# THE SELBERG TRACE FORMULA FOR MODULAR CORRESPONDENCES

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## 0. Introduction

In this paper we compute the trace formula for a modular correspondence on a general cofinite Fuchsian group, in the space of discrete eigenfunctions of the Laplace-Beltrami operator. Similar computations have been done by Akiyama and Tanigawa [A-T] and, in a representation-theoretic setting, by Hoffmann [Ho1, Ho2]. Selberg suggested that such a formula could be obtained already during the 1950's; see [S1, pp. 444–446, 460–462, 504–505].

In the present paper, we deal exclusively with the case of trivial automorphy character and weight zero. In part two, a continuation of this paper, we will extend this to the case of more general multiplier systems.

In contrast to [A-T, Ho1, Ho2], our computations are modelled on Selberg's classical Göttingen lectures [S1, pp. 626–674]. The style in this paper is thus completely classical and explicit, and our treatment includes careful error estimates. We have striven to make our general trace formula as explicit and "ready–to–apply" as possible. As a consequence, along the way, the reader will have to stand a fairly involved apparatus of notation. However, at the end, a lot of cancellation will occur, making the final trace formula a lot simpler than what might be feared in the course of reading.

The final form of the trace formula is given in Section 8, cf. p. 42 and Theorem 8.3. In the last section, we specialize this formula to the case of an arbitrary Hecke operator on $\mathrm{PSL}(2,\mathbf{Z})$, cf. p. 54. The corresponding computation for $\Gamma = \Gamma_0(N)$ ($N$ square free) will appear in a paper which is now in preparation.

This work is based on notes of Hejhal, where the trace formula for Hecke operators of prime order on $\mathrm{PSL}(2,\mathbf{Z})$ is computed, see [H3].

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## 1. Preliminaries

Let $\Gamma \subset \mathrm{PSL}(2, \mathbf{R})$ be a *cofinite* Fuchsian group acting on the Poincaré upper half-plane $\mathcal{H}$. In this section we draw attention to some useful facts about the cusps of $\Gamma$ and their mapping properties under elements from the commensurator of $\Gamma$. Notice that, to start with, $\Gamma$ may have no cusp, although many of the statements below then are pointless.

For general facts about Fuchsian groups, see [L], [K], [M, chapter 1], [Sh, chapter 1].

Begin with $\alpha \in \mathrm{Comm}(\Gamma)$. Thus

$$[\Gamma : \Gamma \cap \alpha^{-1}\Gamma\alpha] < \infty$$
$$[\alpha^{-1}\Gamma\alpha : \Gamma \cap \alpha^{-1}\Gamma\alpha] < \infty$$

We let

$$d = [\Gamma : \Gamma \cap \alpha^{-1}\Gamma\alpha]$$

See [Sh, p. 51] for the definition of $\mathrm{Comm}(\Gamma)$. Cf. [M, p. 69].

Using $[A \cap G : A \cap H] \leq [G : H]$ for general groups, one checks that $\mathrm{Comm}(\Gamma)$ is a group.

**Lemma 1.1.** *Any two elements $\alpha'$, $\alpha''$ in $\Gamma\alpha\Gamma$ can be related*

$$\alpha' = g\alpha''h, \qquad g, h \in \Gamma.$$

*Proof.* Trivial. □

**Lemma 1.2.** *Let $\xi$ be any element of $\Gamma\alpha\Gamma$. We then have*

$$[\Gamma : \Gamma \cap \xi^{-1}\Gamma\xi] = d$$
$$[\Gamma : \Gamma \cap \xi\Gamma\xi^{-1}] = d.$$

*Proof.* First check that both indices are preserved under $\xi \to g\xi h$. Therefore we may assume $\xi = \alpha$. Now just review [H1, p. 469, Prop. 2.12]. □

*Remark 1.3.* Also, by conjugation with $\xi$ we see that

$$[\xi^{-1}\Gamma\xi : \Gamma \cap \xi^{-1}\Gamma\xi] = [\xi\Gamma\xi^{-1} : \Gamma \cap \xi\Gamma\xi^{-1}] = d.$$

**Lemma 1.4.** *Keep $\epsilon_i \in \Gamma$, and use $\bigsqcup$ to denote disjoint union. Then*

$$\Gamma = \bigsqcup_{i=1}^{d} \left(\Gamma \cap \alpha^{-1}\Gamma\alpha\right) \epsilon_i \quad \textit{iff} \quad \Gamma\alpha\Gamma = \bigsqcup_{i=1}^{d} \Gamma\alpha\epsilon_i$$

*Proof.* See [H1, p. 466, Prop. 2.7] (cf. [M, p. 69, Lemma 2.7.1(4)]). □

**Lemma 1.5.** *Let $\Gamma_1$ be a subgroup of $\Gamma$ of finite index $e$. Then $\Gamma$ and $\Gamma_1$ have the same set of cusps. Let $\eta$ be any cusp of $\Gamma$, such that the stabilizer $\Gamma_\eta$ equals $[P] := \{P^n \mid n \in \mathbf{Z}\}$. Then:*

$$(\Gamma_1)_\eta = [P^m]$$

*for a uniquely determined $m \in \{1, .., e\}$.*

*Proof.* If $\eta$ is a cusp of $\Gamma_1$, clearly it is a cusp of $\Gamma$.

Conversely, let $\eta$ be a cusp of $\Gamma$, with $\Gamma_\eta = [P]$. Look at the cosets

$$\Gamma_1 P,\ \Gamma_1 P^2,\ ...,\ \Gamma_1 P^{e+1}.$$

There must be at least one duplicate; $\Gamma_1 P^i = \Gamma_1 P^j$ for some $i < j$. We then get $\Gamma_1 P^{j-i} = \Gamma_1$, i.e. $P^{j-i} \in \Gamma_1$. Hence $\eta$ is a cusp of $\Gamma_1$.

Now since $\Gamma_1 \subset \Gamma$, every $T \in \Gamma_1$ fixing $\eta$ has to be of the form $P^n$ $(n \in \mathbf{Z})$. Hence

$$(\Gamma_1)_\eta = \{P^n \mid n \in \mathbf{Z},\ P^n \in \Gamma_1\}$$

The set of admissible $n$ is an ideal, hence a principal ideal. So:

$$(\Gamma_1)_\eta = [P^m], \quad \text{for some } m \in \mathbf{Z}^+.$$

But in the above computation with cosets, $1 \leq j - i \leq e$, hence $m \leq e$. □

**Lemma 1.6.** *Take $\xi \in \Gamma\alpha\Gamma$. Let $\eta$ be any cusp of $\Gamma$. Then $\xi(\eta)$ is a cusp of $\Gamma$.*

*Proof.* Let $\Gamma_\eta = [P]$, and apply Lemma 1.5 with $\Gamma_1 = \Gamma \cap \xi^{-1}\Gamma\xi$. Thus $P^m \in \Gamma \cap \xi^{-1}\Gamma\xi$ for some $m \in \{1, .., d\}$. This gives $\xi P^m \xi^{-1} \in \Gamma$. This element is parabolic, with fixpoint $\xi(\eta)$. □

**Lemma 1.7.** *Let $\xi \in \Gamma\alpha\Gamma$ and $T \in \Gamma$ be given. Then*

$$\begin{aligned}\{n \in \mathbf{Z} \mid \xi T^n \xi^{-1} \in \Gamma\} &= \{n \in \mathbf{Z} \mid \Gamma\xi T^n = \Gamma\xi\} \\ &= \{n \in \mathbf{Z} \mid (\Gamma \cap \xi^{-1}\Gamma\xi)T^n = \Gamma \cap \xi^{-1}\Gamma\xi\} \\ &= (m)\mathbf{Z}\end{aligned}$$

*for a uniquely determined $m \in \{1, ..., d\}$.*

*Proof.* The three $n-$sets are clearly equal. They are also clearly $\mathbf{Z}-$ideals, hence principal. To conclude, just use Lemma 1.2 and the *minimality* characterization of $m$. □

**Lemma 1.8.** *The groups $\Gamma$, $\omega^{-1}\Gamma\omega$, $\Gamma \cap \omega^{-1}\Gamma\omega$ have identical sets of cusps for any $\omega \in Comm(\Gamma)$.*

*Proof.* By Lemma 1.2, Remark 1.3 (taking $\alpha = \xi = \omega$), $\Gamma \cap \omega^{-1}\Gamma\omega$ is a subgroup of finite index both in $\Gamma$ and in $\omega^{-1}\Gamma\omega$. Now apply Lemma 1.5. □

It is important to keep in mind that $\mathrm{PSL}(2, \mathbf{R})$ (and hence $\Gamma$) consists of Möbius transformations, not matrices. However, throughout this paper we will let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (where $a, b, c, d \in \mathbf{R}$, $ad - bc = 1$) denote an element of $\mathrm{PSL}(2, \mathbf{R})$, *not* of $\mathrm{SL}(2, \mathbf{R})$. Since we never need to look at $\mathrm{SL}(2, \mathbf{R})$, this will cause no confusion, as long as we remember that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ denote the same object.

In the following, it will be convenient to use the customary notation

$$\mathbf{a}(\gamma) := \begin{pmatrix} \sqrt{\gamma} & 0 \\ 0 & 1/\sqrt{\gamma} \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{R}), \quad \gamma > 0 \qquad \text{and}$$

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{R}), \quad S^r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{R}), \qquad r \in \mathbf{R}.$$

We notice the relation

$$S^r \mathbf{a}(\gamma) = \mathbf{a}(\gamma) S^{r/\gamma} \tag{1.1}$$

(corresponding simply to $r + \gamma z = \gamma(z + r/\gamma)$).

From now on, we assume that $\Gamma \setminus \mathcal{H}$ has at least one cusp. We will use the same notation as in [H2, p. 268], which we now describe: Let $\mathcal{F} \subset \mathcal{H}$ be a canonical fundamental domain for $\Gamma \setminus \mathcal{H}$, and let $\eta_1, ..., \eta_\kappa$ (where $\kappa \geq 1$) be the vertices at infinity of $\mathcal{F}$. We stress that $\eta_1, ..., \eta_\kappa$ are $\Gamma-$inequivalent: Each cusp can be written as $A\eta_j$ for some $A \in \Gamma$, $j \in \{1, ..., \kappa\}$, where $j$ is *uniquely* determined.

For each $j \in \{1, ..., \kappa\}$ we choose $N_j \in \mathrm{PSL}(2, \mathbf{R})$ such that $N_j(\eta_j) = \infty$ and such that the stabilizer $\Gamma_{\eta_j}$ is $[T_j]$, where $T_j := N_j^{-1} S^{-1} N_j$, and also

$$\begin{aligned} N_j(\mathcal{F}) \bigcap \{z \in \mathcal{H} \mid \mathrm{Im}\, z > Y\} \\ = \{z \in \mathcal{H} \mid 0 < \mathrm{Re}\, z < 1,\ \mathrm{Im}\, z > Y\}, \end{aligned} \tag{1.2}$$

for all $Y$ large enough. For such $Y$, the corresponding cuspidal region in $\mathcal{F}$ is called $\mathcal{C}_{jY}$ :

$$\mathcal{C}_{jY} = N_j^{-1}\,[(0,1) \times (Y, \infty)] \subset \mathcal{F}. \tag{1.3}$$

We then let

$$\mathcal{F}_Y = \mathcal{F} - \bigcup_{j=1}^{\kappa} \mathcal{C}_{jY}. \tag{1.4}$$

This is a bounded region.

**Lemma 1.9.** *If $T \in PSL(2, \mathbf{R})$ fixes $\infty$, then $T = S^r \mathbf{a}(\gamma)$ for some uniquely determined $r \in \mathbf{R}$, $\gamma > 0$. $T$ is parabolic if and only if $\gamma = 1$ and $r \neq 0$.*

*Proof.* $T$ fixes $\infty$ if and only if it has the form $T = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, where $a, b, d \in \mathbf{R}$, $ad = 1$. Since $S^r \mathbf{a}(\gamma) = \begin{pmatrix} \sqrt{\gamma} & r/\sqrt{\gamma} \\ 0 & 1/\sqrt{\gamma} \end{pmatrix}$, we have equality in $\mathrm{PSL}(2, \mathbf{R})$ if and only if $\gamma = a^2$, $r = ba$. For the second part, recall that $T$ is parabolic if and only if its trace is $\pm 2$ and $T \neq \mathrm{Id}$. $\square$

**Lemma 1.10.** *Given $y > 0$, $x \neq 0$ real. Then:*

$$\{A \in PSL(2, \mathbf{R}) \mid A S^y A^{-1} = S^x\} = \begin{cases} \emptyset & \text{if } x < 0 \\ \{S^r \mathbf{a}(\frac{x}{y}) \mid r \in \mathbf{R}\} & \text{if } x > 0 \end{cases}$$

*Proof.* If $AS^yA^{-1} = S^x$, then $S^x$ fixes $A(\infty)$, hence $A(\infty) = \infty$, and by Lemma 1.9, $A = S^r\mathbf{a}(\gamma)$ for some $r \in \mathbf{R}$, $\gamma > 0$. For such $A$ we have, using (1.1),

$$AS^yA^{-1} = S^r\mathbf{a}(\gamma)S^y\mathbf{a}(\gamma^{-1})S^{-r} = S^{y\gamma}\mathbf{a}(\gamma)\mathbf{a}(\gamma^{-1}) = S^{y\gamma},$$

and the assertion follows. $\square$

Now take $\xi \in \Gamma\alpha\Gamma$ and $j \in \{1, ..., \kappa\}$. Assume

$$T_j^{\mathtt{y}} \in \xi^{-1}\Gamma\xi \quad (\mathtt{y} \in \mathbf{Z}^+) \quad \text{and} \quad \xi(\eta_j) = \eta_k.$$

Then, for some $\mathtt{x} \in \mathbf{Z} - \{0\}$ :

$$\xi T_j^{\mathtt{y}} \xi^{-1} = T_k^{\mathtt{x}}, \qquad \text{since} \quad \xi T_j^{\mathtt{y}} \xi^{-1} \in \Gamma_{\eta_k}.$$

We might call $\mathtt{x}, \mathtt{y}$ "intertwining parameters". We will use this intertwining to force the form of $\xi$.

