in Lemma 4.2.4 on p. 184 (with $h/4$ in place of M).

Then, for any $W_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in [S] \setminus N_k\Gamma$ which gives a non-zero contribution to the right hand side in (5.20), there must exist some $x \in [\alpha, \beta]$ for which $\Psi(W_0(x+iy,0)) > 0$, and thus, by (5.16), $\text{Im } W_0(x+iy) > h/4$. Hence by Lemma 2.2.7 on p. 148 and (if $\gamma \geqq \frac{1}{2}$) Lemma 4.2.4, we have

$$A \leqq |c| \leqq \frac{2}{\sqrt{hy}}, \qquad \text{where } A = \begin{cases} 1 & \text{if } \gamma < \frac{1}{2} \\ \max\left(1, \frac{y^{\max(\gamma_2, \gamma-\gamma_1)}}{2(\beta-\alpha)}\right) & \text{if } \gamma \geqq \frac{1}{2}, \end{cases} \tag{5.21}$$

and

$$-\frac{d}{c} \in \left[\alpha - \frac{4(\beta-\alpha)}{\sqrt{h}}, \beta + \frac{4(\beta-\alpha)}{\sqrt{h}}\right]. \tag{5.22}$$

From the proof of Lemma 2.2.7 (cf. (2.27)) we also see that

$$\left|x+\frac{d}{c}\right| \leqq \frac{2\sqrt{y}}{\sqrt{h}|c|}. \tag{5.23}$$

Furthermore, from $\Psi(W_0(x+iy,0))>0$ and (5.17), (5.1), it follows that

$$\begin{aligned}
&2\arg\big(c(x+iy)+d\big) \in [-\eta,\eta]+2\pi\mathbb{Z} \\
&\Longrightarrow cx+d\neq 0 \,\wedge\, \arctan\left(\frac{|c|y}{|cx+d|}\right) \leqq \frac{\eta}{2} \qquad \left\{\begin{matrix}\text{We have } \tan(\eta/2) < \eta, \\ \text{since } \eta\in(0,\tfrac{1}{10}).\end{matrix}\right\} \\
&\Longrightarrow \frac{|c|y}{|cx+d|} < \eta \Longrightarrow y < \eta\left|x+\frac{d}{c}\right|.
\end{aligned}$$

Using (5.23) we now get $y<\eta\frac{2\sqrt{y}}{\sqrt{h}|c|}$, and hence,

$$|c| < \frac{2\eta}{\sqrt{hy}}. \tag{5.24}$$

We also note that for any $W_0=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in[S]\setminus N_k\Gamma$ which gives a non-zero contribution to the right hand side in (5.20), we certainly have

$$\int_\alpha^\beta \Psi\left(W_0(x+iy,0)\right)\,dx \leqq \frac{4\sqrt{y}}{\sqrt{h}|c|},$$

since $0\leqq\Psi(p)\leqq 1$ holds a priori, and $\Psi(W_0(x+iy,0))=0$ whenever (5.23) fails to hold.

In conclusion, we find that (5.20) is bounded from above by

$$\frac{1}{\beta-\alpha}\sum_{W_0}\frac{4\sqrt{y}}{\sqrt{h}|c|},$$

where the sum is taken over all $W_0=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in[S]\setminus N_k\Gamma$ satisfying $A\leqq|c|<2\eta/\sqrt{hy}$ and (5.22). If $A\geqq 2\eta/\sqrt{hy}$ then the sum is empty. If $A<2\eta/\sqrt{hy}$ we get the following upper bound, by Lemma 2.2.3 on p. 145,

$$\begin{aligned}
&\frac{4\sqrt{y}}{(\beta-\alpha)\sqrt{h}}\left(2(\beta-\alpha)\Big(1+8/\sqrt{h}\Big)\frac{2\eta}{\sqrt{hy}}+A^{-1}\right) \\
&=\frac{16\eta}{h}\Big(1+8/\sqrt{h}\Big)+\frac{4\sqrt{y}}{(\beta-\alpha)\sqrt{h}}A^{-1} \\
&\leqq\frac{16\eta}{h}\Big(1+8/\sqrt{h}\Big)+\frac{4}{\sqrt{h}}\left\{\begin{matrix} y^{\frac12-\gamma} & \text{if } \gamma<\frac12 \\ 2y^{\frac12-\max(\gamma_2,\gamma-\gamma_1)} & \text{if } \gamma\geqq\frac12\end{matrix}\right\}.
\end{aligned}$$

(Cf. (5.21).) Notice here that the second term tends to 0 as $y\to 0$ (cf. (4.9), if $\gamma\geqq\frac12$). Recall (5.14). Hence, if y is sufficiently small and $\alpha\in\mathcal{G}(y)$, $\beta-\alpha\geqq y^\gamma$, then for each $k\in\{1,\ldots,\kappa\}$ the expression in (5.20) is $\leqq\varepsilon/\kappa$. This implies (5.13), since $f\equiv\sum_{k=1}^{\kappa}\Theta_k$. $\square$

Proof of Theorem 5.2.1. Let $\Gamma,\gamma,\mathcal{G}(y)$ be given as in the statement of the theorem. (If $\gamma<\frac12$, we let $\mathcal{G}(y)=\mathbb{R}$ for all y.) We denote by $C_c(\boldsymbol{M})$ the space of all continuous functions $f:\boldsymbol{M}\to\mathbb{C}$ which have compact support; as usual, we will view the elements of $C_c(\boldsymbol{M})$ as Γ-invariant functions on $T_1\mathcal{H}$. We define a norm on $C_c(\boldsymbol{M})$ through $\|f\|:=\sup_{p\in\boldsymbol{M}}|f(p)|$.

Now fix some function $f_0\in C_c(\boldsymbol{M})$. *Assume that (5.11) does* not *hold for* $f=f_0$. This will be shown to lead to a contradiction.

Explicitly, the assumption just made means that there exist some $\varepsilon_0 > 0$, a sequence $y_1 > y_2 > y_3 > \ldots$ with $\lim_{n\to\infty} y_n = 0$, and some sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ such that, for all $n \geqq 1$:

$$\begin{aligned}&\alpha_n \in \mathcal{G}(y_n), \qquad \beta_n - \alpha_n \geqq y_n^{\gamma}, \qquad \text{and}\\ &\left|\frac{1}{\beta_n - \alpha_n}\int_{\alpha_n}^{\beta_n} f_0(x+iy_n,0)\,dx \right.\\ &\qquad\qquad\left. - \frac{1}{2\pi\mu(\mathcal{F})}\int_{\mathcal{F}}\int_0^{2\pi} f_0(z,\theta)\,d\theta\,d\mu(z)\right| > \varepsilon_0.\end{aligned} \tag{5.25}$$

For each $n \geqq 1$, we define a linear functional Λ_n on $C_c(\boldsymbol{M})$ through

$$\Lambda_n f := \frac{1}{\beta_n - \alpha_n}\int_{\alpha_n}^{\beta_n} f(x+iy_n,0)\,dx, \qquad \text{for all } f \in C_0(\boldsymbol{M}). \tag{5.26}$$

Clearly, Λ_n is bounded, of norm $||\Lambda_n|| \leqq 1$.

By passing to a subsequence, we may assume[19] that $\{\Lambda_n\}$ is weakly convergent, i.e. the limit

$$\Lambda f = \lim_{n\to\infty} \Lambda_n f \tag{5.27}$$

exists for every $f \in C_c(\boldsymbol{M})$. Then Λ must be a linear functional of norm $||\Lambda|| \leqq 1$ on $C_c(\boldsymbol{M})$.

