SOME EXPLICIT COVERING MAPS, PRELIMINARY NOTE

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In [3] we presented, among other things, a new algorithm for numerical computation of modular forms. This algorithm is a generalization of Hejhal [1] to the case of cofinite Fuchsian groups with several cusps.

In this note we present the results obtained by using this algorithm on those cases which were left unresolved in Richards [2].

1. The quintic family on p.83

Let ω be a root of $\omega^5 - 2\omega^4 + 2\omega^3 - 3\omega^2 + 3 = 0$.

The following curve + covering correspond to the five-family on p.83 in [2]. More specifically, if we take ω as the *real* root of $\omega^5 - 2\omega^4 + 2\omega^3 - 3\omega^2 + 3 = 0$, then we obtain a covering which is described by the permutations $\sigma_\infty = (1234567)$, $\sigma_0 = (146573)$. After "shifting $0 \leftrightarrow \infty$", this is the same as G^7_{6e} on p.83 in [2].

Elliptic curve:

$$\begin{aligned} y^2 =& x^3 + \big(46731\omega^4 - 92715\omega^3 - 75252\omega^2 + 114750\omega + 76914\big)x \\ &+ \big(2781282\omega^4 + 100422\omega^3 - 8378580\omega^2 + 1382262\omega + 4534278\big) \end{aligned}$$

Covering map:

$$\begin{aligned} &\big(-8828061696\omega^4 + 35191314432\omega^3 - 35402946048\omega^2 - 12607197696\omega \\ &\qquad + 24488801280\big)J \\ &= \big(-\omega^2 + \omega + 1\big)x^2y \\ &\quad + \big(7\omega^4 + 49\omega^3 - 49\omega^2 - 63\omega\big)x^3 \\ &\quad + \big(-980\omega^4 + 126\omega^3 + 1624\omega^2 + 490\omega - 168\big)xy \\ &\quad + \big(23058\omega^4 - 43470\omega^3 + 11151\omega^2 + 17577\omega - 15498\big)x^2 \\ &\quad + \big(81686\omega^4 - 281974\omega^3 + 252580\omega^2 + 103869\omega - 181182\big)y \\ &\quad + \big(3556917\omega^4 + 8906121\omega^3 - 31288005\omega^2 + 3119130\omega + 20680569\big)x \\ &\quad + \big(-448511061\omega^4 + 1637507517\omega^3 - 1540687182\omega^2 - 612353133\omega + 1076095890\big). \end{aligned}$$

The map branches above $J = \infty$ at ∞ with order 7, above $J = 0$ at $(-73\omega^4 + 82\omega^3 - 69\omega^2 + 79\omega + 165, -972\omega^4 + 2052\omega^3 - 4644\omega^2 + 2268\omega + 5076)$ with order 6, and at three points above $J = 1$, with order 2 at each.

(Each of these last three points has an x-coordinate which satisfies the equation $x^3 + (219\omega^4 - 390\omega^3 - 369\omega^2 + 123\omega - 63)x^2 + (49677\omega^4 + 10923\omega^3 - 172788\omega^2 - 3942\omega + 76014)x + (113072727\omega^4 - 309627783\omega^3 + 198611430\omega^2 + 127006017\omega - 155703711) = 0$.)

2. The first sextic family on p.123

Let ω be a root of $\omega^6 - 2\omega^4 - 4\omega^3 + 12 = 0$.

The following curve + covering correspond to the first six-family on p.123 in [2]. More specifically, if we take ω as the complex root $\big(-1.38... - i \cdot 0.907...\big)$ of $\omega^6 - 2\omega^4 - 4\omega^3 + 12 = 0$, then we obtain a covering which is described by the permutations $\sigma_\infty = (12345678)$, $\sigma_0 = (165784)(23)$. After "shifting $0 \leftrightarrow \infty$", this is the same as G^8_{34a} on p.123 in [2].