For a general $\xi \in \Gamma\alpha\Gamma$, and given $j$, we can find $\mathtt{y} \in \mathbf{Z}^+$ such that $T_j^{\mathtt{y}} \in \xi^{-1}\Gamma\xi$ by Lemma 1.7, and $A \in \Gamma$, $k \in \{1, ..., \kappa\}$ such that $\xi(\eta_j) = A\eta_k$, by Lemma 1.6. Our conclusion then is $A^{-1}\xi T_j^{\mathtt{y}}\xi^{-1}A = T_k^{\mathtt{x}}$ for some $\mathtt{x} \in \mathbf{Z} - \{0\}$. This means

$$A^{-1}\xi\left(N_j^{-1}S^{-\mathtt{y}}N_j\right)\xi^{-1}A = N_k^{-1}S^{-\mathtt{x}}N_k, \qquad \text{i.e.}$$

$$\left(N_kA^{-1}\xi N_j^{-1}\right)S^{\mathtt{y}}\left(N_j\xi^{-1}AN_k^{-1}\right) = S^{\mathtt{x}}.$$

Hence by Lemma 1.10:

$$\mathtt{x} > 0 \quad \text{and} \quad N_kA^{-1}\xi N_j^{-1} = S^r\mathbf{a}(\frac{\mathtt{x}}{\mathtt{y}}), \qquad r \in \mathbf{R}.$$

Now solve for $\xi$. We get:

**Lemma 1.11.** *Take $\xi \in \Gamma\alpha\Gamma$ and $j \in \{1, ..., \kappa\}$. There then exist $A_{\xi j} \in \Gamma$, $k(\xi, j) \in \{1, ..., \kappa\}$, $r_{\xi j} \in \mathbf{R}$ and $\gamma_{\xi j} > 0$ such that*

$$\xi = A_{\xi j}N_{k(\xi,j)}^{-1}S^{r_{\xi j}}\mathbf{a}(\gamma_{\xi j})N_j.$$

*Here $k(\xi, j)$ and $\gamma_{\xi j}$ are uniquely determined, while $r_{\xi j}$ is uniquely determined modulo $\mathbf{Z}$. In this situation, $\gamma_{\xi j} \in \mathbf{Q}^+$, and the cusp $\xi(\eta_j)$ is $\Gamma-$equivalent to $\eta_{k(\xi,j)}$. (The denominator of $\gamma_{\xi j}$ will be at most d.)*

*Proof.* We have proved all of this except uniqueness. We assume that $l, s, \sigma$ do the same job as $k, r, \gamma$, and get:

$$\Gamma N_k^{-1}S^r\mathbf{a}(\gamma)N_j = \Gamma N_l^{-1}S^s\mathbf{a}(\sigma)N_j$$

From this,

$$N_k^{-1}S^r\mathbf{a}(\gamma/\sigma)S^{-s}N_l \in \Gamma$$

Notice that this map takes $\eta_l$ to $\eta_k$, hence these cusps are $\Gamma-$equivalent, i.e. $l = k$, and our transformation lies in $\Gamma_{\eta_k}$. Hence for some $m \in \mathbf{Z}$

$$N_k^{-1}S^r\mathbf{a}(\gamma/\sigma)S^{-s}N_k = T_k^m,$$

hence (using $T_k^m = N_k^{-1}S^{-m}N_k$)

$$S^r\mathbf{a}(\gamma/\sigma)S^{-s} = S^{-m}$$

and thus $\gamma = \sigma$ and $r - s = -m$. Hence: $l = k$, $\sigma = \gamma$ and $r - s \in \mathbf{Z}$, as asserted.

Conversely, any $\mathbf{Z}-$shift of $r$ is indeed admissible, as is seen from

$$N_k^{-1}S^{r+m} = T_k^{-m}N_k^{-1}S^r, \qquad \forall m \in \mathbf{Z}. \tag{1.5}$$

□

*Remark 1.12.* From the proof it is clear that $k(\xi, j), r_{\xi j}, \gamma_{\xi j}$ are uniquely determined (resp. uniquely determined modulo $\mathbf{Z}$) already by $j$ and *the left coset* $\Gamma\xi$.

For $\xi$ and $j$ as in Lemma 1.11 we now define $x_{\xi j}$ and $y_{\xi j}$ by

$$\gamma_{\xi j} = \frac{x_{\xi j}}{y_{\xi j}}, \qquad x_{\xi j},\, y_{\xi j} \in \mathbf{Z}^+,\ (x_{\xi j}, y_{\xi j}) = 1. \tag{1.6}$$

These equal the intertwining parameters $\mathtt{x}$, $\mathtt{y}$ from above, reduced by their largest common divisor. We then have:

**Lemma 1.13.** *Take $\xi \in \Gamma\alpha\Gamma$, $j \in \{1, ..., \kappa\}$ and let $k = k(\xi, j)$, $r = r_{\xi j}$, $x = x_{\xi j}$ and $y = y_{\xi j}$ as above, so that*

$$\xi = A_{\xi j} N_k^{-1} S^r \mathbf{a}(\frac{x}{y}) N_j \qquad (\textit{with } A_{\xi j} \in \Gamma).$$

*Then:*

$$N_k^{-1} S^{r+v/y} \mathbf{a}(\frac{x}{y}) N_j \in \Gamma\alpha\Gamma$$

*for* every $v \in \mathbf{Z}$. *Two such elements determine the same left $\Gamma-$coset if and only if $v_1 \equiv v_2 \mod y\mathbf{Z}$.*

*Proof.* Since $(x, y) = 1$, we have $v = \mu y + \lambda x$ for some $\mu$, $\lambda \in \mathbf{Z}$. Then

$$\begin{aligned}
&N_k^{-1} S^{r+v/y} \mathbf{a}(\frac{x}{y}) N_j \\
&= N_k^{-1} S^{r+\mu+\lambda x/y} \mathbf{a}(\frac{x}{y}) N_j \\
&= N_k^{-1} S^{r+\mu} \mathbf{a}(\frac{x}{y}) S^{\lambda} N_j \quad \{\text{by } (1.1)\} \\
&= T_k^{-\mu} N_k^{-1} S^r \mathbf{a}(\frac{x}{y}) N_j T_j^{-\lambda} \quad \{\text{via } (1.5) \text{ and analog}\} \\
&= T_k^{-\mu} \left(A_{\xi j}^{-1} \xi\right) T_j^{-\lambda} \qquad \in \Gamma\alpha\Gamma.
\end{aligned}$$

Now suppose $v_1, v_2 \in \mathbf{Z}$ and

$$\Gamma N_k^{-1} S^{r+v_1/y} \mathbf{a}(\frac{x}{y}) N_j = \Gamma N_k^{-1} S^{r+v_2/y} \mathbf{a}(\frac{x}{y}) N_j. \tag{1.7}$$

Then from the uniqueness part of Lemma 1.11 (cf. Remark 1.12), it follows that $v_1 \equiv v_2 \mod y$. Conversely, if $v_1 \equiv v_2 \mod y$, then clearly (1.7) holds, as is seen from (1.5). $\square$

## 2. The operator $\mathcal{M}$ and the spectral decomposition of $L_2(\Gamma \setminus \mathcal{H})$

We will be working on the function space $L_2(\Gamma \setminus \mathcal{H})$, and represent its elements by measurable functions $f : \mathcal{H} \to \mathbf{C}$ satisfying

$$f(Tz) = f(z) \quad \text{for each } T \in \Gamma,\ z \in \mathcal{H}; \tag{2.1}$$

$$\int_{\mathcal{F}} |f(z)|^2 \, d\mu(z) < \infty. \tag{2.2}$$

Here $d\mu(z)$ is the hyperbolic area element; $d\mu(z) = y^{-2}\, dx\, dy$. The inner product in $L_2(\Gamma \setminus \mathcal{H})$ is then given by $\langle f_1, f_2 \rangle = \int_{\mathcal{F}} f_1(z)\overline{f_2(z)}\, d\mu(z)$.

The elements of other function spaces on $\Gamma \setminus \mathcal{H}$ (e.g. $C(\Gamma \setminus \mathcal{H})$) are represented similarly.

We take $\alpha \in \mathrm{Comm}(\Gamma)$ as before, and assume (see Lemma 1.4)

$$\Gamma\alpha\Gamma = \bigsqcup_{i=1}^{d} \Gamma\alpha_i \qquad (\alpha_i \in \Gamma\alpha\Gamma). \tag{2.3}$$

Then for $f \in L_2(\Gamma \setminus \mathcal{H})$ or $f \in C(\Gamma \setminus \mathcal{H})$ we define

$$(\mathcal{M}f)(z) = \sum_{i=1}^{d} f(\alpha_i z) \tag{2.4}$$

**Lemma 2.1.** *(a) $(\mathcal{M}f)(z)$ doesn't depend on the choice of $\alpha_i$ in* (2.3).

*(b) $(\mathcal{M}f)(Tz) = (\mathcal{M}f)(z)$ for each $T \in \Gamma$.*

*(c) $\mathcal{M}$ is a bounded linear operator $L_2(\Gamma \setminus \mathcal{H}) \to L_2(\Gamma \setminus \mathcal{H})$.*

*Proof.* (a) is trivial, and (b) follows at once from (a). Hence in (c), we only have to prove the boundedness of $\mathcal{M}$.

We fix the choice of $\alpha_i$:s in (2.3), and take $i \in \{1, ..., d\}$ and $f \in L_2(\Gamma \setminus \mathcal{H})$. Notice that $\alpha_i\mathcal{F}$ is a fundamental domain for $\alpha_i\Gamma\alpha_i^{-1}$. Now by Lemma 1.2, Remark 1.3 we can construct one fundamental domain $\mathcal{G}$ for $\left(\Gamma \cap \alpha_i\Gamma\alpha_i^{-1}\right)$ consisting of $d$ $\alpha_i\Gamma\alpha_i^{-1}-$translates of $\alpha_i\mathcal{F}$, including $\alpha_i\mathcal{F}$ itself, and another one, $\mathcal{G}'$, consisting of $d$ $\Gamma-$translates of $\mathcal{F}$. Using the $\Gamma-$invariance of $f$ we get

$$\int_{\mathcal{F}} |f(\alpha_i z)|^2 \, d\mu(z) = \int_{\alpha_i\mathcal{F}} |f(z)|^2 \, d\mu(z) \leq \int_{\mathcal{G}} |f(z)|^2 \, d\mu(z)$$
$$= \int_{\mathcal{G}'} |f(z)|^2 \, d\mu(z) = d \cdot \int_{\mathcal{F}} |f(z)|^2 \, d\mu(z).$$

Using this inequality for each $i \in \{1, ..., d\}$, (c) follows. □

Since $\mathrm{Comm}(\Gamma)$ is a group we have $\alpha^{-1} \in \mathrm{Comm}(\Gamma)$, and the same constuction as above can be carried out for this element. Thus for

$$\Gamma\alpha^{-1}\Gamma = \bigsqcup_{i=1}^{d} \Gamma\beta_i \qquad (\beta_i \in \Gamma\alpha^{-1}\Gamma) \tag{2.5}$$

and $f \in L_2(\Gamma \setminus \mathcal{H})$ or $f \in C(\Gamma \setminus \mathcal{H})$ we define

$$(\mathcal{M}^* f)(z) = \sum_{i=1}^{d} f(\beta_i z)$$

That we indeed get the same $d$ in (2.5) as in (2.3) follows from Lemma 1.4 applied to $\alpha$ and $\alpha^{-1}$, and Lemma 1.2.

Of course, the analog of Lemma 2.1 holds for $\mathcal{M}^*$.

**Lemma 2.2.** *For $f, g \in L_2(\Gamma \setminus \mathcal{H})$ we have:*

$$\langle \mathcal{M} f, g\rangle = \langle f, \mathcal{M}^* g\rangle$$

*Proof.* As in [H1, pp. 470-472], but simplified by $\chi \equiv 1$, $m = 0$. □

**Corollary 2.3.** *The norm of the operator $\mathcal{M} : L_2(\Gamma \setminus \mathcal{H}) \to L_2(\Gamma \setminus \mathcal{H})$ is $d$.*

*Proof.* We will prove $\|\mathcal{M} f\|_{L_2(\Gamma\setminus\mathcal{H})} \leq d \cdot \|f\|_{L_2(\Gamma\setminus\mathcal{H})}$ for each $f$ from $L_2(\Gamma \setminus \mathcal{H})$. By the continuity of $\mathcal{M}$, cf. Lemma 2.1(c), we need only consider those $f$ that satisfy $|f|^2 \in L_2(\Gamma \setminus \mathcal{H})$. Then

$$|(\mathcal{M} f)(z)|^2 = \left|\sum_{i=1}^{d} f(\alpha_i z)\right|^2 \leq d \cdot \sum_{i=1}^{d} |f(\alpha_i z)|^2 = d \cdot \left(\mathcal{M}\left(|f|^2\right)\right)(z)$$

for each $z \in \mathcal{H}$, and hence

$$\begin{aligned}\|\mathcal{M} f\|^2_{L_2(\Gamma\setminus\mathcal{H})} &\leq d \cdot \int_{\mathcal{F}} \mathcal{M}\left(|f|^2\right) d\mu = d \cdot \left\langle \mathcal{M}\left(|f|^2\right), 1\right\rangle \\ &= d \cdot \left\langle |f|^2, \mathcal{M}^*(1)\right\rangle = d^2 \cdot \|f\|^2_{L_2(\Gamma\setminus\mathcal{H})},\end{aligned}$$

using Lemma 2.2.

Conversely, take $f \equiv$ constant to see that this is the best possible bound. □

*Remark 2.4.* The proof of Lemma 2.1(c) yields only $\|\mathcal{M} f\|_{L_2(\Gamma\setminus\mathcal{H})} \leq d^{3/2} \cdot \|f\|_{L_2(\Gamma\setminus\mathcal{H})}$.

Let the $L_2$ eigenfunctions of the Laplace–Beltrami operator on $\Gamma \setminus \mathcal{H}$ be $\phi_0, \phi_1, \ldots$ We take these to be orthonormal and ordered with increasing eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ [H2, pp. 140(Th. 11.11), 298(line 6)]. Thus

$$\phi_n \in C^\infty \cap L_2(\Gamma \setminus \mathcal{H});$$

$$D\phi_n(z) + \lambda_n \phi_n(z) \equiv 0, \qquad \text{where } D = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right);$$

$$\langle \phi_n, \phi_m \rangle = \delta_{nm}.$$

Now $A$, *the discrete spectrum,* is defined as the closed subspace of $L_2(\Gamma \setminus \mathcal{H})$ spanned by all $\phi_n$:s. In general, we don't know if $A$ is infinite dimensional or not. For any $\lambda \geq 0$ one defines $A(\Gamma \setminus \mathcal{H}, \lambda)$ as the finite dimensional subspace of $L_2(\Gamma \setminus \mathcal{H})$ spanned by $\{\phi_n \mid \lambda_n = \lambda\}$. One also defines $A_0(\lambda)$ to be the subspace spanned by the *cusp forms* in $A(\Gamma \setminus \mathcal{H}, \lambda)$, and $A_1(\lambda)$ as its orthogonal complement in $A(\Gamma \setminus \mathcal{H}, \lambda)$. Cf. [H2, p. 282]. Notice that $\lambda \geq 1/4 \Longrightarrow A_1(\lambda) = 0$.

To describe the full spectral decomposition of $L_2(\Gamma \setminus \mathcal{H})$ we need the *Eisenstein series.* We now recall from [H2] some basic facts about these.