Clearly, we have $\Lambda f \geqq 0$ for each non-negative function $f \in C_c(\boldsymbol{M})$, i.e. Λ is a *positive* functional. Hence by Riesz Representation Theorem (cf., e.g., [R5, Theorem 2.14]), there is a unique regular Borel measure ω on $\boldsymbol{M}$ such that

$$\Lambda f = \int_{\boldsymbol{M}} f\,d\omega \qquad \text{for each } f \in C_c(\boldsymbol{M}).$$

We have $\omega(\boldsymbol{M}) \leqq 1$, since $||\Lambda|| \leqq 1$ (cf. [R5, p. 41(1)]). On the other hand, for any given $\varepsilon > 0$ we have, if we take $f \in C_c(\boldsymbol{M})$ to be a function satisfying the conditions in Lemma 5.2.3,

$$\begin{aligned}\omega(\boldsymbol{M}) \geqq \int_{\boldsymbol{M}} f\,d\omega = \Lambda f = \lim_{n\to\infty}\Lambda_n f&\\ = \lim_{n\to\infty}\frac{1}{\beta_n - \alpha_n}\int_{\alpha_n}^{\beta_n} f(x+iy_n,0)\,dx \geqq 1-\varepsilon.&\end{aligned}$$

Hence $\omega(\boldsymbol{M}) = 1$, i.e. ω is a probability measure.

Furthermore, for any given numbers $0 < h < B$ and $\varepsilon > 0$ we have, if we now take $f \in C_c(\boldsymbol{M})$ to be a function satisfying the conditions in Lemma 5.2.4,

$$\omega(\mathfrak{S}_{h,B}) = \int_{\mathfrak{S}_{h,B}} d\omega \leqq \int_{\boldsymbol{M}} f\,d\omega = \lim_{n\to\infty}\frac{1}{\beta_n - \alpha_n}\int_{\alpha_n}^{\beta_n} f(x+iy_n,0)\,dx \leqq \varepsilon.$$

Hence $\omega(\mathfrak{S}_{h,B}) = 0$, and using $\mathfrak{S} = \cup_{m=2}^{\infty}\mathfrak{S}_{1/m,m}$ we obtain

$$\omega(\mathfrak{S}) = 0.$$

We claim that the measure ω is invariant under the horocycle flow. To prove this, it suffices to prove that $\Lambda(f\circ\psi_t) = \Lambda f$ for all $f \in C_c(\boldsymbol{M})$ and all $t \in \mathbb{R}$. But notice that

---

[19] Cf. [R5, Theorem 11.28], with $X = C_c(\boldsymbol{M})$. Notice that $C_c(\boldsymbol{M})$ is separable; cf. [C, pp. 225(ex. 8), 348(D.33)] and also [D3, (7.4.4)].

$$\begin{aligned}\Lambda_n(f\circ\psi_t) &= \frac{1}{\beta_n-\alpha_n}\int_{\alpha_n}^{\beta_n}[f\circ\psi_t](x+iy_n,0)\,dx\\ &= \frac{1}{\beta_n-\alpha_n}\int_{\alpha_n}^{\beta_n} f(x+ty_n+iy_n,0)\,dx\\ &= \frac{1}{\beta_n-\alpha_n}\int_{\alpha_n+ty_n}^{\beta_n+ty_n} f(x+iy_n,0)\,dx.\end{aligned}$$

Hence, for fixed f, t, we have

$$\begin{aligned}&|\Lambda(f\circ\psi_t)-\Lambda f| = \lim_{n\to\infty}|\Lambda_n(f\circ\psi_t)-\Lambda_n f|\\ &\leqq \limsup_{n\to\infty}\frac{1}{\beta_n-\alpha_n}\left[\int_{\alpha_n}^{\alpha_n+ty_n}+\int_{\beta_n}^{\beta_n+ty_n}\right]|f(x+iy_n,0)|\,|dx|\\ &\leqq \limsup_{n\to\infty}\frac{2ty_n\,||f||}{\beta_n-\alpha_n}=0,\end{aligned}$$

since $\beta_n-\alpha_n \geqq y_n^{\gamma}$, $\gamma<1$. This proves our claim.

By Dani's classification of invariant measures, Proposition 5.1.1, we can now conclude that $\omega=(2\pi\mu(\mathcal{F}))^{-1}\nu$. Hence, in particular, we have for the function f_0 that we started with:

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\beta_n-\alpha_n}\int_{\alpha_n}^{\beta_n} f_0(x+iy_n,0)\,dx = \lim_{n\to\infty}\Lambda_n f_0=\Lambda f_0=\int_{\boldsymbol{M}} f_0\,d\omega\\ &=\frac{1}{2\pi\mu(\mathcal{F})}\int_{\boldsymbol{M}} f_0\,d\nu=\frac{1}{2\pi\mu(\mathcal{F})}\int_{\mathcal{F}}\int_0^{2\pi} f_0(z,\theta)\,d\theta\,d\mu(z).\end{aligned}$$

(Cf. (5.7).) This is a contradiction against (5.25).

Hence, we have proved that (5.11) must hold for any function $f\in C_c(\boldsymbol{M})$. To complete the proof, we now have to show how to treat arbitrary functions $f\in C(\boldsymbol{M})$ restricted only by (5.10). This is very similar to the second part of the proof of Theorem 2.3.1 (pp. 160–161) but slightly easier, since we never need to bother about smoothness of any kind (beyond mere continuity).

Specifically, we consider any $f\in C(\boldsymbol{M})$ satisfying (5.10), and let $\varepsilon>0$ be given. We fix M to be a number so large that

$$M>\max(B_0,10) \qquad\text{and}\qquad \frac{C_1C}{\sqrt{M}}+\frac{40\kappa C}{\sqrt{M}}\leqq\varepsilon,$$

where C and C_1 are as in (5.10) and (2.65), respectively. We then let $F:\mathbb{R}^+\to[0,1]$ be a continuous function such that $F(y)=1$ for $0<y\leqq M$ and $F(y)=0$ for $M+1\leqq y$, and define

$$f_1(z,\theta)=F(\mathcal{Y}_\Gamma(z))\cdot f(z,\theta).$$

Then $f_1\in C_c(\boldsymbol{M})$, since $f_1(z,\theta)=0$ whenever $(z,\theta)\in\widetilde{\mathcal{F}}-\widetilde{\mathcal{F}}_{M+1}$. In view of (5.10), we have

$$|f(z,\theta)-f_1(z,\theta)|\leqq C\sqrt{\lfloor\mathcal{Y}_\Gamma(z)\rfloor_M}\qquad\forall(z,\theta)\in T_1\mathcal{H}.$$

Since f_1 has compact support, it follows from what we have proved above that f_1 satisfies (5.11)! By mimicing the last paragraphs in the proof of Theorem 2.3.1

(where, if $\gamma \geqq \frac{1}{2}$, we use Proposition 4.2.3 instead of Proposition 2.2.5), we now get that

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy, 0)\, dx - \frac{1}{2\pi\mu(\mathcal{F})} \int_{\mathcal{F}} \int_0^{2\pi} f(z, \theta)\, d\theta d\mu(z) \right| \leqq 2\varepsilon$$

for all sufficiently small y, and all $\alpha \in \mathcal{G}(y)$ and $\beta \geqq \alpha + y^\gamma$. Since ε was arbitrary, this concludes the proof of Theorem 5.2.1. □ □ □

Remark 5.2.5. There is a general equidistribution theorem by Dani and Margulis, [DM, Theorem 3], involving an arbitrary Ad-unipotent one-parameter flow on a connected Lie group; its proof depends on Ratner's results in [R2]. A special case of this theorem implies a result which is similar to Theorem 5.2.1 above.