Elliptic curve:

$$\begin{aligned}
y^2 =& x^3 + \big(46356785310\omega^5 + 47752376283\omega^4 - 193850186100\omega^3 \\
&\quad - 320211910974\omega^2 + 203893282908\omega + 468169535277\big)x \\
&\quad + \big(2590629438042846\omega^5 + 30542148080241648\omega^4 - 22867232747465688\omega^3 \\
&\quad - 116861853075383046\omega^2 - 36469396346710764\omega + 218730331465850598\big)
\end{aligned}$$

Covering map:

$$\begin{aligned}
&\Big(15367367447279817517425 6\omega^5 + 197841773568331874216736\omega^4 \\
&\quad - 659688819159461814802080\omega^3 - 1201904498342629360544976\omega^2 \\
&\quad + 608010100506825088767552\omega + 1825168869396830089474128\Big)J \\
&= (19\omega^5 - 19\omega^4 - 34\omega^3 + 9\omega^2 - 64\omega + 137)x^4 \\
&\quad + \big(-280\omega^5 + 348\omega^4 + 1376\omega^3 - 1904\omega^2 - 912\omega + 1368)x^2y \\
&\quad + \big(183376\omega^5 + 180776\omega^4 - 767648\omega^3 - 1228976\omega^2 + 825400\omega + 1783212)x^3 \\
&\quad + \big(198460944\omega^5 + 20197680\omega^4 - 750351072\omega^3 - 716749104\omega^2 \\
&\qquad + 1189249632\omega + 731523216)xy \\
&\quad + \big(-178781021346\omega^5 - 70321346388\omega^4 + 698459009028\omega^3 \\
&\qquad + 830735261994\omega^2 - 981828345528\omega - 1019069462970)x^2 \\
&\quad + \big(64672033900632\omega^5 + 114509040556524\omega^4 - 291114902634816\omega^3 \\
&\qquad - 616762404291888\omega^2 + 202214285106672\omega + 983992861336824)y \\
&\quad + \big(147628708983966864\omega^5 - 101976580260901260\omega^4 - 507656693759555520\omega^3 \\
&\qquad - 117731362144338696\omega^2 + 1085571776204633256\omega - 264184596755378052)x \\
&\quad + \big(369424247462040817928430\omega^5 + 605881678023932628443070\omega^4 \\
&\qquad - 1642107002348679515901540\omega^3 - 3351887922303485029844790\omega^2 \\
&\qquad + 1237914225648260975983920\omega + 5287667183324166639043530).
\end{aligned}$$

The map branches above $J = \infty$ at ∞ with order 8, above $J = 0$ at $(11942\omega^5 - 40369\omega^4 - 27196\omega^3 + 104523\omega^2 + 142966\omega - 243285, 31536999\omega^5 - 14328144\omega^4 - 111666762\omega^3 - 51624972\omega^2 + 219099816\omega - 4922316)$ with order 6 and at $(-52894\omega^5 + 42179\omega^4 + 179444\omega^3 + 22155\omega^2 - 398618\omega + 133635, 31536999\omega^5 - 14328144\omega^4 - 111666762\omega^3 - 51624972\omega^2 + 219099816\omega - 4922316)$ with order 2, and at three points above $J = 1$, with order 2 at each. (Each of these last three points has an x-coordinate which satisfies the equation $x^3 + (32331\omega^5 - 36087\omega^4 - 105234\omega^3 + 23049\omega^2 + 261348\omega - 152883)x^2 + (18888680814\omega^5 + 52799862609\omega^4 - 93381964140\omega^3 - 248859578958\omega^2 + 25823840616\omega + 421111820331)x + (7318039641833655\omega^5 - 26524959566549772\omega^4 - 15895501060022682\omega^3 + 70394479302353433\omega^2 + 90680154040106628\omega - 161422752938790693) = 0$.)

3. The second sextic family on p.123

Let ω be a root of $\omega^6 - 2\omega^5 + \omega^4 - 4\omega^3 + 5\omega^2 - 6\omega + 3 = 0$.

The following curve + covering correspond to the second six-family on p. 123 in [2]. More specifically, if we take ω as the complex root $\big(0.403... - i \cdot 0.907...\big)$ of $u^6 - 2\omega^5 + u^4 - 4\omega^3 + 5\omega^2 - 6\omega + 3 = 0$, then we obtain a covering which is described by the permutations $\sigma_\infty = (12345678)$, $\sigma_0 = (16578432)$. After "shifting $0 \leftrightarrow \infty$", this is the same as G^8_{35a} on p.123 in [2].