For $j \in \{1, .., \kappa\}$, $z \in \mathcal{H}$ and $\operatorname{Re} s > 1$, $E_j(z, s)$ is defined by the absolutely convergent sum

$$E_j(z,s) = \sum_{W_0 \in [T_j] \setminus \Gamma} (\operatorname{Im} N_j W_0 z)^s. \tag{2.6}$$

This function has a meromorphic continuation in $s$ to all of $\mathbf{C}$, and

$$E_j(Tz, s) = E_j(z, s), \qquad \forall\, T \in \Gamma,\ z \in \mathcal{H}; \tag{2.7}$$

$$E_j(z,s) \text{ is } C^\infty \text{ on } \mathcal{H} \times (\mathbf{C} - \{\text{poles}\});$$

$$D_z E_j(z,s) + s(1-s) E_j(z,s) = 0 \quad \text{ on } \mathcal{H} \times (\mathbf{C} - \{\text{poles}\}).$$

All poles of $E_j(z, s)$ in $\operatorname{Re} s \geq \frac{1}{2}$ are simple and restricted to the segment $(\frac{1}{2}, 1]$. In particular, the line $\operatorname{Re} s = \frac{1}{2}$ contains no pole. $a \in (\frac{1}{2}, 1]$ is a pole of some $E_j$ ($j \in \{1, ..., \kappa\}$) if and only if $A_1(a(1-a)) \neq 0$ [H2, pp. 78, 284, 297(E)]. For such a pole $a$, the functions $f_j(z) = \operatorname{Res}_{s=a} E_j(z, s)$, $j = 1, ..., \kappa$ span $A_1(a(1-a))$ [H2, p. 288 (3.19)]. As in [H2], we order these poles $a$ in a list

$$1 = s_0 > s_1 \geq s_2 \geq ... \geq s_{M_e} > \frac{1}{2}, \tag{2.8}$$

each $a$ occurring with multiplicity $|\{j \mid s_j = a\}| = \dim A_1(a(1-a))$.

We also need to recall the asymptotical behaviour of the Eisenstein series near cusps. Take $j, k \in \{1, .., \kappa\}$ and let $B$ be a compact subset of $\{\mathrm{Re}\, s > 1\}$. Then

$$E_j(z,s) = \delta_{jk} \cdot (\mathrm{Im}\, N_k z)^s + \varphi_{jk}(s) \cdot (\mathrm{Im}\, N_k z)^{1-s} + O(e^{-2\pi \cdot \mathrm{Im}\, N_k z}) \tag{2.9}$$

as $\mathrm{Im}\, N_k z \to \infty$ (i.e. $z \to \eta_k$ cuspidally), uniformly over $s \in B$. Cf. [H2, p. 280]. Here $\varphi_{jk}(s)$ is a meromorphic function in all of $\mathbf{C}$. All its poles in $\mathrm{Re}\, s \geq \frac{1}{2}$ are simple, and restricted to the set $\{s_0, ..., s_{M_e}\} \subset (\frac{1}{2}, 1]$, as for the Eisenstein series. Also, equation (2.9) extends to the left of $\mathrm{Re}\, s = 1$ [H2, p. 297(F)].

The basic result of the spectral analysis of $D$ on $\Gamma\backslash\mathcal{H}$ is that $L_2(\Gamma\backslash\mathcal{H})$ decomposes as a direct orthogonal sum $A \oplus \mathcal{E}$, where $A$, the discrete spectrum, is as above, and $\mathcal{E}$, *the continuous spectrum*, is spanned by the functions $E_j(z, \frac{1}{2} + it)$, $t \geq 0$, $j \in \{1, .., \kappa\}$ (see [H2, pp. 235, 291 (3.22), 317 (line 4)]).

It follows at once from (2.4) that $\mathcal{M}$ commutes with $D$. Hence by Lemma 2.1, $\mathcal{M}$ maps any discrete eigenfunction $\phi_n$ into $A(\Gamma \setminus \mathcal{H}, \lambda_n)$, and thus $\mathcal{M}[A] \subset A$. Similarly, $\mathcal{M}^*[A] \subset A$. Since $\mathcal{E} = A^\perp$, one also gets $\mathcal{M}[\mathcal{E}] \subset \mathcal{E}$. In fact, if $e \in \mathcal{E}$, then for all $a \in A$, $\langle \mathcal{M}e, a\rangle = \langle e, \mathcal{M}^* a\rangle = 0$, hence $\mathcal{M}e \in \mathcal{E}$. Similarly, $\mathcal{M}^*[\mathcal{E}] \subset \mathcal{E}$.

The trace formula will involve the trace Tr $\mathcal{M}_{|A(\Gamma\backslash\mathcal{H},\lambda)}$. Concerning this, we have

**Lemma 2.5.** *Take $\lambda \in \{\lambda_n \mid n \in \mathbf{Z}\}$. Then $\mathcal{M}_{|A(\Gamma\backslash\mathcal{H},\lambda)}$ is a linear operator on $A(\Gamma \setminus \mathcal{H}, \lambda)$, and if $\phi^{(1)}, ..., \phi^{(m)}$ is* any *orthonormal basis in $A(\Gamma \setminus \mathcal{H}, \lambda)$, then*

$$Tr\ \mathcal{M}_{|A(\Gamma\backslash\mathcal{H},\lambda)} = \sum_{n=1}^{m} \int_{\mathcal{F}} \left(\mathcal{M}\phi^{(n)}(z)\right) \overline{\phi^{(n)}(z)}\, d\mu(z).$$

*Proof.* That $\mathcal{M}$ maps $A(\Gamma \setminus \mathcal{H}, \lambda)$ into $A(\Gamma \setminus \mathcal{H}, \lambda)$ was seen above. Assume $\mathcal{M}\phi^{(n)} = \sum_{j=1}^{m} a_{nj}\phi^{(j)}$, for $n = 1, ..., m$. Then

$$\sum_{n=1}^{m} \int_{\mathcal{F}} \left(\mathcal{M}\phi^{(n)}(z)\right) \overline{\phi^{(n)}(z)}\, d\mu(z) = \sum_{n=1}^{m}\sum_{j=1}^{m} a_{nj} \int_{\mathcal{F}} \phi^{(j)}(z)\overline{\phi^{(n)}(z)}\, d\mu(z)$$

$$= \sum_{n=1}^{m}\sum_{j=1}^{m} a_{nj}\delta_{nj} = \sum_{n=1}^{m} a_{nn} = \mathrm{Tr}\ \mathcal{M}_{|A(\Gamma\backslash\mathcal{H},\lambda)}.$$

$\square$

The fact $\mathcal{M}[\mathcal{E}] \subset \mathcal{E}$ can be seen more explicitly by the next lemma, which gives the precise action of $\mathcal{M}$ on the Eisenstein series.

*From now on, we fix $\alpha_1, ..., \alpha_d$ in* (2.3) *once and for all.* For each $i \in \{1, ..., d\}$, $j \in \{1, ..., \kappa\}$ we have a decomposition

$$\alpha_i = A_{ij} N_{k(i,j)}^{-1} S^{r_{ij}} \mathbf{a}(\gamma_{ij}) N_j \qquad (\text{with } A_{ij} \in \Gamma), \tag{2.10}$$

obtained by taking $\xi = \alpha_i$ in Lemma 1.11. For $k, j \in \{1, ..., \kappa\}$ we define

$$\xi_{kj}(s) = \sum_{\substack{i=1 \\ k(i,j)=k}}^{d} \gamma_{ij}^s, \tag{2.11}$$

where the sum is taken over those $i \in \{1, .., d\}$ that satisfy $k(i, j) = k$. We then have:

**Lemma 2.6.**

$$\mathcal{M} E_k(z, s) = \sum_{j=1}^{\kappa} \xi_{kj}(s) \cdot E_j(z, s)$$

*for each* $k \in \{1, .., \kappa\}$, $z \in \mathcal{H}$, $s \in \mathbf{C} - \{poles\}$.

*Proof.* To simplify notation here and later we will write (as in [H2, p. 268]), given $j \in \{1, ..., \kappa\}$,

$$\begin{aligned} &\text{for } z \in \mathcal{H}: \qquad z_j = x_j + iy_j := N_j z \quad (x_j \in \mathbf{R},\ y_j > 0) \\ &\text{for } w \in \mathcal{H}: \qquad w_j = u_j + iv_j := N_j w \quad (u_j \in \mathbf{R},\ v_j > 0) \end{aligned}$$

For $i \in \{1, ..., d\}$, $j \in \{1, ..., \kappa\}$, and a *fixed* $s$ (Re $s > 1$), we study $E_k(\alpha_i z, s)$ as $y_j \to \infty$ (i.e. $z \to \eta_j$ cuspidally).

$$\begin{aligned} E_k(\alpha_i z, s) &= E_k(N_{k(i,j)}^{-1} S^{r_{ij}} \mathbf{a}(\gamma_{ij}) z_j, s) \quad \{\text{by } (2.10),(2.7)\} \\ &= E_k\left(N_{k(i,j)}^{-1}\left[\gamma_{ij} \cdot z_j + r_{ij}\right], s\right) \\ &= \delta_{k,k(i,j)} \cdot (\gamma_{ij} \cdot y_j)^s + \varphi_{k,k(i,j)}(s) \cdot (\gamma_{ij} \cdot y_j)^{1-s} \\ &\qquad + O\left(e^{-2\pi \gamma_{ij} \cdot y_j}\right). \qquad \{\text{by } (2.9)\} \end{aligned}$$

Hence from (2.4):

$$\begin{aligned} \mathcal{M} E_k(z, s) = &\left(\sum_{i=1}^{d} \delta_{k,k(i,j)} \cdot \gamma_{ij}^s\right) \cdot y_j^s + \left(\sum_{i=1}^{d} \varphi_{k,k(i,j)}(s) \cdot \gamma_{ij}^{1-s}\right) \cdot y_j^{1-s} \\ &+ O\left(e^{-\epsilon \cdot y_j}\right) \quad = \quad \xi_{kj}(s) \cdot y_j^s + O\left(y_j^{1-\mathrm{Re}\, s}\right), \end{aligned}$$

as $y_j \to \infty$. Now define

$$f(z) = \mathcal{M}E_k(z,s) - \sum_{j=1}^{\kappa} \xi_{kj}(s) \cdot E_j(z,s).$$

Clearly, $f \in C^\infty(\mathcal{H})$, $Df(z) + s(1-s)f(z) \equiv 0$, and $f$ is $\Gamma-$invariant. Also, by one more application of (2.9),

$$f(z) = O(y_j^{1-\mathrm{Re}\, s}),$$

as $y_j \to \infty$, for each $j \in \{1, ..., \kappa\}$. Thus $f \in L_2(\Gamma \setminus \mathcal{H})$. Together, these facts imply $f \equiv 0$ (cf. [H2, pp. 69(bottom)–71, 282 item 2]). Hence our assertion is true for $\mathrm{Re}\, s > 1$, and therefore for any $s \in \mathbf{C} - \{\text{poles}\}$, by analytic continuation. □

*Remark 2.7.* By exactly the same "cusp bookkeeping" manipulations, one can also study the $0-$order term in the Fourier expansion of any $\mathcal{M}\phi_n$. Cf. [H2, p. 282 (3.9)]. In particular, if $\phi_n$ is a cusp form, so is $\mathcal{M}\phi_n$ (this much follows directly from $\mathcal{M}D = D\mathcal{M}$ and Lemma 1.6).

## 3. Initial Development of the trace formula

From now on, we will assume that $\alpha \notin \Gamma$.

We recall some useful geometric expressions, involving the hyperbolic distance $d = d(z,w)$ on $\mathcal{H} \times \mathcal{H}$.

$$\frac{|z-w|^2}{\mathrm{Im}\, z \cdot \mathrm{Im}\, w} = 4\sinh^2(\frac{d}{2}) \tag{3.1}$$

$$\frac{|z-\overline{w}|^2}{\mathrm{Im}\, z \cdot \mathrm{Im}\, w} = \frac{|z-w|^2}{\mathrm{Im}\, z \cdot \mathrm{Im}\, w} + 4 = 4\cosh^2(\frac{d}{2}) \tag{3.2}$$

We now start to develop the trace formula, along the lines of [H2, pp. 166–210, 303–315]. One difference, however, is that, here, we can make use of known spectral properties of $L_2(\Gamma \setminus \mathcal{H})$ from the start.

We fix $\delta > 0$, and define our class of admissible functions $h(r)$ by:

$$\left\{ \begin{array}{l} h(r) \text{ is analytic in the strip } |\mathrm{Im}\, r| \le \frac{1}{2} + \delta \\ h(-r) = h(r) \\ h(r) \in \mathbf{R} \text{ for } r \in \mathbf{R} \\ h(r) = O(e^{-15|r|}) \text{ when } |\mathrm{Im}\, r| \le \frac{1}{2} + \delta \end{array} \right\} \tag{3.3}$$

We then form

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\cdot e^{-iru}\,dr$$

$$Q(x) = g(u) \qquad \text{for } x \geq 0,\ x = e^u + e^{-u} - 2,\ u \in \mathbf{R} \tag{3.4}$$

$$\Phi(t) = -\frac{1}{\pi}\int_t^{\infty} \frac{Q'(x)}{\sqrt{x-t}}\,dx = -\frac{2}{\pi}\int_0^{\infty} Q'(t+u^2)\,du, \qquad t \geq 0 \tag{3.5}$$

$$k(z,w) = \Phi\left[\frac{|z-w|^2}{\operatorname{Im} z\cdot \operatorname{Im} w}\right], \qquad \text{for } z,w \in \mathcal{H}. \tag{3.6}$$

Notice that $k(z,w)$ (for fixed $h(r)$) only depends on the hyperbolic distance $d(z,w)$, by (3.1). $k(z,w)$ is the inverse Selberg/Harish-Chandra transform of $h(r)$. To see its significance, cf. (3.12) below, and (in a different case than ours) [H1, pp. 8(Prop 3.1), 16(e)].

For $h$ as in (3.3), the transforms $g$, $Q$ and $\Phi$ will satisfy (cf. [H2, pp. 168–169], noticing that $\mathbf{R}^+ := [0,\infty)$):

$$g \text{ is even and holomorphic for } |\operatorname{Im} u| < 15$$

$$g^{(k)}(u) = O_k\left(e^{-(\frac{1}{2}+\delta)|u|}\right) \text{ for } u \in \mathbf{R},\ k \geq 0.$$

$$Q \in C^\infty(\mathbf{R}^+), \quad Q^{(j)}(x) = O_j\left((x+4)^{-j-\frac{1}{2}-\delta}\right), \text{ for } j \geq 0. \tag{3.7}$$

$$\Phi \in C^\infty(\mathbf{R}^+), \quad \Phi^{(j)}(t) = O_j\left((t+4)^{-j-1-\delta}\right), \text{ for } j \geq 0. \tag{3.8}$$

In studying convergence and majorants, it is useful to pass to a wider class of real $\Phi(t)$:s than those obtained from $h(r)$ as above. Following [H2, p. 100(top)], we define for $\gamma > 1$

$$\mathfrak{C}_\gamma = \{\Phi \in C^4(\mathbf{R}^+) \mid \Phi \text{ real-valued},$$
$$|\Phi^{(j)}(t)| \leq A(t+4)^{-\gamma-j} \text{ for } 0 \leq j \leq 4\}.$$

Thus for $\Phi(t)$ obtained from an admissible $h(r)$, we have $\Phi \in \mathfrak{C}_{1+\delta}$.