Specifically, in the particular case where "G/Γ" and "U" are taken to be $\mathrm{SL}(2, \mathbb{R})/\overline{\Gamma}$ and $\{S^t \mid t \in \mathbb{R}\}$ respectively, the theorem by Dani and Margulis is equivalent to the following statement[20]:

Let f be any bounded continuous function on $\boldsymbol{M} = \Gamma \backslash T_1\mathcal{H}$. Let K be a compact subset of $\boldsymbol{M}$ and let $\varepsilon > 0$ be given. Then there exists a compact subset C of $\mathfrak{S}$ for which the following holds: For any compact subset F of $K - C$ there exists a $T_0 \geqq 0$ such that for all $p \in F$ and $T > T_0$,

$$\left| \frac{1}{T} \int_0^T f(\psi_t p)\, dt - \frac{1}{\nu(\boldsymbol{M})} \int_{\boldsymbol{M}} f\, d\nu \right| < \varepsilon.$$

To see the connection with Theorem 5.2.1, notice that in the above situation we have, by taking $p = \pi(\alpha + iy, 0)$ and $T = (\beta - \alpha)/y$,

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta f(x + iy, 0)\, dx - \frac{1}{\nu(\boldsymbol{M})} \int_{\boldsymbol{M}} f\, d\nu \right| < \varepsilon, \tag{5.28}$$

anytime $\pi(\alpha + iy, 0) \in F$ and $\beta - \alpha > T_0 y$.

Also in the above result, given ε, we may take K as the closure of $\pi(\widetilde{\mathcal{F}}_B)$ in $\boldsymbol{M}$, where B is so large that $\nu(K) > (1 - \varepsilon)\nu(\boldsymbol{M})$. Then, since C is compact and $\nu(C) \leqq \nu(\mathfrak{S}) = 0$, it is possible to take F to be a "nice" regular compact set of volume $\nu(F) > (1 - 2\varepsilon)\nu(\boldsymbol{M})$. Then, since the *full* closed horocycle $\{(x + iy, 0) \mid x \in [0, 1]\}$ is well known to become equidistributed on $\boldsymbol{M}$ as $y \to 0$ (by, e.g., [S1]), we have, for all sufficiently small y,

$$m\big(\{\alpha \in [0, 1] \mid \pi(\alpha + iy, 0) \in F\}\big) > 1 - 3\varepsilon.$$

The conclusion is that *whenever y is sufficiently small (as depends on Γ, f, ε),* (5.28) *holds for "ε-almost all" α, and all $\beta > \alpha + T_0(\Gamma, f, \varepsilon) \cdot y$.*

---

[20] Concerning $\overline{\Gamma}$, cf. the proof of Proposition 5.1.1. Also, in [DM, Theorem 3], note that *any* proper closed subgroup H of $G = \mathrm{SL}(2, \mathbb{R})$ for which $X(H, U) \neq \emptyset$ and for which $H \cap \overline{\Gamma}$ is a lattice in H, must be of the form $H = T^{-1}N_j^{-1}U'N_jT$ for some $j \in \{1, ..., \kappa\}$ and $T \in \overline{\Gamma}$, where either $U' = U$ or $U' = \{\pm S^t \mid t \in \mathbb{R}\}$. For such H, we get $X(H, U)\overline{\Gamma}/\overline{\Gamma} = PN_j\overline{\Gamma}/\overline{\Gamma}$, where P is the subgroup of all upper triangular matrices in G. Finally, we alter the set-up in [DM, Theorem 3] through the inverse map $g \to g^{-1}$, and use the identification $\overline{\Gamma} \setminus G = \boldsymbol{M}$.

6. Appendix 1, proof of an approximation result

On p. 161, in the proof of Theorem 2.3.1, we used the following result:

Given any function $f_0(z) \in C_c(\Gamma \setminus \mathcal{H})$, *and any* $\varepsilon > 0$, *there is some* $f_1(z) \in C^\infty(\mathcal{H}) \cap C_c(\Gamma \setminus \mathcal{H})$ *such that* $|f_1(z) - f_0(z)| \leqq \varepsilon$ *for all* $z \in \mathcal{H}$.

We now give a simple proof of this.

Since $f_0(z)$ has compact support on $\Gamma \setminus \mathcal{H}$, there is some number $B > B_0$ such that $f_0(z) = 0$ for all $z \in \mathcal{F} - \mathcal{F}_B$. We then have, on all of $\mathcal{H}$: $\mathcal{Y}_\Gamma(z) \geqq B \implies f_0(z) = 0$. This is proved by considering the $\mathcal{F}$-pullback of z and using (2.8) in Lemma 2.1.2.

Let $\rho(z,w)$ denote hyperbolic distance. We can now find some number $\rho_0 > 0$ such that

$$\forall z, w \in \mathcal{H}: \qquad \rho(z,w) \leqq \rho_0 \implies |f_0(z) - f_0(w)| \leqq \varepsilon.$$

(For assume that this is *not* possible. Then there are sequences $\{z_n\}_{n=1}^\infty$, $\{w_n\}_{n=1}^\infty$ such that $|f_0(z_n) - f_0(w_n)| > \varepsilon$ for all n, and $\rho(z_n, w_n) \to 0$ as $n \to 0$. For each n, replacing $\langle z_n, w_n \rangle$ by $\langle T(z_n), T(w_n) \rangle$ or $\langle T(w_n), T(z_n) \rangle$ for some suitable $T \in \Gamma$, we can make $f_0(z_n) \neq 0$ and $z_n \in \mathcal{F}$ hold. Then all points z_n must lie in $\mathcal{F}_B$, and passing to a subsequence, we can make z_n converge to some point in $\overline{\mathcal{F}_B}$ as $n \to \infty$. This quickly leads to a contradiction.)

We choose $\Phi(t)$ to be an arbitrary non-negative function in $C_c^\infty([0,\infty))$, not identically 0, with support such that $k(z,w) = \Phi\big(\frac{|z-w|^2}{\operatorname{Im}(z)\operatorname{Im}(w)}\big)$ vanishes whenever $\rho(z,w) \geqq \rho_0$. By multiplying Φ with a constant, we can also ensure that $\int_{\mathcal{H}} k(i,w)\, d\mu(w) = 1$, and hence[21] $\int_{\mathcal{H}} k(z,w)\, d\mu(w) = 1$ for all $z \in \mathcal{H}$. We define

$$f_1(z) = \int_{\mathcal{H}} k(z,w)\, f_0(w)\, d\mu(w) = \int_{\{w\,:\,\rho(z,w)<\rho_0\}} k(z,w)\, f_0(w)\, d\mu(w). \tag{6.1}$$

We then have $f_1(z) \in C^\infty(\mathcal{H}) \cap C_c(\Gamma \setminus \mathcal{H})$. In fact, $f_1(z) \in C^\infty(\mathcal{H})$ follows easily by differentiation under the integration sign in (6.1), and Γ-invariance follows from [H2, pp. 13–14]. To prove that $f_1(z)$ has compact support on $\Gamma \setminus \mathcal{H}$, we let $B_1 = e^{\rho_0} B$, and consider any $k \in \{1, ..., \kappa\}$, $z \in \mathcal{C}_{kB_1}$. Then for all points $w \in \mathcal{H}$ such that $\rho(z,w) \leqq \rho_0$, we get $\operatorname{Im} N_k(w) \geqq e^{-\rho_0} B_1 = B$ (cf. (2.11) and (2.2)), and hence $\mathcal{Y}_\Gamma(w) \geqq B$ and $f_0(w) = 0$. Hence by (6.1), $f_1(z) = 0$. Since this holds for all $z \in \cup_{k=1}^{\kappa} \mathcal{C}_{kB_1}$, we have $f_1 \in C_c(\Gamma \setminus \mathcal{H})$, as claimed.