Elliptic curve:

$$\begin{aligned} y^2 =& x^3 + \big(-179208\omega^5/7 + 402771\omega^4/7 + 528960\omega^3/7 + 1295472\omega^2/7 \\ &\qquad - 23376\omega + 346632/7\big)x \\ &\quad + \big(116402564\omega^5/7 - 134318118\omega^4/7 + 56683392\omega^3/7 - 373043014\omega^2/7 \\ &\qquad + 47822676\omega - 468182962/7\big) \end{aligned}$$

Covering map:

$$\begin{aligned} &\Big(-263105134234560\omega^5 + 16556429668032\omega^4 - 291712540383936\omega^3 \\ &\quad + 438834127664448\omega^2 - 574585015276224\omega + 472720212922752\Big) J \\ &= \Big(-617\omega^5/7 + 1391\omega^4/7 + 397\omega^3/7 + 3461\omega^2/7 - 215\omega + 3926/7\Big)x^4 \\ &\quad + \Big(-47888\omega^5/7 + 98656\omega^4/7 + 23392\omega^3/7 + 251912\omega^2/7 - 16528\omega + 285312/7\Big)x^2y \\ &\quad + \Big(-585024\omega^5/7 + 1463072\omega^4/7 + 492536\omega^3/7 + 3541724\omega^2/7 - 206132\omega \\ &\qquad + 4021732/7\Big)x^3 \\ &\quad + \Big(7072480\omega^5 - 4832720\omega^4 + 4333552\omega^3 - 19688272\omega^2 \\ &\qquad + 15955376\omega - 21851328\Big)xy \\ &\quad + \Big(252474744\omega^5 - 81095850\omega^4 + 227771730\omega^3 - 538416150\omega^2 \\ &\qquad + 558923184\omega - 589497288\Big)x^2 \\ &\quad + \Big(810991792\omega^5/7 + 3599407816\omega^4/7 + 3818502832\omega^3/7 + 5211546488\omega^2/7 \\ &\qquad + 191919776\omega + 6144076584/7\Big)y \\ &\quad + \Big(469214176436\omega^5/7 + 92379179896\omega^4/7 + 617627779240\omega^3/7 - 563500830012\omega^2/7 \\ &\qquad + 144327281744\omega - 589100849780/7\Big)x \\ &\quad + \Big(38650730959389\omega^5/7 + 14508257135833\omega^4/7 + 56395287846799\omega^3/7 \\ &\qquad - 34009162651991\omega^2/7 + 11773825587725\omega - 34162302386104/7\Big) \end{aligned}$$

The map branches above $J = \infty$ at ∞ with order 8, above $J = 0$ at $(-178\omega^5/7 + 64\omega^4/7 - 62\omega^3/7 + 1101\omega^2/7 + 104\omega + 1642/7, 324\omega^5 + 432\omega^4 - 1458\omega^3 - 4050\omega^2 - 5238\omega - 1350)$ with order 8, and at two points above $J = 1$, with order 2 at each. (The x-coordinates of these last two points are the two solutions to the equation $x^2 + \Big(-148\omega^5/7 + 124\omega^4/7 - 128\omega^3/7 + 822\omega^2/7 + 44\omega + 1504/7\Big)x + \Big(-209148\omega^5/7 + 436271\omega^4/7 + 359432\omega^3/7 + 1300172\omega^2/7 - 45740\omega + 798964/7\Big) = 0$.)

4. The heptic family on p.124

We are still working on finding a "nice" presentation in this case. However, we have completed the computation of modular forms, and by use of Newton's algorithm we obtained numerical approximations correct to ~ 2900 digits of an algebraic presentation of the curve and the covering map over a number field, and applied the LLL-algorithm. In particular, we have convincing evidence that the field of definition will be generated by the irreducible polynomial

$$\omega^7 - \omega^6 + 16\omega^5 + 16\omega^4 + 2\omega^3 - 38\omega^2 - 30\omega - 6.$$

In particular, the field of definition has degree 7, i.e. not smaller than expected.

5. The 13-family on p.124

The same remarks apply in this case. In particular, we have an explicit irreducible polynomial of degree 13 generating the field of definition.

References

[1] D. A. Hejhal, On eigenfunctions of the Laplacian for Hecke triangle groups, in *Emerging Applications of Number Theory* (edited by D. Hejhal, J. Friedman, M. Gutzwiller, A. Odlyzko), IMA Series No. 109, Springer-Verlag, 1999, pp. 291–315.

[2] M. J. Richards, *Computing Covering Maps for subgroups of* $\Gamma(1)$ *with genus* ≥ 1, Thesis, University of Oxford, 1996.

[3] B. Selander and A. Strömbergsson, Sextic coverings of genus two which are branched at three points, preprint, available from www.math.uu.se/∼andreas/papers.html.