For $z,w \in \mathcal{H}$ one also forms

$$K(z,w) = \sum_{T\in\Gamma} k(z,Tw). \tag{3.9}$$

For any $\Phi \in \mathfrak{C}_{1+\delta}$, this sum is uniformly absolutely convergent for $(z,w)$ in $\mathcal{H}\times\mathcal{H}$ compacta, and [thus] $K$ is $\Gamma\times\Gamma-$invariant. Also, when $w$ approaches a cusp cuspidally, $K(z,w) \to 0$, uniformly over $z$ in $\mathcal{H}$ compacta. Cf. [H2, pp. 100, 174 (13.8), 306 (item 5)]. Thus for any $f \in L_2(\Gamma \setminus \mathcal{H})$, the function

$$Lf(z) \equiv \int_{\mathcal{F}} K(z,w) f(w)\, d\mu(w)$$

will be well defined, and in fact in $C(\Gamma \setminus \mathcal{H})$. We do *not* assert that $Lf \in L_2(\Gamma \setminus \mathcal{H})$.

Notice that, for any $\Phi \in \mathfrak{C}_{1+\delta}$ and $f \in L_2(\Gamma \setminus \mathcal{H})$, standard "unfolding" yields

$$Lf(z) = \int_{\mathcal{H}} k(z,w) f(w)\, d\mu(w), \tag{3.10}$$

with absolute convergence for each $z \in \mathcal{H}$. The necessary manipulations are justified by use of majorants, which also proves the absolute convergence: We replace $\Phi(t)$ by $A(t+4)^{-1-\delta}$, and use that the new $K(z,w)$ is again bounded for $w \in \mathcal{F}$, $z$ fixed. Together with $f \in L_2(\Gamma \setminus \mathcal{H}) \subset L_1(\Gamma \setminus \mathcal{H})$, this yields

$$\int_{\mathcal{F}} \sum_{T \in \Gamma} k^{[M]}(z, Tw)\, |f(w)|\, d\mu(w) < \infty, \tag{3.11}$$

$$\text{where} \qquad k^{[M]}(z,w) = A \left( \frac{|z-w|^2}{\mathrm{Im}\, z \mathrm{Im}\, w} + 4 \right)^{-1-\delta},$$

as required.

Using the absolute convergence in (3.10), we can extend the argument in [H1, pp. 8–10]. Cf. [H1, p. 16(e)]. Hence, if $\Phi(t)$ comes from an admissible $h(r)$, then for any $L_2$ eigenfunction $\phi_n$ we have

$$L\phi_n(z) = \int_{\mathcal{H}} k(z,w) \phi_n(w)\, d\mu(w) = h(r_n) \cdot \phi_n(z), \tag{3.12}$$

In this equation, $r_n$ is defined from $\lambda_n$ by

$$r_n^2 + \frac{1}{4} = \lambda_n, \qquad r_n \in [-\frac{i}{2}, 0] \cup [0, \infty). \tag{3.13}$$

**Lemma 3.1.** *Take $\Phi \in \mathfrak{C}_{1+\delta}$. Then $L\mathcal{M} = \mathcal{M}L$ on $L_2(\Gamma \setminus \mathcal{H})$.*

*Proof.* Take $f \in L_2(\Gamma \setminus \mathcal{H})$ and $\Phi \in \mathfrak{C}_{1+\delta}$. Notice that for each $i \in \{1, ..., d\}$ and each $z \in \mathcal{H}$, (3.11) gives

$$\int_{\mathcal{H}} k^{[M]}(z,w) |f(w)|\, d\mu(w) < \infty, \qquad \text{and}$$

$$\int_{\mathcal{H}} k^{[M]}(z,w) |f(\alpha_i w)|\, d\mu(w) = \int_{\mathcal{H}} k^{[M]}(z, \alpha_i^{-1} w) |f(w)|\, d\mu(w)$$

$$= \int_{\mathcal{H}} k^{[M]}(\alpha_i z, w) |f(w)|\, d\mu(w) < \infty.$$

Using this, (3.10) and Lemma 2.1(c), the proof of $L\mathcal{M}f(z) = \mathcal{M}Lf(z)$ is just like [H1, pp. 473-474(top)]. Notice that we do *not* assert $L\mathcal{M}f \in L_2(\Gamma \setminus \mathcal{H})$, only $L\mathcal{M}f \in C(\Gamma \setminus \mathcal{H})$. $\square$

For $h,\ g,\ Q,\ \Phi$ as above and $z, w \in \mathcal{H}$, one also defines

$$H(z,w) = \frac{1}{4\pi} \sum_{l=1}^{\kappa} \int_{-\infty}^{\infty} h(t) E_l(z, \frac{1}{2} + it) \overline{E_l(w, \frac{1}{2} + it)}\, dt \tag{3.14}$$

$$K_0(z,w) = K(z,w) - H(z,w). \tag{3.15}$$

The integrals in (3.14) are uniformly absolutely convergent for $(z,w)$ in $\mathcal{H} \times \mathcal{H}$ compacta, cf. [H2, pp. 301(a,d), 315(ii)]. $H$ and $K_0$ are clearly $\Gamma \times \Gamma-$invariant.

In [H2, pp. 176, 188–190, 306 (item 7, 8)] it is proved that $K_0(z,w)$ lies in $L_2(\Gamma \setminus \mathcal{H} \times \Gamma \setminus \mathcal{H})$, and that it has a spectral expansion

$$K_0(z,w) = \sum_{n \geq 0} h(r_n) \phi_n(z) \overline{\phi_n(w)} \qquad (r_n \text{ is as in (3.13)}), \tag{3.16}$$

with uniform absolute convergence on $\mathcal{H} \times \mathcal{H}$ compacta, and

$$\sum_{n \geq 0} |h(r_n) \phi_n(z) \phi_n(w)| = O\left( (\text{Im } N_k z)^{1-\beta} \, (\text{Im } N_j w)^{1-\beta} \right), \tag{3.17}$$

for $\text{Im } N_k z \geq 1$, $\text{Im } N_j w \geq 1$ (hence, in particular, whenever $z \to \eta_k$, $w \to \eta_j$ cuspidally). Here $\beta$ is any number strictly between $\frac{1}{2}$ and $s_{M_e}$ (cf. (2.8)), see [H2, p. 169].

Now to $K_0(z,w)$ there corresponds a bounded operator $\mathcal{L}$, mapping $f \in L_2(\Gamma \setminus \mathcal{H})$ to $\mathcal{L}f(z) = \int_{\mathcal{F}} K_0(z,w) f(w)\, d\mu(w) \in L_2(\Gamma \setminus \mathcal{H})$.

**Lemma 3.2.** $\mathcal{L}\mathcal{M} = \mathcal{M}\mathcal{L}$ *on* $L_2(\Gamma \setminus \mathcal{H})$.

*Proof.* It suffices to prove this on $A_f$ and on $\mathcal{E}$ separately. Here $A_f$ is the space of *finite* linear combinations of eigenfunctions $\phi_n$.

We know that $\mathcal{L} = L$ on $A_f$. See (3.12) and [H2, pp. 176(vi), 306(item 7)]. Recall Lemma 3.1 and Lemma 2.5, part one. We thus get, for any $a \in A_f$,

$$\mathcal{L}\mathcal{M}a = L[\mathcal{M}a] = \mathcal{M}[La] = \mathcal{M}[\mathcal{L}a].$$

Choose any $e \in \mathcal{E}$. We have seen that $\mathcal{M}[\mathcal{E}] \subset \mathcal{E}$. By [H2, pp. 188(e), 306(item 8)], $\mathcal{L}[\mathcal{E}] = 0$. Hence

$$\mathcal{L}[\mathcal{M}e] = \mathcal{L}[\text{element of } \mathcal{E}] = 0, \qquad \mathcal{M}[\mathcal{L}e] = \mathcal{M}[0] = 0.$$

$\square$

The trace formula for the modular correspondence $\mathcal{M}$ will now be obtained by considering Tr $[\mathcal{M}\mathcal{L}]$. We need to set up the appropriate kernel function to permit an integration. For $z, w \in \mathcal{H}$ we let

$$K_{\mathcal{M}}(z,w) = \mathcal{M}_z K(z,w) = \sum_{i=1}^{d} \sum_{T \in \Gamma} k(\alpha_i z, Tw);$$

$$\begin{aligned} H_{\mathcal{M}}(z,w) &= \mathcal{M}_z H(z,w) \\ &= \frac{1}{4\pi} \sum_{l=1}^{\kappa} \int_{-\infty}^{\infty} h(t) \left( \mathcal{M} E_l(z, \frac{1}{2} + it) \right) \overline{E_l(w, \frac{1}{2} + it)}\, dt; \end{aligned} \tag{3.18}$$

$$K_{0\mathcal{M}}(z,w) = \mathcal{M}_z K_0(z,w) = K_{\mathcal{M}}(z,w) - H_{\mathcal{M}}(z,w). \tag{3.19}$$

Then for any $f \in L_2(\Gamma \setminus \mathcal{H})$ :

$$\begin{aligned} \mathcal{M}[\mathcal{L}f](z) = \sum_{i=1}^{d} [\mathcal{L}f](\alpha_i z) &= \sum_{i=1}^{d} \int_{\mathcal{F}} K_0(\alpha_i z, w) f(w)\, d\mu(w) \\ &= \int_{\mathcal{F}} K_{0\mathcal{M}}(z,w) f(w)\, d\mu(w), \end{aligned}$$

i.e. $K_{0\mathcal{M}}(z,w)$ is the kernel function for the operator $\mathcal{M}\mathcal{L} = \mathcal{L}\mathcal{M}$. By (3.16),

$$K_{0\mathcal{M}}(z,w) = \sum_{n \geq 0} h(r_n) (\mathcal{M}\phi_n(z)) \overline{\phi_n(w)}, \tag{3.20}$$

with uniform absolute convergence on $\mathcal{H} \times \mathcal{H}$ compacta.

**Lemma 3.3.**

$$\int_{\mathcal{F}} K_{0\mathcal{M}}(z,z)\, d\mu(z) = \sum_{n \geq 0} h(r_n) \int_{\mathcal{F}} (\mathcal{M}\phi_n(z)) \overline{\phi_n(z)}\, d\mu(z),$$

*with absolute convergence.*

*Proof.* By (2.4) and the beautiful convergence of (3.20) on $\mathcal{H} \times \mathcal{H}$ compacta, it is enough to fix $i \in \{1, ..., d\}$, a cusp $\eta_j$ and a large $Y > 0$, and then prove

$$\int_{\mathcal{C}_{jY}} \sum_{n \geq 0} |h(r_n)| \cdot \left| \phi_n(\alpha_i z) \cdot \overline{\phi_n(z)} \right|\, d\mu(z) < \infty \tag{3.21}$$

(recall (1.2), (1.3)). By (2.10) and (3.17):

$$\begin{aligned}\sum_{n\geq 0}\left|h(r_n)\phi_n(\alpha_i z)\overline{\phi_n(z)}\right| &= \sum_{n\geq 0}\left|h(r_n)\phi_n\left(N_{k(i,j)}^{-1}S^{r_{ij}}\mathbf{a}(\gamma_{ij})z_j\right)\overline{\phi_n(z)}\right| \\ &= \sum_{n\geq 0}\left|h(r_n)\phi_n\left(N_{k(i,j)}^{-1}\left[\gamma_{ij}\cdot z_j + r_{ij}\right]\right)\overline{\phi_n(z)}\right| \\ &= O\left((\mathrm{Im}\,\gamma_{ij}z_j)^{1-\beta}\cdot(\mathrm{Im}\,N_j z)^{1-\beta}\right) \\ &= O\left(y_j^{2-2\beta}\right),\end{aligned}$$

as $y_j \to \infty$. From this, (3.21) follows (since $\beta > \frac{1}{2}$). $\square$

From Lemma 3.3 and Lemma 2.5 we deduce

$$\begin{aligned}\sum_{\lambda\geq 0} h(r_\lambda)\cdot \mathrm{Tr}\ \mathcal{M}_{|A(\Gamma\backslash\mathcal{H},\lambda)} &= \int_{\mathcal{F}} K_{0\mathcal{M}}(z,z)\,d\mu(z) \\ &= \int_{\mathcal{F}} \left(K_{\mathcal{M}}(z,z) - H_{\mathcal{M}}(z,z)\right)\,d\mu(z), \qquad (3.22)\end{aligned}$$

where the definition of $r_\lambda$ is similar to (3.13). In this relation, notice that

$$K_{\mathcal{M}}(z,w) = \sum_{i=1}^{d}\sum_{T\in\Gamma} k(\alpha_i z, Tw) = \sum_{T\in\Gamma\alpha^{-1}\Gamma} k(z,Tw), \qquad (3.23)$$

since $\Gamma\alpha^{-1}\Gamma = (\Gamma\alpha\Gamma)^{-1} = \bigsqcup_{i=1}^{d}\alpha_i^{-1}\Gamma$.

The problem now shifts to calculating the right hand side of (3.22) in group-theoretic language. This will involve, as in the ordinary Selberg trace formula, some *careful manipulations* and *regrouping* of terms. These manipulations will be carried out in the framework of $\Phi \in \mathfrak{C}_{1+\delta}$, and we shall always take care to assure proper types of majorized convergence.

**Definition.** We define the subsets $M$, $G$, $F_j$ of $\Gamma\alpha^{-1}\Gamma$ by:

$$M = \{T \in \Gamma\alpha^{-1}\Gamma \mid T \text{ has no cusp [of } \Gamma\text{] as a fixpoint}\};$$
$$G = \Gamma\alpha^{-1}\Gamma - M;$$
$$F_j = \{T \in \Gamma\alpha^{-1}\Gamma \mid T \text{ fixes } \eta_j\}, \qquad j \in \{1,..,\kappa\}.$$

(Remember that the cusps $\eta_j$ have already been fixed.)

For each $T \in \mathrm{PSL}(2,\mathbf{R})$ we also define

$$Z_\Gamma(T) = \{\sigma\in\Gamma \mid \sigma T = T\sigma\},$$

the centralizer of $T$ in $\Gamma$.

Notice that $M$ and $G$ are closed under conjugation with any $\Gamma-$element.

**Lemma 3.4.** *Take $j, k \in \{1, ..., \kappa\}$ and $\Phi \in \mathfrak{C}_{1+\delta}$. Then:*

$$\sum_{T \in \Gamma - \delta_{jk}[T_k]} |k(z, Tw)| = O\left((\mathrm{Im}\, N_k z)^{-\delta} \cdot (\mathrm{Im}\, N_j w)^{-\delta}\right)$$

*as $\mathrm{Im}\, N_k z$, $\mathrm{Im}\, N_j w \to \infty$. Here the summation is taken over all $T \in \Gamma$ if $k \neq j$, and over $T \in \Gamma - [T_k]$ if $k = j$.*

*Proof.* This is exactly like [H2, pp. 92–93]. We first assume $k \neq j$. From (3.6) and $\Phi \in \mathfrak{C}_{1+\delta}$ we get a majorizing sum:

$$\sum_{T \in \Gamma} |k(z, Tw)| = O(1) \sum_{T \in \Gamma} \left( \frac{\mathrm{Im}\, z \cdot \mathrm{Im}\, Tw}{|z - \overline{Tw}|^2} \right)^{1+\delta}$$
$$= O(1) \sum_{T \in \Gamma} \left( \frac{\mathrm{Im}\, N_k z \cdot \mathrm{Im}\, N_k Tw}{|N_k z - \overline{N_k Tw}|^2} \right)^{1+\delta}$$

(cf. (3.2)). Now choose representatives $W_0 \in \Gamma$ such that $\Gamma$ decomposes into disjoint cosets, $\Gamma = \bigsqcup_{W_0} [T_k] W_0$. Thus writing $T = T_k^n W_0$, we get

$$O(1) \sum_{W_0} \sum_{n \in \mathbf{Z}} y_k^{1+\delta} \left( \frac{\mathrm{Im}\, N_k T_k^n W_0 w}{|z_k - \overline{N_k T_k^n W_0 w}|^2} \right)^{1+\delta}$$
$$\left\{ \text{use } N_k T_k^n = S^{-n} N_k \right\}$$
$$= O(1) \sum_{W_0} \sum_{n \in \mathbf{Z}} y_k^{1+\delta} \left( \frac{\mathrm{Im}\, N_k W_0 w}{|z_k - \overline{N_k W_0 w} + n|^2} \right)^{1+\delta}.$$

The sum over $n$ is now treated as in [H2, p. 92]. For $y_k \to \infty$ we get

$$O(1) \sum_{W_0} y_k^{1+\delta} \cdot (\mathrm{Im}\, N_k W_0 w)^{1+\delta} (y_k + \mathrm{Im}\, N_k W_0 w)^{1-2(1+\delta)}$$
$$= O(1) y_k^{-\delta} \cdot \sum_{W_0} (\mathrm{Im}\, N_k W_0 w)^{1+\delta}.$$

By (2.6), (2.9) (remember $k \neq j$), the $W_0-$sum is $O((\mathrm{Im}\, N_j w)^{-\delta})$ as $\mathrm{Im}\, N_j w \to \infty$, and we are done.