Finally, for all $z \in \mathcal{H}$ we have:

$$\begin{aligned} |f_1(z) - f_0(z)| &= \left| \int_{\mathcal{H}} k(z,w)\, (f_0(w) - f_0(z))\; d\mu(w) \right| \\ &\leqq \int_{\{w\,:\,\rho(z,w)<\rho_0\}} k(z,w)\; |f_0(w) - f_0(z)|\; d\mu(w) \\ &\leqq \int_{\{w\,:\,\rho(z,w)<\rho_0\}} k(z,w)\, \varepsilon\, d\mu(w) = \varepsilon. \end{aligned}$$

This conludes the proof. □

---

[21] Cf. eg. [H2, p. 8(Prop. 3.1, with $f \equiv 1$)].

7. APPENDIX 2, SOME REMARKS ON THE FAREY SERIES

We remarked on p. 207 that Corollary 4.6.4, which concerns the asymptotic distribution of Farey points, can be proved in an alternative, comparatively elementary way. In this appendix we will justify this claim. We will also discuss a natural question closely related to Corollary 4.6.4, which will be seen to be directly connected with the Riemann hypothesis.

To start with, we keep $\delta < 1$. We will show that in this case, Corollary 4.6.4 is a trivial consequence of a result by Niederreiter, [N2]. (Notice that the arguments in [N2] are completely elementary, except for the use of the estimate $M(x) := \sum_{n\leqq x}\mu(n) = O(x/\log^2 x)$.)

We introduce the following notation, for $N \in \mathbb{Z}^+$, $\alpha \in [0,1]$:

$$\Phi(N) := \big|\mathfrak{F}_N - \{0\}\big| = \sum_{n=1}^{N}\phi(n) \quad \text{(where } \phi(n) \text{ is Euler's function)};$$

$$A_N(\alpha) := \big|\mathfrak{F}_N \cap (0,\alpha]\big|; \tag{7.1}$$

$$D_N(\alpha) := A_N(\alpha) - \Phi(N)\alpha. \tag{7.2}$$

By [N2, (4)][22] we have $\left|\frac{A_N(\alpha)}{\Phi(N)} - \alpha\right| = O(\Phi(N)^{-1/2})$ for all $N \in \mathbb{Z}^+$, $\alpha \in [0,1]$. (The implied constant is absolute.) Using the well known fact that $\Phi(N) \sim \frac{3}{\pi^2}N^2$ as $N \to \infty$ (cf. [L3, p. 120]), the same estimate can also be stated as

$$|D_N(\alpha)| = O(N), \qquad \forall N \in \mathbb{Z}^+,\ \alpha \in [0,1]. \tag{7.3}$$

We now have, for all $N \in \mathbb{Z}^+$, $\alpha, \beta \in [0,1]$, $\alpha < \beta$:

$$\begin{aligned} \frac{1}{N^2(\beta-\alpha)} \cdot \Big|\mathfrak{F}_N \cap (\alpha,\beta]\Big| &= \frac{A_N(\beta) - A_N(\alpha)}{N^2(\beta-\alpha)} \\ &= \frac{\Phi(N)}{N^2} + \frac{D_N(\beta) - D_N(\alpha)}{N^2(\beta-\alpha)} \\ &= \frac{\Phi(N)}{N^2} + O\left(\frac{1}{N(\beta-\alpha)}\right). \end{aligned} \tag{7.4}$$

Clearly, this tends to $\frac{3}{\pi^2}$ uniformly as $N \to \infty$ so long as $\beta - \alpha \geqq N^{-\delta}$. This concludes the proof of Corollary 4.6.4 when $\delta < 1$. (Of course, the difference between "$(\alpha,\beta]$" in (7.4) and "$[\alpha,\beta]$" in (4.92) is immaterial, since $\mathfrak{F} \cap (\alpha,\beta]$ and $\mathfrak{F} \cap [\alpha,\beta]$ differ by at most one point, and the denominator $N^2(\beta-\alpha)$ is $\geqq N^{2-\delta} \geqq N$.) □

We now turn to the case $\delta \in [1,2)$. Our proof of Corollary 4.6.4 in this case is based on a precise estimate by Neville [N1, Thms. 8, 9] of the number of Farey points in $\mathfrak{F}_N$ lying between two given, consecutive Farey points in $\mathfrak{F}_M$ $(M < N)$.

We let δ_1, δ_2 be fixed as in Corollary 4.6.4. Also, we choose, once and for all, some number $\delta_3 \in \big(\max(\delta_2, \delta - \delta_1), 1\big)$.

We temporarily fix some $N \in \mathbb{Z}^+$, $\alpha \in \mathbb{G}(N)$ and $\beta \in [0,1]$ such that $\beta - \alpha \geqq N^{-\delta}$. We assume that N is so large that

$$\begin{cases} N^{\max(\delta_2,\delta-\delta_1)-\delta_3} \leqq \frac{1}{100} & \text{(and thus } N^{\delta_1} \geqq N^{-\delta+\delta_1+\delta_3} \geqq 100); \\ N^{1-\delta_3} \geqq 2. \end{cases} \tag{7.5}$$

---

[22] Notice that the notation in [N2] differs from ours; for our application, "N" in [N2] should be taken to equal our $\Phi(N)$.

We then let $\frac{h_1}{k_1} < \frac{h_2}{k_2} < ... < \frac{h_m}{k_m}$ be the subsequence of consecutive fractions *in* $\mathfrak{F}_{[N^{\delta_3}]}$ selected so that $\frac{h_1}{k_1} \leqq \alpha < \frac{h_2}{k_2}$ and $\frac{h_{m-1}}{k_{m-1}} < \beta \leqq \frac{h_m}{k_m}$. Clearly then, $m \geqq 2$.

Our first aim is to show that all differences $\frac{h_{j+1}}{k_{j+1}} - \frac{h_j}{k_j}$ are small compared to $\beta - \alpha$. For this, we will use the assumption that $\alpha \in \mathbb{G}(N)$.

For each $j \in \{1, ..., m-1\}$ we know by [HW, Thm. 30] that $k_j + k_{j+1} > [N^{\delta_3}]$ (i.e., $k_j + k_{j+1} > N^{\delta_3}$); hence

$$\max(k_j, k_{j+1}) > \frac{1}{2} N^{\delta_3}, \quad \text{and} \tag{7.6}$$

$$\frac{h_{j+1}}{k_{j+1}} - \frac{h_j}{k_j} = \frac{h_{j+1}k_j - h_j k_{j+1}}{k_j k_{j+1}} = \frac{1}{k_j k_{j+1}} < \frac{2}{N^{\delta_3} \min(k_j, k_{j+1})}. \tag{7.7}$$

(Cf. [HW, Thm. 28].)

We now claim:

$$k_j \geqq \frac{N^{-\max(\delta_2, \delta - \delta_1)}}{2(\beta - \alpha)} \qquad \text{for each } \; j \in \{1, ..., m\}. \tag{7.8}$$

If $k_j \geqq [N^{\delta_1}]$, then (7.8) is clear from $[N^{\delta_1}] > \frac{1}{2} N^{\delta_1}$ (cf. (7.5)) and $\beta - \alpha \geqq N^{-\delta}$.

Next, assume $k_j < [N^{\delta_1}]$. Then $k_j < N^{\delta_1} \leqq N^{\delta_2} \leqq \frac{1}{100} N^{\delta_3}$ (cf. (7.5)). If $j = m$, we now obtain from (7.6), (7.7) that $\frac{h_m}{k_m} - \frac{h_{m-1}}{k_{m-1}} < 2/(N^{\delta_3} k_m)$; if $j = 1$ we similarly get $\frac{h_2}{k_2} - \frac{h_1}{k_1} < 2/(N^{\delta_3} k_1)$. Recall $\alpha < \frac{h_2}{k_2}$ and $\frac{h_{m-1}}{k_{m-1}} < \beta$. Hence, whatever the value of $j \in \{1, ..., m\}$ is, we certainly have

$$\left| \frac{h_j}{k_j} - \alpha \right| < (\beta - \alpha) + \frac{2}{N^{\delta_3} k_j}. \tag{7.9}$$

On the other hand, $k_j < [N^{\delta_1}]$ implies $\frac{h_j}{k_j} \in \mathfrak{F}_{[N^{\delta_1}]}$, and hence, using $\alpha \in \mathbb{G}(N)$ and (4.93),

$$\left| \frac{h_j}{k_j} - \alpha \right| > \frac{N^{-\delta_2}}{k_j}.$$

Combining this with (7.9) and $N^{-\delta_3} < \frac{1}{100} N^{-\delta_2}$, we get $(1 - \frac{1}{50}) N^{-\delta_2}/k_j < \beta - \alpha$, i.e. (7.8) holds.

Now, for all $j \in \{1, ..., m-1\}$, (7.8) and (7.7) imply that

$$\frac{h_{j+1}}{k_{j+1}} - \frac{h_j}{k_j} < 4(\beta - \alpha) N^{\max(\delta_2, \delta - \delta_1) - \delta_3}. \tag{7.10}$$

In particular, adding these inequalities over $j = 1, ..., m-1$ and using $\beta - \alpha \leqq \frac{h_m}{k_m} - \frac{h_1}{k_1}$ and (7.5), we see that $\beta - \alpha < \frac{4}{100}(\beta - \alpha)(m-1)$, i.e.

$$m > 26. \tag{7.11}$$

We will now count the number of $\mathfrak{F}_N$-points in each interval $(\frac{h_j}{k_j}, \frac{h_{j+1}}{k_{j+1}})$ separately. By [N1, Thms. 8,9][23] we have

$$\begin{aligned}&\left|\mathfrak{F}_N \cap \left(\tfrac{h_j}{k_j}, \tfrac{h_{j+1}}{k_{j+1}}\right)\right| \\ &= \frac{3}{\pi^2}\frac{N^2}{k_j k_{j+1}}\left(1 + O(1)\max\left(\frac{\log(N/k_j)}{N/k_j}, \frac{\log(N/k_{j+1})}{N/k_{j+1}}\right)\right) \\ &= \frac{3}{\pi^2}\frac{N^2}{k_j k_{j+1}}\left(1 + O(N^{\delta_3 - 1 + \varepsilon})\right). \qquad \left\{\text{Since } k_j, k_{j+1} \leqq [N^{\delta_3}].\right\}\end{aligned}$$

(Here ε denotes some arbitrary, fixed number strictly between 0 and $1 - \delta_3$. In the first "big O", the implied constant is absolute; in the second one it depends only on ε, δ_3.) Adding one more Farey point does not change the right hand side, since $\frac{N^2}{k_j k_{j+1}} N^{\delta_3 - 1 + \varepsilon} \geq N^{1-\delta_3+\varepsilon} \geqq 1$. Using $\frac{1}{k_j k_{j+1}} = \frac{h_{j+1}}{k_{j+1}} - \frac{h_j}{k_j}$, we get

$$\left|\mathfrak{F}_N \cap \left[\tfrac{h_j}{k_j}, \tfrac{h_{j+1}}{k_{j+1}}\right)\right| = \frac{3}{\pi^2}N^2\left(\frac{h_{j+1}}{k_{j+1}} - \frac{h_j}{k_j}\right)\left(1 + O(N^{\delta_3 - 1 + \varepsilon})\right).$$

Adding this over $j = 1, 2, ..., m-1$ (and adding yet another Farey point), we obtain

$$\left|\mathfrak{F}_N \cap \left[\tfrac{h_1}{k_1}, \tfrac{h_m}{k_m}\right]\right| = \frac{3}{\pi^2}N^2\left(\frac{h_m}{k_m} - \frac{h_1}{k_1}\right)\left(1 + O(N^{\delta_3 - 1 + \varepsilon})\right).$$

Exactly the same type of formula can also be obtained for $\left|\mathfrak{F}_N \cap \left[\frac{h_2}{k_2}, \frac{h_{m-1}}{k_{m-1}}\right]\right|$ (since $m - 1 \geqq 3$; cf. (7.11)). Hence, since $\left[\frac{h_2}{k_2}, \frac{h_{m-1}}{k_{m-1}}\right] \subset [\alpha, \beta] \subset \left[\frac{h_1}{k_1}, \frac{h_m}{k_m}\right]$, we have:

$$\begin{aligned}&\frac{3}{\pi^2}N^2\left(\frac{h_{m-1}}{k_{m-1}} - \frac{h_2}{k_2}\right)\left(1 + O(N^{\delta_3 - 1 + \varepsilon})\right) \\ &\qquad \leqq \left|\mathfrak{F}_N \cap [\alpha, \beta]\right| \leqq \frac{3}{\pi^2}N^2\left(\frac{h_m}{k_m} - \frac{h_1}{k_1}\right)\left(1 + O(N^{\delta_3 - 1 + \varepsilon})\right).\end{aligned}$$

But it follows from (7.10) (with $j = 1$ and $j = m - 1$) and $\frac{h_1}{k_1} \leqq \alpha < \frac{h_2}{k_2}$, $\frac{h_{m-1}}{k_{m-1}} < \beta \leqq \frac{h_m}{k_m}$ that

$$\left|\frac{h_1}{k_1} - \alpha\right|, \left|\frac{h_2}{k_2} - \alpha\right|, \left|\frac{h_{m-1}}{k_{m-1}} - \beta\right|, \left|\frac{h_m}{k_m} - \beta\right| < 4(\beta - \alpha)N^{\max(\delta_2, \delta - \delta_1) - \delta_3}.$$

Hence:

$$\left|\mathfrak{F}_N \cap [\alpha, \beta]\right| = \frac{3}{\pi^2}N^2(\beta - \alpha)\left(1 + O(N^{\max(\delta_2, \delta - \delta_1) - \delta_3})\right)\left(1 + O(N^{\delta_3 - 1 + \varepsilon})\right).$$

This holds for all positive integers N satisfying (7.5), and all $\alpha \in \mathbb{G}(N)$, $\beta \in [0, 1]$ such that $\beta - \alpha \geqq N^{-\delta}$, and the implied constants depend only on ε, δ_3. (The same formula can also be obtained in the case $\alpha \in [0, 1]$, $\beta \in \mathbb{G}(N)$, $\beta - \alpha \geqq N^{-\delta}$, by repeating the above proof with a small number of very simple changes around (7.9).) Clearly, Corollary 4.6.4 (for $\delta \in [1, 2)$) follows from this. $\square$

We will conclude this appendix by a discussion of a question which arises quite naturally when contemplating the two proofs given above.

---

[23] Note that Theorem 9 in [N1] holds uniformly for all real numbers $l, m \geqq 1 + \varepsilon$. This condition is fulfilled in our application; $N/k_j, N/k_{j+1} \geqq N^{1-\delta_3} \geqq 2$, cf. (7.5). (A minor remark: "$\Phi(l, l) = \Phi(l) = ...$" in [N1, p. 137, line 21] should be corrected to "$\Phi(l, l) = \Phi(l) - 1 = ...$".)