In the case $k = j$ we get, by exactly the same manipulations,

$$\sum_{T \in \Gamma - [T_k]} |k(z, Tw)| = O(1) y_k^{-\delta} \sum_{W_0 \notin [T_k]} (\mathrm{Im}\, N_k W_0 w)^{1+\delta},$$

where, again, $\Gamma = \bigsqcup_{W_0} [T_k] W_0$. The $W_0-$sum is $O((\mathrm{Im}\, N_k w)^{-\delta})$ as $\mathrm{Im}\, N_k w \to \infty$, again by (2.6), (2.9). $\square$

**Lemma 3.5.** *Take $j \in \{1, ..., \kappa\}$, $\Phi \in \mathfrak{C}_{1+\delta}$. Then:*

$$\sum_{T \in \Gamma\alpha^{-1}\Gamma - F_j} |k(z, Tz)| = O\left((\operatorname{Im} N_j z)^{-2\delta}\right)$$

*as* $\operatorname{Im} N_j z \to \infty$. *In particular (cf.* (1.3)*),*

$$\int_{\mathcal{C}_{jY}} \sum_{T \in \Gamma\alpha^{-1}\Gamma - F_j} |k(z, Tz)| \, d\mu(z) < \infty.$$

*Proof.* The second statement follows trivially from the first. Since $\Gamma\alpha^{-1}\Gamma = \bigsqcup_{i=1}^{d} \alpha_i^{-1}\Gamma$, it is clearly enough to prove the result for the analogous sum over $\left(\alpha_i^{-1}\Gamma - F_j\right)$, $i$ fixed. Recall the decomposition (2.10), and let

$$\xi = N_{k(i,j)}^{-1} S^{r_{ij}} \mathbf{a}(\gamma_{ij}) N_j.$$

Then $\Gamma\alpha_i = \Gamma\xi$, thus $\alpha_i^{-1}\Gamma = \xi^{-1}\Gamma$, and we have to prove

$$\sum_{\substack{V \in \Gamma \\ \xi^{-1}V \notin F_j}} \left|k(z, \xi^{-1}Vz)\right| = O\left((\operatorname{Im} N_j z)^{-2\delta}\right)$$

as $\operatorname{Im} N_j z \to \infty$. Here

$$\begin{aligned} \sum_V \left|k(z, \xi^{-1}Vz)\right| &= \sum_V |k(\xi z, Vz)| \\ &= \sum_V \left|k\left(N_{k(i,j)}^{-1} S^{r_{ij}} \mathbf{a}(\gamma_{ij}) z_j, Vz\right)\right| \\ &= \sum_{\substack{V \in \Gamma \\ \xi^{-1}V \notin F_j}} \left|k\left(N_{k(i,j)}^{-1}\left[\gamma_{ij} \cdot z_j + r_{ij}\right], Vz\right)\right|. \end{aligned} \tag{3.24}$$

If $k(i,j) \neq j$, this is $O\left(\left(\gamma_{ij} y_j\right)^{-\delta} \cdot y_j^{-\delta}\right)$ as $y_j \to \infty$, by Lemma 3.4 (with $k = k(i,j) \neq j$). If $k(i,j) = j$ we get the same result, for then $\xi^{-1}$ fixes $\eta_j$, and hence all $V \in [T_j]$ are *excluded* in (3.24). In both cases we are done. $\square$

Now $M \subset \Gamma\alpha^{-1}\Gamma - \bigcup_{j=1}^{\kappa} F_j$, and we thus get for $\Phi \in \mathfrak{C}_{1+\delta}$, by Lemma 3.5:

$$\int_{\mathcal{F}} \sum_{T \in M} |k(z, Tz)| \, d\mu(z) < \infty. \tag{3.25}$$

Hence (in the case of admissible $h(r)$) we may continue from (3.22):

$$\int_{\mathcal{F}} \left(K_{\mathcal{M}}(z,z) - H_{\mathcal{M}}(z,z)\right)\, d\mu(z) = \sum_{T\in M} \int_{\mathcal{F}} k(z,Tz)\, d\mu(z)$$

$$+\int_{\mathcal{F}} \left(\sum_{T\in G} k(z,Tz) - H_{\mathcal{M}}(z,z)\right) d\mu(z). \tag{3.26}$$

Recall that $\alpha \notin \Gamma$.

**Theorem 3.6.** *$M$ contains only hyperbolic and elliptic elements. For $T \in M$ hyperbolic, let $N(T)$ denote the (unique) number $\gamma > 1$ making $T$ conjugate to $\mathbf{a}(\gamma)$ (in $PSL(2,\mathbf{R})$), and let $T_0$ be a generator of the centralizer $Z_\Gamma(T)$, which is cyclic and nontrivial. For $T \in M$ elliptic, let $M_T = |Z_\Gamma(T)|$, and let $\theta(T)$ denote the (unique) angle $\theta \in (0,\pi)$ making $T$ conjugate (in $PSL(2,\mathbf{R})$) to $\mathbf{k}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then, for admissible $h(r)$ :*

$$\sum_{T\in M} \int_{\mathcal{F}} k(z,Tz)\, d\mu(z) = \sum_{\substack{\{T\}\subset M,\\ \textit{hyperbolic}}} \frac{\ln N(T_0)}{N(T)^{1/2} - N(T)^{-1/2}} \cdot g(\ln N(T))$$

$$+ \sum_{\substack{\{T\}\subset \Gamma\alpha^{-1}\Gamma,\\ \textit{elliptic}}} \frac{1}{2M_T \cdot \sin\theta(T)} \cdot \int_{-\infty}^{\infty} \frac{e^{-2\theta(T)r}}{1+e^{-2\pi r}} \cdot h(r)\, dr,$$

*The $\{T\}$ sums are taken over distinct $\Gamma-$conjugacy classes. The various sums and integrals are all convergent with good majorants. The elliptic sum is finite.*

*Proof.* Due to (3.25), this is entirely similar to [H1, pp. 480–486, $m = 0$] (even though $\Phi$ there is taken to be of compact support). For the first statement, notice that Id $\notin M$ is obvious from $\alpha \notin \Gamma$, and if some $T \in M$ were parabolic, then $Z_\Gamma(T) = \{\mathrm{Id}\}$, and we would get a contradiction as on p. 485. The proof that $Z_\Gamma(T)$ is nontrivial for $T \in M$ hyperbolic is similar, cf. p. 483. Concerning the computation of the elliptic term, cf. [H1, pp. 351 (5.3), 449 (remark 9.4)] and [H2, pp. 389–396].

To get good majorants for all sums and integrals in the above equation, we take $\Phi(t) = (t+4)^{-1-\delta}$ in (3.25) and [H1, pp. 480–485]. Cf. [H1, p. 416 (6.18)]. In particular, [H1, p. 486, Prop. 3.1] extends to our case.

If $T \in \mathrm{PSL}(2,\mathbf{R})$ is elliptic, then $Z_\Gamma(T)$ consists exactly of those elliptic elements *in* $\Gamma$ that have the same fixpoint as $T$, plus the identity. Hence $M_T$ is bounded from above by the largest order of the elliptic points of $\Gamma$, which can be read off from the signature of $\Gamma$. Using this, [H1, p. 486, Prop. 3.1(b)] implies that the elliptic sum is finite.

This can also be seen more elementarily; the proof of [H1, p. 487, Prop. 3.2] goes through, with only one change: To see that $\beta_k(\overline{\mathcal{F}})$ has nonempty intersection only with a *finite* number of polygons $L(\overline{\mathcal{F}})$ ($L \in \Gamma$) (closures are taken in $\mathcal{H}$, *not* in $\overline{\mathcal{H}}$) we use that $\beta_k \in \Gamma\alpha^{-1}\Gamma$ takes cusps to cusps (Lemma 1.6 applied to $\alpha^{-1} \in \mathrm{Comm}(\Gamma)$). To study the situation near $\beta_k(\eta_j)$ in $\beta_k(\overline{\mathcal{F}})$, use an auxiliary conjugation to take $\beta_k(\eta_j) = \infty$. $\square$

In the remaining integral in (3.26), we partition $\mathcal{F}$ into its $\kappa$ cuspidal regions and the remaining part, as in (1.3), (1.4). We will keep $Y$ so large that (1.2) holds for all $j$, and so that the regions $\mathcal{C}_{jY}$ are mutually disjoint. Then our integral is:

$$\begin{aligned}
&\int_{\mathcal{F}} \left( \sum_{T\in G} k(z,Tz) - H_{\mathcal{M}}(z,z) \right) d\mu(z) \\
&\quad = \int_{\mathcal{F}_Y} \left( \sum_{T\in G} k(z,Tz) - H_{\mathcal{M}}(z,z) \right) d\mu(z) \\
&\quad\quad + \sum_{j=1}^{\kappa} \int_{\mathcal{C}_{jY}} \left( \sum_{T\in G} k(z,Tz) - H_{\mathcal{M}}(z,z) \right) d\mu(z) \\
&\quad = -\int_{\mathcal{F}_Y} H_{\mathcal{M}}(z,z)\, d\mu(z) \\
&\quad\quad + \sum_{T\in G} \int_{\mathcal{F}_Y} k(z,Tz)\, d\mu(z) + \sum_{j=1}^{\kappa} \sum_{T\in G-F_j} \int_{\mathcal{C}_{jY}} k(z,Tz)\, d\mu(z) \\
&\quad\quad + \sum_{j=1}^{\kappa} \int_{\mathcal{C}_{jY}} \left( \sum_{T\in F_j} k(z,Tz) - H_{\mathcal{M}}(z,z) \right) d\mu(z).
\end{aligned} \tag{3.27}$$

The change of order in the $\int_{\mathcal{C}_{jY}}$ –integrals is justified by Lemma 3.5. For the $\int_{\mathcal{F}_Y}$ –integral we use that *on* $\mathcal{H}$ *compacta*, $H_{\mathcal{M}}(z,z)$ is uniformly bounded, and $\sum_{T\in G} k(z,Tz)$ is uniformly absolutely convergent (via $G \subset \Gamma\alpha^{-1}\Gamma = \bigsqcup_{i=1}^{d} \alpha_i^{-1}\Gamma$ and (3.9)).

We now fix one $\Gamma-$conjugacy class $C \subset G$. We propose to collect the contribution from $C$ to the *fifth line* in (3.27). Notice that in this line we have nice majorized convergence for any $\Phi \in \mathfrak{C}_{1+\delta}$, due to Lemma 3.5. The computation below will be carried out for any such $\Phi$.

By the definition of $G$, each element of $C$ fixes a cusp, and by $\Gamma-$conjugation we can find a $T \in C$ such that $T$ fixes some $\eta_k$. Keep $T$ and $\eta_k$ fixed.

*We first assume $T$ to be hyperbolic.*

Then in $C = \{T\} = \{A^{-1}TA \mid A \in \Gamma\}$, all $A^{-1}TA$ :s are different. In fact, any $A \in \Gamma - \{\mathrm{Id}\}$ commuting with $T$ would have to be a hyperbolic map fixing both fixpoints of $T$, contradicting $\Gamma_{\eta_k} = \{T_k^n \mid n \in \mathbf{Z}\}$. We now get three cases, depending on what the *other* fixpoint of $T$ is. (the first is $\eta_k$).

Case 1:The other fixpoint of $T$ ($T$ hyperbolic) is a cusp equivalent to $\eta_k$.

We then write this fixpoint as $V\eta_k$, where $V \in \Gamma - \{\mathrm{Id}\}$. Then the contribution from $C = \{T\}$ to the fifth line in (3.27) is

$$\sum_{A\in\Gamma} \int_{\mathcal{F}_Y} k(z, A^{-1}TAz)\, d\mu(z) + \sum_{A\in\Gamma-[T_k]-V[T_k]} \int_{\mathcal{C}_{kY}} k(z, A^{-1}TAz)\, d\mu(z)$$
$$+ \sum_{j\neq k} \sum_{A\in\Gamma} \int_{\mathcal{C}_{jY}} k(z, A^{-1}TAz)\, d\mu(z)$$
$$= \sum_{A\in\Gamma} \int_{A(\mathcal{F}_Y)} k(z, Tz)\, d\mu(z) + \sum_{A\in\Gamma-[T_k]-V[T_k]} \int_{A(\mathcal{C}_{kY})} k(z, Tz)\, d\mu(z)$$
$$+ \sum_{j\neq k} \sum_{A\in\Gamma} \int_{A(\mathcal{C}_{jY})} k(z, Tz)\, d\mu(z).$$

By looking at the tiling and the Riemann surface $\Gamma \setminus \mathcal{H}$ (and neglecting certain sets of measure zero), we see that this is

$$\int_{(*)_{T,Y}} k(z, Tz)\, d\mu(z),$$

where $(*)_{T,Y}$ is the region

$$\mathcal{H} - N_k^{-1}\{z \in \mathcal{H} \mid y > Y\} - VN_k^{-1}\{z \in \mathcal{H} \mid y > Y\},$$

i.e. all of $\mathcal{H}$ except a "horoball" at $\eta_k$ and one at $V\eta_k$. These two horoballs will be disjoint, by our assumptions on $Y$.

Case 2:The other fixpoint of $T$ ($T$ hyperbolic) is a cusp equivalent to $\eta_l$, where $l \neq k$.

Write this fixpoint as $V\eta_l$, $V \in \Gamma$.