Let us order the elements in $\mathfrak{F}_N$ as $0 = r_0 < r_1 < \ldots < r_{\Phi(N)} = 1$. Notice that the above proof for $\delta \in [1,2)$ is of a much more "local" nature than the one for $\delta \in (0,1)$, in so far as the proof for $\delta \in [1,2)$ only uses properties of those Farey points which lie in a small neighbourhood of $[\alpha,\beta]$. In particular, if the Farey points in $[\alpha,\beta]$ are $r_m, r_{m+1}, \ldots, r_n$, say, the proof for $\delta \in [1,2)$ leaves open the possibility that *all* these points may be far displaced versus the corresponding points $\frac{m}{\Phi(N)}, \frac{m+1}{\Phi(N)}, \ldots, \frac{n}{\Phi(N)}$ in a uniformly distributed sequence.

In view of this, it is only natural to ask whether our method for $\delta \in (0,1)$ —where we used the overestimate $\left|D_N(\beta) - D_N(\alpha)\right| \leqq \left|D_N(\alpha)\right| + \left|D_N(\beta)\right|$— can be made to work in some reasonable way also for $\delta \in [1,2)$. We will show that in a certain sense, this is *not* possible for any $\delta > \frac{3}{2}$, and that it is possible for *all* $\delta < \frac{3}{2}$ *if and only if the Riemann hypothesis is true.*

Of course, the key question is whether "$O(N)$" in (7.3) can be replaced by "$O(N^{\nu})$" for some $\nu < 1$, when we restrict attention to "generic" α. (Some kind of genericity assumption is certainly required here, since by [N2, (4)], we know that there is a constant $c > 0$ such that $\sup_{\alpha\in[0,1]} |D_N(\alpha)| \geqq cN$ holds for all $N \in \mathbb{Z}^+$.[24])

We will agree to call a number $\nu \in \mathbb{R}$ *Fd-nice* ("Fd" as in "Farey distribution") if the following holds:

$$\begin{cases} \text{There exists a sequence of subsets } \mathbf{G}(1), \mathbf{G}(2), \ldots \subset [0,1] \\ \text{and some } \varepsilon > 0 \text{ such that} \\ \qquad m(\mathbf{G}(N)) = 1 - O(N^{-\varepsilon}) \qquad \text{as}^{25}\ N \to \infty, \\ \text{and} \\ \qquad \left|D_N(\alpha)\right| = O(N^{\nu}) \\ \text{uniformly for all } N \in \mathbb{Z}^+,\ \alpha \in \mathbf{G}(N). \end{cases}$$

Notice that if ν is Fd-nice, with $\mathbf{G}(1), \mathbf{G}(2), \ldots$ as a corresponding sequence of sets, then the same computation as in (7.4) immediately shows that for any fixed exponent $\delta < 2 - \nu$ we have

$$\frac{1}{N^2(\beta-\alpha)} \cdot \left|\mathfrak{F}_N \cap (\alpha,\beta]\right| \to \frac{3}{\pi^2}$$

uniformly as $N \to \infty$ so long as $\alpha, \beta \in \mathbf{G}(N)$ and $\beta - \alpha \geqq N^{-\delta}$.

Fact. *Let ν_0 be the infimum of all Fd-nice numbers. Let $\beta_0 \in [\frac{1}{2}, 1]$ be the supremum of the real parts of the zeros of the Riemann zeta function. Then*

$$\nu_0 = \beta_0. \tag{7.12}$$

Before proving this fact, we remind the reader of the famous, closely related result by Franel [F1][26], which asserts that β_0 equals the infimum of all exponents θ for which

$$Q = \sum_{m=1}^{\Phi(N)} \left(r_m - \frac{m}{\Phi(N)} \right)^2 = O\left(N^{-2+2\theta}\right) \qquad \text{as } N \to \infty.$$

---

[24] In fact, by a recent result by Dress, [D4], we have $\sup_{\alpha\in[0,1]} |D_N(\alpha)| = \Phi(N)/N$ for all $N \in \mathbb{Z}^+$.

[25] This is a reasonable requirement, since $m(\mathbb{G}(N)) = 1 - O(N^{-\varepsilon})$ clearly holds for the sets $\mathbb{G}(N)$ in Corollary 4.6.4; cf. (4.94).

[26] Cf. also [L1], [L2, VII. Teil, Kap. 13], [Z2].

A statement equivalent to this is that β_0 is the infimum of all exponents θ for which

$$I = \int_0^1 D_N(\alpha)^2\, d\alpha = O(N^{2\theta}) \qquad \text{as } N \to \infty. \tag{7.13}$$

(That the two statements are indeed equivalent follows from the identity $Q = \Phi(N)^{-1}(I - \frac{1}{3})$, cf. [L2, Satz 489][27].)

We now turn to the proof of the above fact, (7.12). We define, for any $\nu \in \mathbb{R}$,

$$M_{N,\nu} := \bigl\{\alpha \in [0,1] \quad \big| \quad |D_N(\alpha)| \geqq N^\nu\bigr\}.$$

Then (7.13) (with $\theta = \beta_0 + \varepsilon$) implies that $m(M_{N,\nu}) \cdot N^{2\nu} = O(N^{2(\beta_0+\varepsilon)})$ as $N \to \infty$ (for any fixed $\varepsilon > 0$), i.e.

$$m(M_{N,\nu}) = O(N^{2(\beta_0 - \nu + \varepsilon)}). \tag{7.14}$$

If $\nu > \beta_0$ and $0 < \varepsilon < \nu - \beta_0$, the exponent in (7.14) is negative. This shows that each $\nu > \beta_0$ is Fd-nice (we can take $\mathbf{G}(N) = [0,1] - M_{N,\nu}$). Hence:

$$\nu_0 \leqq \beta_0.$$

It remains to prove that *no* number $\nu < \beta_0$ is Fd-nice. Specifically, we will show that

$$\limsup_{N\to\infty}\ m(M_{N,\nu}) > 0 \qquad \text{if } \nu < \beta_0 \text{ and } \beta_0 > \tfrac{1}{2}; \tag{7.15}$$

$$\limsup_{N\to\infty}\ m(M_{N,\nu}) \cdot \log N > 0 \qquad \text{if } \nu < \beta_0 \text{ and } \beta_0 = \tfrac{1}{2}. \tag{7.16}$$

Clearly, given $\nu < \beta_0$, (7.15), (7.16) imply that no number *less than* ν is Fd-nice; and since (7.15), (7.16) hold for *all* $\nu < \beta_0$, we obtain $\nu_0 \geqq \beta_0$, as desired.

Our proof of (7.15), (7.16) is based on a more precise version of (7.13) and a lower bound on the integral $\int_0^1 |D_N(\alpha)|\, d\alpha$.

We let $M(X)$ be the sum function of the Möbius function, i.e., $M(X) = \sum_{n \leqq X} \mu(n)$. It is well known that for all $\varepsilon > 0$,

$$M(X) = O(X^{\beta_0+\varepsilon}) \qquad \text{as } X \to \infty; \tag{7.17}$$

$$M(X) = \Omega(X^{\beta_0-\varepsilon}) \qquad \text{as } X \to \infty. \tag{7.18}$$

(This can be proved using the same methods as in [L2, VII. Teil, Kap. 12].)

We first assume $\beta_0 > \frac{1}{2}$, and let $\nu < \beta_0$ be given. We fix some number $\beta_1 \in \bigl(\max(\frac{1}{2}, \nu), \beta_0\bigr)$, and let $1 = N_1 < N_2 < \ldots$ be the sequence of integers where $|M(N)|/N^{\beta_1}$ takes its successive maxima. It follows from (7.18) that the sequence $\{N_j\}$ is infinite, and that

$$\lim_{j\to\infty} |M(N_j)|/N_j^{\beta_1} = \infty. \tag{7.19}$$

Furthermore, we have

$$|M(X)| \leqq \left(\frac{X}{N_j}\right)^{\beta_1} \bigl|M(N_j)\bigr| \qquad \text{for all } j \in \mathbb{Z}^+,\ X \in [0, N_j]. \tag{7.20}$$

To see this, notice that $M(0) = 0$, that $M(X) = M([X])$, and that if $X \in [1, N_j]$, then $|M([X])|/[X]^{\beta_1} \leqq |M(N_j)|/N_j^{\beta_1}$ by the definition of $\{N_j\}$.

---

[27] Notice that our $D_N(\alpha)$ equals $G(\alpha, N) - \frac{1}{2}$ in the notation of [L2, pp. 167–172]. Also notice that $G(1-\alpha, N) = -G(\alpha, N)$ for all $\alpha \in [0,1] - \mathfrak{F}_N$, since the Farey series $\mathfrak{F}_N$ is symmetric about the point $\frac{1}{2}$. Hence, $\int_0^1 G(\alpha, N)\, d\alpha = 0$.

Now to obtain a lower bound on $\int_0^1 |D_N(\alpha)|\,d\alpha$ we will use the formula (cf. [L2, p. 167(bottom)])

$$M(N) = \sum_{n=1}^{\Phi(N)} \cos(2\pi r_n),$$

where (again) $0 = r_0 < r_1 < ... < r_{\Phi(N)} = 1$ are all the Farey points in $\mathfrak{F}_N$. Using (7.1) and (7.2), this formula can be rewritten as follows:

$$\begin{aligned} M(N) &= \int_0^1 \cos(2\pi\alpha)\,dA_N(\alpha) \\ &= \big[\cos(2\pi\alpha)A_N(\alpha)\big]_0^1 + 2\pi \int_0^1 \sin(2\pi\alpha)A_N(\alpha)\,d\alpha \\ &= A_N(1) - A_N(0) + 2\pi \int_0^1 \sin(2\pi\alpha)\big(D_N(\alpha) + \Phi(N)\alpha\big)\,d\alpha. \end{aligned}$$

Notice here that $A_N(1) = \Phi(N)$, $A_N(0) = 0$ and $2\pi \int_0^1 \sin(2\pi\alpha)\alpha\,d\alpha = -1$. Hence

$$M(N) = 2\pi \int_0^1 \sin(2\pi\alpha) D_N(\alpha)\,d\alpha,$$

and

$$\int_0^1 |D_N(\alpha)|\,d\alpha \geqq \frac{1}{2\pi}|M(N)|. \tag{7.21}$$

We now turn to $\int_0^1 D_N(\alpha)^2\,d\alpha$. For each $N \in \mathbb{Z}^+$ we have

$$\int_0^1 D_N(\alpha)^2\,d\alpha = \frac{1}{4} + \frac{1}{12}\sum_{a=1}^{N}\sum_{b=1}^{N} M\left(\frac{N}{a}\right) M\left(\frac{N}{b}\right) \frac{(a,b)^2}{ab}. \tag{7.22}$$

Cf. [L2, Satz 488] and footnote 27 above.

Hence, if $N \in \{N_j\}$, (7.20) implies that[28]

$$\begin{aligned} \int_0^1 D_N(\alpha)^2\,d\alpha &\leqq \frac{1}{4} + \frac{M(N)^2}{12} \sum_{a=1}^{N}\sum_{b=1}^{N} \frac{1}{a^{\beta_1}} \frac{1}{b^{\beta_1}} \frac{(a,b)^2}{ab} \\ &\leqq \frac{1}{4} + \frac{M(N)^2}{12} \sum_{a,b=1}^{\infty} \frac{(a,b)^2}{(ab)^{1+\beta_1}} \\ &= \frac{1}{4} + \frac{M(N)^2}{12} \sum_{\substack{a_1,b_1,c=1 \\ (a_1,b_1)=1}}^{\infty} \frac{c^2}{(a_1 c b_1 c)^{1+\beta_1}}. \end{aligned}$$

The triple sum is clearly convergent, since $\beta_1 > \frac{1}{2}$. Note that $|M(N)| \geqq 1$, since $N \in \{N_j\}$. Hence there is some constant $C = C(\beta_1) > 0$ such that

$$\int_0^1 D_N(\alpha)^2\,d\alpha \leqq C \cdot M(N)^2, \qquad \forall N \in \{N_j\}. \tag{7.23}$$

---

[28] We now imitate [L2, p. 175].

Together, (7.21) and (7.23) imply that for each $N \in \{N_j\}$,

$$\begin{aligned} \frac{1}{2\pi}\big|M(N)\big| &\leqq \int_0^1 |D_N(\alpha)|\,d\alpha \leqq \int_{[0,1]-M_{N,\nu}} N^\nu\,d\alpha + \int_{M_{N,\nu}} |D_N(\alpha)|\,d\alpha \\ &\leqq N^\nu + \sqrt{\int_{M_{N,\nu}} d\alpha}\sqrt{\int_{M_{N,\nu}} D_N(\alpha)^2\,d\alpha} \\ &\leqq N^\nu + \sqrt{m(M_{N,\nu})}\,\sqrt{C}\,|M(N)|. \qquad (7.24) \end{aligned}$$

But for all sufficiently large j we have $N_j^\nu \leqq \frac{1}{4\pi}|M(N_j)|$, because of $\nu < \beta_1$ and (7.19). For these j, (7.24) clearly implies that $m(M_{N_j,\nu}) \geqq ((4\pi)^2C)^{-1}$, and (7.15) is proved.

In the case $\nu < \beta_0 = \frac{1}{2}$, the above proof can be modified as follows. We write $C = \limsup_{N\to\infty} |M(N)|/\sqrt{N}$. If $C = \infty$, then we let $1 = N_1 < N_2 < \dots$ be the sequence of integers where $|M(N)|/\sqrt{N}$ takes its successive maxima. In the other case, $C < \infty$, we know from [T1, Thm. 14.26(B)] that $C > 0$ holds, and we let $N_1 < N_2 < \dots$ be the sequence of all integers N for which $|M(N)|/\sqrt{N} > C/2$.

In both cases, it follows that the sequence $N_1, N_2, \dots$ is infinite, and that

$$\inf_{j\in\mathbb{Z}^+} |M(N_j)|/\sqrt{N_j} > 0. \qquad (7.25)$$

Furthermore, there is some (absolute) constant $C_1 > 0$ such that

$$|M(X)| \leqq C_1\sqrt{\frac{X}{N_j}}\,\big|M(N_j)\big| \qquad \text{for all } j \in \mathbb{Z}^+,\ X \in [0, N_j]. \qquad (7.26)$$

(If $0 < C < \infty$ we can take $C_1 = (2/C)\sup_{X>0}|M(X)|/\sqrt{X}$; if $C = \infty$ we can take $C_1 = 1$.)

(7.26) and (7.22) imply that for all $N \in \{N_j\}$,

$$\begin{aligned} \int_0^1 D_N(\alpha)^2\,d\alpha &\leqq \frac{1}{4} + \frac{C_1^2M(N)^2}{12}\sum_{a=1}^{N}\sum_{b=1}^{N}\frac{1}{\sqrt{a}}\frac{1}{\sqrt{b}}\frac{(a,b)^2}{ab} \\ &\leqq \frac{1}{4} + \frac{C_1^2M(N)^2}{12}\sum_{c=1}^{N}\sum_{\substack{a_1,b_1=1\\(a_1,b_1)=1}}^{\infty}\frac{c^2}{(a_1cb_1c)^{3/2}} \\ &\leqq C_2\cdot M(N)^2\cdot\log(2N), \end{aligned}$$

for some (absolute) constant $C_2 > 0$.