The contribution from $C = \{T\}$ to the fifth line in (3.27) is then, as above

$$\begin{aligned}
&\sum_{A\in\Gamma} \int_{A(\mathcal{F}_Y)} k(z,Tz)\,d\mu(z) + \sum_{A\in\Gamma-[T_k]} \int_{A(\mathcal{C}_{kY})} k(z,Tz)\,d\mu(z) \\
&+ \sum_{A\in\Gamma-V[T_l]} \int_{A(\mathcal{C}_{lY})} k(z,Tz)\,d\mu(z) + \sum_{j\notin\{k,l\}} \sum_{A\in\Gamma} \int_{A(\mathcal{C}_{jY})} k(z,Tz)\,d\mu(z) \\
&= \int_{(*)_{T,Y}} k(z,Tz)\,d\mu(z),
\end{aligned}$$

where $(*)_{T,Y}$ is the region

$$\mathcal{H} - N_k^{-1}\{z \in \mathcal{H} \mid y > Y\} - VN_l^{-1}\{z \in \mathcal{H} \mid y > Y\},$$

i.e. all of $\mathcal{H}$ minus two disjoint horoballs, one at $\eta_k$ and one at $V\eta_l$.

Case 3:The other fixpoint of $T$ ($T$ hyperbolic) is not a cusp.

This will now be shown to be impossible. The contribution from $C = \{T\}$ to the fifth line in (3.27) would here be

$$\begin{aligned}
&\sum_{A\in\Gamma} \int_{A(\mathcal{F}_Y)} k(z,Tz)\,d\mu(z) + \sum_{A\in\Gamma-[T_k]} \int_{A(\mathcal{C}_{kY})} k(z,Tz)\,d\mu(z) \\
&\quad + \sum_{j\neq k} \sum_{A\in\Gamma} \int_{A(\mathcal{C}_{jY})} k(z,Tz)\,d\mu(z) \\
&= \int_R k(z,Tz)\,d\mu(z), \quad \text{where } R = \mathcal{H} - N_k^{-1}\{z \mid y > Y\}.
\end{aligned}$$

After conjugating with a $\mathrm{PSL}(2,\mathbf{R})-$element that maps $\eta_k$ to $\infty$ and the other fixpoint of $T$ to 0, we get:

$$\int_{\mathrm{Im}\, z<Y'} \Phi\left(\frac{|z-\gamma z|^2}{\mathrm{Im}\, z \cdot \mathrm{Im}\, (\gamma z)}\right) d\mu(z),$$

where $Y' > 0$, $\gamma > 0$, $\gamma \neq 1$. However, this is divergent for $\Phi(t) = (t+4)^{-1-\delta}$, thus contradicting the majorized convergence obtained from Lemma 3.5.

*We next assume $T$ to be parabolic ($T\eta_k = \eta_k$ as before)*

Then in $C = \{T\} = \{A^{-1}TA \mid A \in \Gamma\}$, we have $A_1^{-1}TA_1 = A_2^{-1}TA_2$ if and only if $[T_k]A_1 = [T_k]A_2$. We choose representatives $W_0 \in \Gamma$ such that $\Gamma$ decomposes into disjoint cosets; $\Gamma = \bigsqcup_{W_0}[T_k]W_0$.

Now the contribution from $C = \{T\}$ to the fifth line in (3.27) is:

$$\begin{aligned}
&\sum_{W_0} \int_{W_0(\mathcal{F}_Y)} k(z,Tz)\,d\mu(z) + \sum_{W_0 \notin [T_k]} \int_{W_0(\mathcal{C}_{kY})} k(z,Tz)\,d\mu(z) \\
&\quad + \sum_{j \neq k} \sum_{W_0} \int_{W_0(\mathcal{C}_{jY})} k(z,Tz)\,d\mu(z) \\
&= \int_R k(z,Tz)\,d\mu(z), \quad \text{where } R = (\mathcal{F} - \mathcal{C}_{kY}) \bigsqcup \left( \bigsqcup_{W_0 \notin [T_k]} W_0(\mathcal{F}) \right)
\end{aligned}$$

(we assumed one of the $W_0$:s to be Id). But $k(z,Tz)$ is invariant under the shift $z \mapsto T_k^n z$ for all $n \in \mathbf{Z}$, since $T_k^{-n}TT_k^n = T$. Hence we may change $R$ to any region $R'$ *equivalent* to $R$ under $[T_k]$, and in particular to

$$R' = \{z \in \mathcal{H} \mid 0 < x_k < 1,\ y_k < Y\} = N_k^{-1}\left[(0,1) \times (0,Y)\right],$$

i.e. a "horo-sector" at $\eta_k$.

In all of these cases, we may now get rid of the assumption $T\eta_k = \eta_k$ by conjugating $T$ with an arbitrary $\Gamma-$element, keeping track of the horoball/horosector geometry.

The conjugacy manipulations being done (and justified) for general $\Phi \in \mathfrak{C}_{1+\delta}$, we finally switch back to admissible $h(r)$ (see (3.3)). Our result is that (3.27) equals

$$\begin{aligned}
&-\int_{\mathcal{F}_Y} H_{\mathcal{M}}(z,z)\,d\mu(z) + \sum_{\{T\} \subset G} \int_{(*)_{T,Y}} k(z,Tz)\,d\mu(z) \\
&+ \sum_{j=1}^{\kappa} \int_{\mathcal{C}_{jY}} \left( \sum_{T \in F_j} k(z,Tz) - H_{\mathcal{M}}(z,z) \right) d\mu(z), \qquad \text{where}
\end{aligned}$$

$$(*)_{T,Y} = \begin{cases} \mathcal{H} - V_1 N_k^{-1}\{z \mid y > Y\} - V_2 N_l^{-1}\{z \mid y > Y\} \\ V N_k^{-1}\left[(0,1) \times (0,Y)\right] \end{cases}.$$

Here the first case is for $T$ hyperbolic, with fixpoints $V_1\eta_k$, $V_2\eta_l$ ($V_1$, $V_2 \in \Gamma$, $k$ and $l$ may or may not be equal), and the other case is for $T$ parabolic, with fixpoint $V\eta_k$ ($V \in \Gamma$).

Our equality holds for any $Y$ large enough, and hence also as $Y \to \infty$. Concerning the last term in the above expression, we use (3.23) and (3.19) to write, for *each* $j \in \{1, ..., \kappa\}$ :

$$\sum_{T \in F_j} k(z, Tz) \; - \; H_{\mathcal{M}}(z, z) \; = \; K_{0\mathcal{M}}(z, z) \; - \sum_{T \in \Gamma \alpha^{-1} \Gamma - F_j} k(z, Tz).$$

By Lemma 3.3 (cf. (3.20), (3.21)) and Lemma 3.5 we see that this expression has a good $L_1-$majorant near the cusp $\eta_j$ on $\mathcal{F}$. Thus:

$$\lim_{Y \to \infty} \int_{\mathcal{C}_{jY}} \left| \sum_{T \in F_j} k(z, Tz) - H_{\mathcal{M}}(z, z) \right| d\mu(z) \quad = \quad 0.$$

Our result is

$$\int_{\mathcal{F}} \left( \sum_{T \in G} k(z, Tz) - H_{\mathcal{M}}(z, z) \right) d\mu(z)$$

$$= \lim_{Y \to \infty} \left( \sum_{\{T\} \subset G} \int_{(*)_{T,Y}} k(z, Tz) \, d\mu(z) - \int_{\mathcal{F}_Y} H_{\mathcal{M}}(z, z) \, d\mu(z) \right). \tag{3.28}$$

## 4. The contribution from hyperbolic $\{T\} \subset G$

We keep $h(r)$ admissible, see (3.3). Fix $T \in G$ and $Y > 0$ ($Y$ is kept large as on p. 22). We will express the integral coming from $\{T\}$ in (3.28) in terms of $h(r)$. We first *assume $T$ to be hyperbolic.*

Let the fixpoints of $T$ be $V_1 \eta_j$ and $V_2 \eta_k$ ($V_1, V_2 \in \Gamma$). We have to compute

$$\int_{(*)_{T,Y}} k(z, Tz) \, d\mu(z), \qquad \text{where}$$

$$(*)_{T,Y} = \mathcal{H} - V_1 N_j^{-1} \{\operatorname{Im} z > Y\} - V_2 N_k^{-1} \{\operatorname{Im} z > Y\}.$$

Choosing $x \in \mathbf{R}$ appropriately, the map

$$A = S^x \mathbf{a}(1/Y) N_j V_1^{-1}$$

will send $V_1 \eta_j$ to $\infty$ and $V_2 \eta_k$ to 0. Thus $ATA^{-1} = \mathbf{a}(\gamma)$ for some $\gamma > 0$, $\gamma \neq 1$. In fact, with $N(T)$ as in Theorem 3.6, we have $\gamma = N(T)$ if $V_1 \eta_j$ is the attracting fixpoint of $T$, otherwise $\gamma = N(T)^{-1}$. By construction, $A$ maps the horoball $V_1 N_j^{-1} \{\operatorname{Im} z > Y\}$ to $\{\operatorname{Im} z > 1\}$, and $V_2 N_k^{-1} \{\operatorname{Im} z > Y\}$ to a horoball at 0, i.e. to $\{z \mid |z - ir| < r\}$ for some $r > 0$. Here $r \leq 1/2$ by our disjointness assumption above (3.27). Hence our integral is

$$\int_{y \leq 1,\, |z-ir| \geq r} k(z, \gamma z)\, d\mu(z) \tag{4.1}$$

As $Y \to \infty$, note that $r \to 0$ necessarily.

Since any map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{R})$ with $c \neq 0$ maps $\{\mathrm{Im}\, z > Y\}$ to a horoball with Euclidean radius $(2c^2 Y)^{-1}$, and since the lower horoball in (4.1) is actually the image of $\{\mathrm{Im}\, z > Y\}$ under $AV_2N_k^{-1} = S^x\mathbf{a}(1/Y)N_jV_1^{-1}V_2N_k^{-1}$, we find

$$r = \frac{1}{2} \cdot \frac{1}{c^2Y^2}, \tag{4.2}$$

$$\text{where } c \text{ comes from } D := N_jV_1^{-1}V_2N_k^{-1} = \begin{pmatrix} * & * \\ c & * \end{pmatrix}.$$

One easily checks at this point that $c^2$ only depends on the conjugacy class $\{T\} \subset \Gamma\alpha^{-1}\Gamma$. In fact, suppose $T' = \sigma T \sigma^{-1}$, where $\sigma \in \Gamma$. Then the $\langle V_1, V_2, j, k\rangle-$tuple corresponding to $T'$ is $\langle \sigma V_1 T_j^n, \sigma V_2 T_k^m, j, k\rangle$ or $\langle \sigma V_2 T_k^n, \sigma V_1 T_j^m, k, j\rangle$, for some $n, m \in \mathbf{Z}$. The corresponding $c-$value is then taken from

$$\begin{aligned} D' &= N_j \left(\sigma V_1 T_j^n\right)^{-1} \left(\sigma V_2 T_k^m\right) N_k^{-1} = S^n D S^{-m} \qquad \text{or} \\ D'' &= N_k \left(\sigma V_2 T_k^n\right)^{-1} \left(\sigma V_1 T_j^m\right) N_j^{-1} = S^n D^{-1} S^{-m}, \end{aligned}$$

and is thus equal to our original $c$, up to sign.

Let us reexpress $c^2$ in a form more suitable for explicit computation. Since $T$ fixes $V_1\eta_j$ we have (by an analog of Lemma 1.9)

$$T = V_1 N_j^{-1} \mathbf{a}(\gamma) S^r N_j V_1^{-1}, \tag{4.3}$$

for some $r \in \mathbf{R}$. $\gamma$ is here the same as before.

**Lemma 4.1.** *Assume* $T = V_1N_j^{-1}\mathbf{a}(\gamma)S^rN_jV_1^{-1}$ *and let*

$$p = \frac{\gamma r}{1-\gamma} \in \mathbf{R} \subset \overline{\mathcal{H}}.$$

*Then* $p$ *is a cusp for* $N_j\Gamma N_j^{-1}$. *Let* $G$ *be a generator of the stabilizer* $(N_j\Gamma N_j^{-1})_p$, *and define* $\mathbf{c} = \mathbf{c}(N_j\Gamma N_j^{-1}, p)$ *by* $G = \begin{pmatrix} * & * \\ \mathbf{c} & * \end{pmatrix}$, $\mathbf{c} > 0$. *Then our* $c^2$ *from above equals* $\mathbf{c}(N_j\Gamma N_j^{-1}, p)$.

*Proof.* From the definition of $p$ we see that $[\mathbf{a}(\gamma)S^r]\,(p) = \gamma(p+r) = p$. Hence the two fixpoints of $T$ are $V_1N_j^{-1}(\infty) = V_1\eta_j$ and $V_1N_j^{-1}(p)$. Hence $V_1N_j^{-1}(p) = V_2\eta_k$, that is

$$p = N_jV_1^{-1}V_2N_k^{-1}(\infty) = D(\infty).$$

Also, trivially, $D^{-1}\left(N_j\Gamma N_j^{-1}\right)D = N_k\Gamma N_k^{-1}$, and this group has $\left(N_k\Gamma N_k^{-1}\right)_\infty = [S]$. Thus $p = D(\infty)$ is a cusp for $N_j\Gamma N_j^{-1}$, and $D^{-1}\left(N_j\Gamma N_j^{-1}\right)_p D = [S]$. Hence $G = DS^{\pm 1}D^{-1}$, and if $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ :

$$\begin{aligned} G &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \pm c \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} * & * \\ \pm c^2 & * \end{pmatrix}, \end{aligned}$$

as asserted. □

*Remark 4.2.* For any Fuchsian group $\Gamma'$ with a cusp $p \in \mathbf{R}$ we may define $\mathbf{c} = \mathbf{c}(\Gamma', p)$ as above, i.e. by $G = \begin{pmatrix} * & * \\ \mathbf{c} & * \end{pmatrix}$, $\mathbf{c} > 0$, where $G$ is a generator of $(\Gamma')_p$. Then

$$\mathbf{c}(\Gamma', p) = \inf\{c > 0 \mid \begin{pmatrix} 1 + cp & -cp^2 \\ c & 1 - cp \end{pmatrix} \in \Gamma'\}.$$

*Proof.* Let $\omega = \begin{pmatrix} 0 & 1 \\ -1 & p \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{R})$. Then $\omega(p) = \infty$, and thus $\infty$ is a cusp for $\omega\Gamma'\omega^{-1}$, and $\omega G\omega^{-1}$ is a generator of $(\omega\Gamma'\omega^{-1})_\infty$. Hence:

$$\omega G\omega^{-1} = S^{\pm x_0}, \quad \text{where } x_0 = \inf\{x > 0 \mid S^{-x} \in \omega\Gamma'\omega^{-1}\};$$
$$G = \omega^{-1}S^{\pm x_0}\omega, \quad \text{where } x_0 = \inf\{x > 0 \mid \omega^{-1}S^{-x}\omega \in \Gamma'\}.$$

Now our claim follows from

$$\begin{aligned} \omega^{-1}S^{-x}\omega &= \begin{pmatrix} p & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & p \end{pmatrix} \\ &= \begin{pmatrix} p & -1 - xp \\ 1 & -x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & p \end{pmatrix} = \begin{pmatrix} 1 + xp & -xp^2 \\ x & 1 - xp \end{pmatrix}. \end{aligned}$$

□

We now compute the integral in (4.1). By (3.6), it is

$$\int\int_{y \leq 1,\, x^2 + (y-r)^2 \geq r^2} \Phi\left(\frac{(1-\gamma)^2}{\gamma} \cdot \frac{(x^2 + y^2)}{y^2}\right) \frac{dx\,dy}{y^2}.$$