The computation (7.24) can now be repeated, yielding

$$\frac{1}{2\pi}\big|M(N)\big| \leqq N^\nu + \sqrt{m(M_{N,\nu})}\,\sqrt{C_2}\,|M(N)|\sqrt{\log(2N)} \quad \text{for all } N \in \{N_j\}.$$

Notice that $N_j^\nu \leqq \frac{1}{4\pi}|M(N_j)|$ for all sufficiently large j, because of $\nu < \frac{1}{2}$ and (7.25). For all these j, we obtain $m(M_{N_j,\nu}) \geqq ((4\pi)^2C_2\log(2N_j))^{-1}$, and (7.16) is proved. □

REFERENCES

[AS] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, 1965.

[B1] C. B. Balogh, Uniform asymptotic expansions of the modified Bessel function of the third kind of large imaginary order, Bull. Amer. Math. Soc., **72** (1966), 40–43.

[B2] C. B. Balogh, Asymptotic expansions of the modified Bessel function of the third kind of imaginary order, SIAM J. Appl. Math. **15** (1967), 1315–1323.

[C] D. L. Cohn, *Measure Theory*, Birkhäuser, Boston, 1980.

[D1] S. G. Dani, Invariant measures of horospherical flows on noncompact homogeneous spaces, Invent. Math. **47** (1978), 101–138.

[D2] S. G. Dani, Invariant measures and minimal sets of horospherical flows, Invent. Math. **64** (1981), 357–385.

[DM] S. G. Dani, G. A. Margulis, Limit distributions of orbits of unipotent flows and values of quadratic forms, Adv. in Soviet Math. **16** (1993), 91–137.

[D3] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, 1960.

[D4] F. Dress, Discrépance des suites de Farey, J. Théor. Nombres Bordeaux **11** (1999), 345–367.

[EHS] C. Epstein, J. L. Hafner, P. Sarnak, Zeros of L-functions attached to Maass forms, Math. Z. **190**, 113–128 (1985); Corrigenda in [H1].

[E] A. Erdélyi, *Higher Transcendental Functions*, vol. 2, McGraw-Hill, 1953.

[F1] J. Franel, Les suites de Farey et le problème des nombres premiers, Göttinger Nachrichten (1924), 198–201.

[F2] H. Furstenberg, The unique ergodicity of the horocycle flow, in *Recent Advances in Topological Dynamics,* Springer-Verlag, 1972, 95–115.

[H1] J. L. Hafner, Some Remarks on Odd Maass Wave Forms (and a correction to "Zeros of L-functions attached to Maass forms", [EHS]), Math. Z. **196**, 129–132 (1987).

[HW] G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, third edition, Oxford University Press, 1954.

[H2] D. A. Hejhal, *The Selberg Trace Formula for PSL*$(2,\mathbb{R})$, Vol. 1, Lecture Notes in Math. **548**, Springer-Verlag, Berlin, 1976.

[H3] D. A. Hejhal, *The Selberg Trace Formula for PSL*$(2,\mathbb{R})$, Vol. 2, Lecture Notes in Math. **1001**, Springer-Verlag, Berlin, 1983.

[H4] D. A. Hejhal, On eigenfunctions of the Laplacian for Hecke triangle groups, in *Emerging Applications of Number Theory* (edited by D. Hejhal, J. Friedman, M. Gutzwiller, A. Odlyzko), IMA Series No. 109, Springer-Verlag, 1999, pp. 291 – 315.

[H5] D. A. Hejhal, On value distribution properties of automorphic functions along closed horocycles, in *XVI-th Rolf Nevanlinna Colloquium,* (edited by I. Laine and O. Martio), deGruyter, 1996, pp. 39–52.

[H6] D. A. Hejhal, On the Uniform Equidistribution of Long Closed Horocycles, Asian J. Math. **4** (2000).

[H7] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin, 1983.

[H8] M. N. Huxley, Scattering Matrices for Congruence Subgroups, in *Modular Forms,* (edited by R. Rankin) Ellis Horwood Ltd., Chichester, 1984, 141–156.

[I1] H. Iwaniec, *Introduction to the Spectral Theory of Automorphic Forms,* Biblioteca de la Revista Matemática Iberoamericana, Madrid, 1995.

[I2] A. E. Ingham, *The Distribution of Prime Numbers,* Cambridge University Press, 1932.

[K] S. Katok, *Fuchsian Groups,* Univ. of Chicago Press, Chicago and London, 1992.

[L1] E. Landau, Bemerkungen zu der vorstehenden Abhandlung von Herrn Franel, Göttinger Nachrichten (1924), 202–206.

[L2] E. Landau, *Vorlesungen über Zahlentheorie,* Vol. II, S. Hirzel, Leipzig, 1927.

[L3] W. J. LeVeque, *Topics in number theory*, Vol. I, Addison-Wesley, Reading, Massachusetts, 1956.

[M1] A. Manning, Dynamics of geodesic and horocycle flows on surfaces of constant negative curvature, in *Ergodic theory, symbolic dynamics, and hyperbolic spaces,* (edited by T. Bedford, M. Keane, C. Series), Oxford Univ. Press, 1991, 71–91.

[M2] J. Marklof, Theta sums, Eisenstein series, and the semiclassical dynamics of a precessing spin, in *Emerging Applications of Number Theory,* (edited by D. A. Hejhal, J. Friedman, M. C. Gutzwiller, A. M. Odlyzko), IMA Vol. Math. Appl. **109**, Springer-Verlag, New York, 1999, 405–450.

[M3] E. J. McShane, *Integration*, Princeton University Press, 1944.

[M4] T. Miyake, *Modular Forms,* Springer-Verlag, New York, 1989.

[N1] E. H. Neville, The structure of Farey series, Proc. London Math. Soc. **51** (1949), 132–144.

[N2] H. Niederreiter, The Distribution of Farey Points, Math. Ann. **201** (1973), 341–345.

[O1] F. W. J. Olver, Error bounds for first approximations in turning-point problems, SIAM J. Appl. Math. **11** (1963), 748–772.

[O2] F. W. J. Olver, Error bounds for asymptotic expansions in turning-point problems, SIAM J. Appl. Math. **12** (1964), 200–214.

[R1] M. Ratner, Raghunathan's topological conjecture and distributions of unipotent flows, Duke Math. J. **63** (1991), 235–280.

[R2] M. Ratner, On Raghunathan's measure conjecture, Ann. of Math., **134** (1991), 545–607.

[R3] M. Ratner, Raghunathan's conjectures for SL$(2, \mathbb{R})$, Israel J. Math., **80** (1992), 1–31.

[R4] V. A. Rohlin, Selected topics in the metric theory of dynamical systems, Uspekhi Mat. Nauk. **4**, (1949) 57–128; also Amer. Math. Soc. Transl. Ser **2**, 49, (1966) 171–240.

[R5] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1987.

[S1] P. Sarnak, Asymptotic behavior of periodic orbits of the horocycle flow and Eisenstein series, Comm. Pure Appl. Math. **34** (1981), 719–739.

[S2] H. Shimizu, On discontinuous groups acting on the product of the upper half planes, Ann. Math., II. Ser. **77**, 33–71 (1963).

[T1] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function,* Oxford University Press, 1986.

[W] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2d ed., Cambridge Univ. Press, Cambridge, 1944.

[Z1] D. Zagier, Eisenstein series and the Riemann zeta function, in *Automorphic Forms, Representation Theory and Arithmetic,* Tata Institute (Bambay), Springer-Verlag, 1981, pp. 275-301.

[Z2] A. Zulauf, The distribution of Farey numbers, J. Reine Angew. Math. **289** (1977), 209–213.