We substitute $x = \rho \cdot \cos v$, $y = \rho \cdot \sin v$ :

$$\begin{aligned}
&\int_0^{\pi} \int_{2r\cdot\sin v}^{1/\sin v} \Phi(\beta/\sin^2 v) \cdot \frac{d\rho\, dv}{\rho \cdot \sin^2 v} \qquad \left\{\text{where } \beta = \frac{(1-\gamma)^2}{\gamma}\right\} \\
&= -\int_0^{\pi} \Phi(\beta/\sin^2 v) \cdot \ln\left(2r\sin^2 v\right) \cdot \frac{dv}{\sin^2 v} \\
&= -2\int_0^{\pi/2} \Phi(\beta/\sin^2 v) \cdot \left(\ln 2r + \ln(\sin^2 v)\right) \cdot \frac{dv}{\sin^2 v}.
\end{aligned}$$

Substituting $w = \cot v$, and then using (3.5), we get

$$\begin{aligned}
&-2\int_0^{\infty} \Phi\left(\beta \cdot (1+w^2)\right) \cdot \left(\ln 2r - \ln(1+w^2)\right)\, dw \\
&= \frac{4}{\pi}\int_0^{\infty}\int_0^{\infty} Q'\left(\beta \cdot (1+w^2) + u^2\right) \cdot \left(\ln 2r - \ln(1+w^2)\right)\, du\, dw.
\end{aligned}$$

The double integral is nicely absolutely convergent, by (3.7). We substitute $w = \beta^{-1/2} \cdot s \cdot \sin\varphi$, $u = s \cdot \cos\varphi$ :

$$\frac{4}{\pi\sqrt{\beta}}\int_0^{\infty}\int_0^{\pi/2} Q'(\beta+s^2) \cdot \left(\ln 2r - \ln\left(1 + \frac{s^2}{\beta}\sin^2\varphi\right)\right) s\, d\varphi\, ds$$

One easily derives from [GR, p. 527 (4.224.9)] that:

$$\int_0^{\pi/2} \ln(1 + A\sin^2\varphi)\, d\varphi = \pi \cdot \ln\frac{1+\sqrt{1+A}}{2}, \qquad \text{for } A \geq 0.$$

Hence we get:

$$\begin{aligned}
&\frac{4}{\sqrt{\beta}} \cdot \int_0^{\infty} Q'(\beta+s^2) \cdot \left[\frac{1}{2}\ln 2r - \ln\left(\frac{1}{2}\left(1+\sqrt{1+\frac{s^2}{\beta}}\right)\right)\right] s\, ds \\
&\qquad\qquad \left\{q = \beta + s^2\right\} \\
&= \frac{2}{\sqrt{\beta}} \cdot \int_{\beta}^{\infty} Q'(q) \cdot \left[\frac{1}{2}\ln 2r - \ln\left(\frac{\sqrt{q}+\sqrt{\beta}}{2\sqrt{\beta}}\right)\right] dq \\
&= -\frac{\ln 2r}{\sqrt{\beta}} \cdot Q(\beta) - \frac{2}{\sqrt{\beta}} \cdot \int_{\beta}^{\infty} Q'(q) \cdot \ln\left(\frac{\sqrt{q}+\sqrt{\beta}}{2\sqrt{\beta}}\right) dq.
\end{aligned}$$

The first term is, via (3.4), $\beta = (1-\gamma)^2/\gamma = \gamma + \gamma^{-1} - 2$ and (4.2):

$$\left(2\ln Y + \ln(c^2)\right) \cdot \frac{g(\ln\gamma)}{|\gamma^{1/2} - \gamma^{-1/2}|}.$$

The second term is, after integrating by parts (using (3.7)):

$$\frac{1}{\sqrt{\beta}}\int_{\beta}^{\infty} Q(q)\cdot\frac{dq}{\sqrt{q}(\sqrt{q}+\sqrt{\beta})}.$$

We substitute $q = e^u + e^{-u} - 2$, and use (3.4) and $\beta = \gamma + \gamma^{-1} - 2$ :

$$\begin{aligned}&\frac{1}{\sqrt{\beta}}\int_{|\ln\gamma|}^{\infty} g(u)\cdot\frac{e^u - e^{-u}}{(e^{u/2}-e^{-u/2})(e^{u/2}-e^{-u/2}+\sqrt{\beta})}\,du\\ &=\frac{1}{\sqrt{\beta}}\int_{|\ln\gamma|}^{\infty} g(u)\cdot\frac{e^{u/2}+e^{-u/2}}{e^{u/2}-e^{-u/2}+\sqrt{\beta}}\,du.\end{aligned}$$

We can now collect our result. For hyperbolic $T \in G$ with fixpoints $V_1\eta_j$, $V_2\eta_k$ we have proved that:

$$\begin{aligned}\int_{(*)_{T,Y}} k(z,Tz)\,d\mu(z) \quad &= \big(2\ln Y + \ln(c^2)\big)\cdot\frac{g(\ln\gamma)}{|\gamma^{1/2}-\gamma^{-1/2}|}\\ &+\frac{1}{|\gamma^{1/2}-\gamma^{-1/2}|}\cdot\int_{|\ln\gamma|}^{\infty} g(u)\cdot\frac{e^{u/2}+e^{-u/2}}{e^{u/2}-e^{-u/2}+|\gamma^{1/2}-\gamma^{-1/2}|}\,du,\end{aligned}\tag{4.4}$$

where $\gamma = N(T)^{\pm 1}$ (the sign obviously doesn't matter), and $c^2$ is given by either (4.2) or Lemma 4.1 (cf. Remark 4.2).

## 5. The contribution from parabolic $\{T\} \subset G$

We now assume $T \in G$ to be parabolic. Via $\Gamma-$conjugation, we may assume its fixpoint to be $\eta_j$ ($j \in \{1, ..., \kappa\}$). It is convenient to define

$$T_j^x := N_j^{-1} S^{-x} N_j \qquad \text{for } x \in \mathbf{R};$$

for $x \in \mathbf{Z}$ this is nothing new.

By Lemma 1.9 applied to $N_j T N_j^{-1}$,

$$T = N_j^{-1} S^v N_j = T_j^{-v} \qquad \text{for some } v \in \mathbf{R} - \{0\}.$$

From $\alpha \notin \Gamma$ it follows that $T \notin \Gamma$, hence $v \notin \mathbf{Z}$. Now $T_j^{-v+n} = T_j^{-v} T_j^n$ lies in $G$ for each $n \in \mathbf{Z}$, and these are pairwise $\Gamma-$inconjugate. We assemble the contribution from all of these elements to (3.28). We may assume $v \in (0, 1)$.

$$
\begin{aligned}
&\sum_{n\in\mathbf{Z}} \int_{N_j^{-1}[(0,1)\times(0,Y)]} k(z, T_j^{-v+n} z)\, d\mu(z) \\
&\qquad \left\{\text{use } T_j^{-v+n} = N_j^{-1} S^{v-n} N_j\right\} \\
&= \sum_{n\in\mathbf{Z}} \int_{(0,1)\times(0,Y)} k(z, z+v-n)\, d\mu(z) \\
&= \sum_{n\in\mathbf{Z}} \int_0^Y \int_0^1 \Phi\left(\frac{(v-n)^2}{y^2}\right) \frac{dx\,dy}{y^2} \\
&= \sum_{n=0}^{\infty} \int_0^Y \Phi\left(\frac{(v+n)^2}{y^2}\right) \frac{dy}{y^2} \\
&\quad + \sum_{n=0}^{\infty} \int_0^Y \Phi\left(\frac{((1-v)+n)^2}{y^2}\right) \frac{dy}{y^2}.
\end{aligned} \tag{5.1}
$$

Look at the first sum. After the substitution $u = (v+n)/y$, it is

$$
\sum_{n=0}^{\infty} \frac{1}{v+n} \int_{(v+n)/Y}^{\infty} \Phi(u^2)\, du = \int_0^{\infty} \Phi(u^2) \left( \sum_{0\le n\le Yu-v} \frac{1}{v+n} \right) du,
$$

for by (3.8), we have nicely majorized convergence. Observe that for *fixed* $v \in (0,1)$ :

$$
\sum_{0\le n\le x-v} \frac{1}{v+n} = \ln x + C_v + O_v\left(\frac{1}{\sqrt{x}}\right), \quad \text{uniformly for } x > 0, \tag{5.2}
$$

where $C_v \in \mathbf{R}$ is a "generalized Euler constant". We get

$$
\int_0^{\infty} \Phi(u^2) \left( \ln(Yu) + C_v + O_v\left(\frac{1}{\sqrt{Yu}}\right)\right) du.
$$

Now mimic the computation in [H2, pp. 204–206]. We get

$$
\begin{aligned}
&\frac{1}{2}\left(\ln Y - \ln 2 + C_v - C\right) g(0) + \frac{1}{8} h(0) \\
&\quad - \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\, dr + O_v\left(\frac{1}{\sqrt{Y}}\right),
\end{aligned}
$$

where $C$ is the usual Euler constant (defined by taking $v = 1$ in (5.2); $C = C_1$). The last sum in (5.1) is entirely similar, since $1 - v \in (0,1)$.

Our final result is:

$$\sum_{n\in\mathbf{Z}} \int_{N_j^{-1}[(0,1)\times(0,Y)]} k(z, T_j^{-v+n} z)\, d\mu(z) \tag{5.3}$$
$$= \left(\ln Y - \ln 2 + \frac{C_v + C_{1-v}}{2} - C\right) g(0) + \frac{1}{4} h(0)$$
$$- \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\, dr + O_v\left(\frac{1}{\sqrt{Y}}\right).$$

We remark here that

$$C_v = -\frac{\Gamma'(v)}{\Gamma(v)} \qquad \text{for } v \in (0,1].$$

This is easily deduced from the well known formula

$$\frac{\Gamma'(v)}{\Gamma(v)} = -C - \frac{1}{v} - \sum_{n=1}^{\infty}\left(\frac{1}{v+n} - \frac{1}{n}\right) \qquad (v \in \mathbf{C} - \{0,-1,-2,...\}).$$

## 6. Enumeration of the $\Gamma$−conjugacy classes in $G$

Recall that for each $i \in \{1,...,d\}$ and $j \in \{1,...,\kappa\}$ we have a decomposition (2.10),

$$\alpha_i = A_{ij} N_{k(i,j)}^{-1} S^{r_{ij}} \mathbf{a}(\gamma_{ij}) N_j \qquad (\text{with } A_{ij} \in \Gamma).$$

Remember that we have *fixed* $\alpha_1, ..., \alpha_d$ in (2.3). When $i$ and $j$ are given, $k(i,j)$ and $\gamma_{ij}$ are uniquely determined, and $r_{ij}$ is uniquely determined mod $\mathbf{Z}$, by Lemma 1.11 (cf. also Remark 1.12). For each $i, j$ we *fix* one choice of $r_{ij}$.

We also define $x_{ij}, y_{ij} \in \mathbf{Z}^+$ by $\gamma_{ij} = x_{ij}/y_{ij}$, $(x_{ij}, y_{ij}) = 1$, as in (1.6).

We will describe (in Lemma 6.2 and Lemma 6.5) the $\Gamma$−conjugacy classes in $G$ in terms of our numbers $k(i,j)$, $r_{ij}$, $x_{ij}$, $y_{ij}$.

**Lemma 6.1.** *Let $i, j$ be given as above. Then there are exactly $y_{ij}$ values $i' \in \{1,...,d\}$ such that $k(i',j) = k(i,j)$, $\gamma_{i'j} = \gamma_{ij}$ and $r_{i'j} \equiv r_{ij}$ mod $y_{ij}^{-1}\mathbf{Z}$.*

*Proof.* Decomposing $\alpha_{i'}$ as above, and using Remark 1.12, we see that $i' \in \{1,...,d\}$ has the required properties if and only if

$$\exists v \in \mathbf{Z}: \qquad \Gamma\alpha_{i'} = \Gamma N_{k(i,j)}^{-1} S^{r_{ij}+v/y_{ij}} \mathbf{a}(\gamma_{ij}) N_j.$$

The map $i' \mapsto \Gamma\alpha_{i'}$ is a *bijection* between $\{1, ..., d\}$ and the set of left $\Gamma-$cosets in $\Gamma\alpha\Gamma$. The assertion now follows from Lemma 1.13 ($\xi = \alpha_i$). $\square$

For any $j \in \{1, ..., \kappa\}$ and $i \in \{1, ..., d\}$ such that $k(i,j) = j$ and $\gamma_{ij} \neq 1$, and any $v \in \mathbf{Z}$, we define

$$T_{jiv} := N_j^{-1}\mathbf{a}(\gamma_{ij}^{-1})S^{-r_{ij}-v/y_{ij}}N_j. \tag{6.1}$$

$T_{jiv}$ is a *hyperbolic* map with $\eta_j$ as one fixpoint (attracting if $\gamma_{ij}^{-1} > 1$, repelling if $\gamma_{ij}^{-1} < 1$). By Lemma 1.13 ($\xi = \alpha_i$), $T_{jiv}^{-1} \in \Gamma\alpha\Gamma$. Thus:

$$T_{jiv} \in G.$$

We will use these $T_{jiv}$:s as representatives for the $\Gamma-$conjugacy classes in $G$. Concerning conjugation with $T_j^m \in \Gamma$ (for any $m \in \mathbf{Z}$) we have:

$$\begin{aligned} T_j^m T_{jiv} T_j^{-m} &= \left(N_j^{-1}S^{-m}N_j\right)\left(N_j^{-1}\mathbf{a}(\gamma_{ij}^{-1})S^{-r_{ij}-v/y_{ij}}N_j\right)\left(N_j^{-1}S^m N_j\right) \\ &\left\{\text{we suppress indices } ij\right\} \\ &= N_j^{-1}S^{-m}\mathbf{a}(\gamma^{-1})S^{-r-v/y}S^m N_j \\ &= N_j^{-1}\mathbf{a}(\gamma^{-1})S^{-m\gamma}S^{-r-v/y}S^m N_j \quad \{\text{via } (1.1)\} \\ &= N_j^{-1}\mathbf{a}(\gamma^{-1})S^{-r-\frac{v-m(y-x)}{y}}N_j \\ &= T_{ji[v-m(y_{ij}-x_{ij})]}. \end{aligned} \tag{6.2}$$

**Lemma 6.2.** *We define $H$ as the set of all triples $\langle j, i, v\rangle$ such that*

*(a)* $j \in \{1, .., \kappa\}, \quad i \in \{1, .., d\}$;
*(b)* $k(i,j) = j$;
*(c)* $y_{ij} > x_{ij}$;
*(d)* $v \in \{1, .., y_{ij} - x_{ij}\}$.

*Then for each* hyperbolic $T \in G$ *there is some triple $\langle j, i, v\rangle \in H$ such that $T$ is $\Gamma-$conjugate to $T_{jiv}$. For each $\langle j, i, v\rangle \in H$, there are exactly $y_{ij}$ triples $\langle j', i', v'\rangle \in H$ (including $\langle j, i, v\rangle$ itself) such that $T_{jiv}$ and $T_{j'i'v'}$ are $\Gamma-$conjugate.*

*Proof.* Take $T \in G$ hyperbolic. From the impossibility of Case 3 on p. 24, we know that the *attracting* fixpoint of $T$ can be written as $A\eta_j$ for some $A \in \Gamma$, $j \in \{1, .., \kappa\}$. Now $A^{-1}TA \in \Gamma\alpha^{-1}\Gamma = \bigsqcup_{i=1}^{d}\alpha_i^{-1}\Gamma$. Hence $A^{-1}TA \in \alpha_i^{-1}\Gamma$ for some $i \in \{1, .., d\}$, and by (2.10)

$$A^{-1}TA = N_j^{-1}\mathbf{a}(\gamma_{ij}^{-1})S^{-r_{ij}}N_{k(i,j)}B \qquad \text{for some } B \in \Gamma.$$

But $A^{-1}TA\eta_j = \eta_j$, hence

$$N_{k(i,j)}B\eta_j = S^{r_{ij}}\mathbf{a}(\gamma_{ij})N_j\eta_j = \infty, \quad \text{i.e. } B\eta_j = \eta_{k(i,j)},$$

from which $k(i,j) = j$ and $B = T_j^n$ for some $n \in \mathbf{Z}$. Conjugating further with $T_j^m$, any $m \in \mathbf{Z}$, we get

$$\begin{aligned} T_j^m \left(A^{-1}TA\right) T_j^{-m} &= T_j^m \left(N_j^{-1}\mathbf{a}(\gamma_{ij}^{-1})S^{-r_{ij}}N_jT_j^n\right) T_j^{-m} \\ &\Big\{\text{mimic the computation in (6.2)}\Big\} \\ &= N_j^{-1}\mathbf{a}(\gamma_{ij}^{-1})S^{-m\gamma_{ij}-r_{ij}-n+m}N_j \\ &\Big\{\text{let } v = ny_{ij} + m(x_{ij} - y_{ij})\Big\} \\ &= N_j^{-1}\mathbf{a}(\gamma_{ij}^{-1})S^{-r_{ij}-v/y_{ij}}N_j \quad = \quad T_{jiv}. \end{aligned}$$

By construction, $\eta_j$ is actually the *attracting* fixpoint of $T_j^m A^{-1}TAT_j^{-m}$, hence $\gamma_{ij}^{-1} > 1$, i.e. $y_{ij} > x_{ij}$. Choosing $m$ appropriately we have $v \in \{1,..,y_{ij} - x_{ij}\}$ and $\langle j,i,v\rangle \in H$, and $T$ is $\Gamma-$conjugate to $T_{jiv}$.

To prove the second statement, assume $AT_{jiv}A^{-1} = T_{j'i'v'}$ for some $A \in \Gamma$ and $\langle j,i,v\rangle$, $\langle j',i',v'\rangle \in H$. Since $\eta_j$ and $\eta_{j'}$ are the *attracting* fixpoints of $T_{jiv}$ and $T_{j'i'v'}$ respectively, we must have $A\eta_j = \eta_{j'}$, and thus $j = j'$ and $A = T_j^n$ for some $n \in \mathbf{Z}$. Observe:

$$\begin{aligned} T_j^nT_{jiv}T_j^{-n} = T_{ji'v'} &\iff T_{ji[v-n(y_{ij}-x_{ij})]} = T_{ji'v'} \quad \{\text{via (6.2)}\} \\ &\iff \begin{cases} \gamma_{ij}^{-1} = \gamma_{i'j}^{-1} \\ -r_{ij} - \dfrac{v - n(y_{ij} - x_{ij})}{y_{ij}} = -r_{i'j} - \dfrac{v'}{y_{i'j}} \end{cases} \quad \{\text{via (6.1)}\} \\ &\iff \begin{cases} x_{ij} = x_{i'j} \\ y_{ij} = y_{i'j} \\ r_{i'j} - r_{ij} = \dfrac{v-v'}{y_{ij}} - \dfrac{y_{ij}-x_{ij}}{y_{ij}}n \end{cases}. \end{aligned}$$

From this we see that if $\langle j,i,v\rangle \in H$ is given, then for each $i' \in \{1,..,d\}$ satisfying $k(i',j) = j$, $\gamma_{i'j} = \gamma_{ij}$ and $r_{i'j} \equiv r_{ij} \pmod{y_{ij}^{-1}}$ there is exactly one $v' \in \{1,..,y_{ij} - x_{ij}\}$ such that $T_{jiv}$ and $T_{ji'v'}$ are $\Gamma-$conjugate, and *there are no other* $\langle j',i',v'\rangle$:s in $H$ making $T_{j'i'v'}$ $\Gamma-$conjugate to $T_{jiv}$. Our assertion is now a direct consequence of Lemma 6.1. $\square$

*Remark 6.3.* By instead considering the *repelling* fixpoint, one proves that *exactly the same statement* holds for the set $H'$ of all triples $\langle j, i, v\rangle$ such that

(a) $j \in \{1, .., \kappa\}, \quad i \in \{1, .., d\}$;
(b) $k(i,j) = j$;
(c) $y_{ij} < x_{ij}$;
(d) $v \in \{1, .., x_{ij} - y_{ij}\}$.

We can now compute the *hyperbolic contribution* to the sum over $\{T\} \subset G$ in (3.28). For $T = T_{jiv}$ in (4.4) we can take $\gamma = \gamma_{ij}^{-1}$, and get $c^2 = \mathbf{c}\left(N_j \Gamma N_j^{-1}, p\right)$, where (cf. (6.1) and Lemma 4.1)

$$p = \frac{\gamma_{ij}^{-1}(-r_{ij} - v/y_{ij})}{1 - \gamma_{ij}^{-1}} = \frac{v + y_{ij} r_{ij}}{y_{ij} - x_{ij}}.$$

Hence by (4.4), Lemma 6.2 and Remark 6.3, the hyperbolic contribution to (3.28) is

$$\frac{1}{2} \sum_{\langle j,i,v\rangle} \frac{1}{y_{ij}} \cdot \left[ \left( 2 \ln Y + \ln \mathbf{c} \left( N_j \Gamma N_j^{-1}, \frac{v + y_{ij} r_{ij}}{y_{ij} - x_{ij}} \right) \right) \cdot \frac{g(\ln \gamma_{ij})}{|\gamma_{ij}^{1/2} - \gamma_{ij}^{-1/2}|} \right.$$
$$\left. + \frac{1}{|\gamma_{ij}^{1/2} - \gamma_{ij}^{-1/2}|} \cdot \int_{|\ln \gamma_{ij}|}^{\infty} g(u) \cdot \frac{e^{u/2} + e^{-u/2}}{e^{u/2} - e^{-u/2} + |\gamma_{ij}^{1/2} - \gamma_{ij}^{-1/2}|}\, du \right],$$

where the sum is taken over all $\langle j, i, v\rangle \in H \cup H'$. Notice that

$$\frac{1}{y_{ij}} \cdot \frac{1}{|\gamma_{ij}^{1/2} - \gamma_{ij}^{-1/2}|} = \frac{\sqrt{\gamma_{ij}}}{|x_{ij} - y_{ij}|}.$$

Now carry out the addition over $v$. Using $\sum_{i,j}^{(h)}$ to denote summation over

$$\{\langle i, j\rangle \mid i \in \{1, ..., d\},\, j \in \{1, ..., \kappa\},\, k(i,j) = j,\, \gamma_{ij} \neq 1\},$$

we can write our result as

$$\sum_{\substack{\{T\}\subset G\\ \text{hyperbolic}}} \int_{(*)_{T,Y}} k(z,Tz)\,d\mu(z) = \sum_{i,j}^{(h)} \sqrt{\gamma_{ij}} \cdot g(\ln\gamma_{ij}) \cdot \ln Y \tag{6.3}$$

$$+\frac{1}{2}\sum_{i,j}^{(h)} \frac{\sqrt{\gamma_{ij}}}{|x_{ij}-y_{ij}|} \cdot \left( \sum_{v=1}^{|x_{ij}-y_{ij}|} \ln \mathbf{c}\left( N_j\Gamma N_j^{-1}, \frac{v+y_{ij}r_{ij}}{y_{ij}-x_{ij}} \right) \right) \cdot g(\ln\gamma_{ij})$$

$$+\frac{1}{2}\sum_{i,j}^{(h)} \sqrt{\gamma_{ij}} \cdot \int_{|\ln\gamma_{ij}|}^{\infty} g(u) \cdot \frac{e^{u/2}+e^{-u/2}}{e^{u/2}-e^{-u/2}+|\gamma_{ij}^{1/2}-\gamma_{ij}^{-1/2}|}\,du$$

*Remark 6.4.* Using (6.2), we see that the sum over $v$ actually could be taken over any set of representatives $v \in \mathbf{Z}$ mod $|x_{ij}-y_{ij}|$. One may also check this more directly at this point: For *any* parabolic fixpoint $p \in \mathbf{R}$ of $N_j\Gamma N_j^{-1}$ one easily verifies $\mathbf{c}(N_j\Gamma N_j^{-1}, p+1) = \mathbf{c}(N_j\Gamma N_j^{-1}, p)$, via $S \in N_j\Gamma N_j^{-1}$.

The corresponding treatment of the *parabolic* conjugacy classes in $G$ is easier. For each $j \in \{1,...,\kappa\}$ and $i \in \{1,...,d\}$ such that $k(i,j) = j$ and $\gamma_{ij} = 1$, (2.10) gives $N_j^{-1}S^{r_{ij}}N_j \in \Gamma\alpha_i \subset \Gamma\alpha\Gamma$, and thus $N_j^{-1}S^{-r_{ij}}N_j$, i.e. $T_j^{r_{ij}}$, lies in $\Gamma\alpha^{-1}\Gamma$. Notice that $r_{ij} \notin \mathbf{Z}$, since $\alpha \notin \Gamma$. Hence for all $n \in \mathbf{Z}$,

$$T_j^{r_{ij}+n} \text{ is parabolic, with fixpoint } \eta_j, \text{ and } T_j^{r_{ij}+n} \in G.$$

**Lemma 6.5.** *For each* parabolic $T \in G$ *there is exactly one triple* $\langle j,i,n\rangle$ *satisfying* $j \in \{1,...,\kappa\}$, $i \in \{1,..,d\}$, $k(i,j) = j$, $\gamma_{ij} = 1$ *and* $n \in \mathbf{Z}$, *such that* $T$ *is* $\Gamma-$*conjugate to* $N_j^{-1}S^{-r_{ij}-n}N_j$, *i.e.* $T_j^{r_{ij}+n}$.

*Proof.* Take $T \in G$ parabolic. Write its fixpoint as $A\eta_j$ ($A \in \Gamma$, $j \in \{1,...,\kappa\}$). As in the proof of Lemma 6.2 we then have $A^{-1}TA \in \alpha_i^{-1}\Gamma$ for some $i \in \{1,...,d\}$ such that $k(i,j) = j$, and

$$A^{-1}TA = N_j^{-1}\mathbf{a}(\gamma_{ij}^{-1})S^{-r_{ij}}N_jT_j^n$$

for some $n \in \mathbf{Z}$. This equals $N_j^{-1}\mathbf{a}(\gamma_{ij}^{-1})S^{-r_{ij}-n}N_j$. But $T$, and hence also $A^{-1}TA$, is parabolic. Thus $\gamma_{ij} = 1$, and the existence is proved.

To prove uniqueness, assume

$$AT_j^{r_{ij}+n}A^{-1} = T_{j'}^{r_{i'j'}+n'} \tag{6.4}$$

for some $A \in \Gamma$ and admissible $\langle j,i,n\rangle$, $\langle j',i',n'\rangle$. By looking at the fixpoint we get $A\eta_j = \eta_{j'}$, hence $j' = j$ and $A = T_j^m$ for some $m \in \mathbf{Z}$. Then by (6.4), $r_{i'j} - r_{ij} \in \mathbf{Z}$, and by decomposing $\alpha_i$, $\alpha_{i'}$ as usual and using (1.5) we see $\Gamma\alpha_i = \Gamma\alpha_{i'}$, hence $i' = i$. Then by (6.4), $n' = n$. Hence $\langle j', i', n'\rangle = \langle j, i, n\rangle$. □

We now compute the *parabolic contribution* to the sum over $\{T\} \subset G$ in (3.28). Look at (5.3). The $v$ there is now $\equiv -r_{ij} \pmod 1$. For $r \in \mathbf{R} - \mathbf{Z}$ we define

$$C(r) = \frac{C_v + C_{1-v}}{2} - C, \qquad \text{where } v \in (0,1) \text{ has } v \equiv -r \pmod 1.$$

We use $\sum_{i,j}^{(p)}$ to denote summation over

$$\{\langle i, j\rangle \mid i \in \{1, ..., d\},\ j \in \{1, ..., \kappa\},\ k(i,j) = j,\ \gamma_{ij} = 1\}.$$

Then our result is:

$$\begin{aligned} \text{(6.5)} \qquad \sum_{\substack{\{T\}\subset G \\ \text{parabolic}}} \int_{(*)_{T,Y}} k(z, Tz)\, d\mu(z) &= {\sum_{i,j}}^{(p)} g(0) \cdot \ln Y \\ &+ {\sum_{i,j}}^{(p)} \Bigg[ (C(r_{ij}) - \ln 2) \cdot g(0) + \frac{1}{4} h(0) \\ &- \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\, dr \Bigg] + O\left(\frac{1}{\sqrt{Y}}\right). \end{aligned}$$

## 7. The contribution from $H_{\mathcal{M}}(z, w)$

Finally, we need to calculate $\int_{\mathcal{F}_Y} H_{\mathcal{M}}(z,z)\, d\mu(z)$ from (3.28). By (3.18) and Lemma 2.6, $H_{\mathcal{M}}(z, w)$ equals

$$\frac{1}{4\pi} \sum_{l=1}^{\kappa} \int_{-\infty}^{\infty} h(t) \left( \sum_{j=1}^{\kappa} \xi_{lj}(\frac{1}{2} + it) E_j(z, \frac{1}{2} + it) \right) \overline{E_l(w, \frac{1}{2} + it)}\, dt.$$

Here $\xi_{lj}(\frac{1}{2} + it)$ is as in (2.11). In particular it is bounded. See [H2, pp. 301 (a,d), 315 (ii)] for good uniform bounds on the Eisenstein series. We get:

$$\begin{aligned} \text{(7.1)} \qquad & \int_{\mathcal{F}_Y} H_{\mathcal{M}}(z,z)\, d\mu(z) \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t) \sum_{j,l} \xi_{lj}(\frac{1}{2} + it) \int_{\mathcal{F}_Y} E_j(z, \frac{1}{2} + it) \overline{E_l(z, \frac{1}{2} + it)}\, d\mu(z)\, dt. \end{aligned}$$

Recall the decomposition of $\mathcal{F}$ into $\kappa$ cuspidal regions and the remaining part, (1.3), (1.4). For each cusp $\eta_j$ and $z \in \mathcal{F}$, we define

$$E_0^{(j)}(z,s) = \begin{cases} E_j(z,s) & \text{if } z \in \mathcal{F}_B \\ E_j(z,s) - \delta_{jk} y_k^s - \varphi_{jk}(s) \cdot y_k^{1-s} & \text{if } z \in \mathcal{C}_{kB} \end{cases} \tag{7.2}$$

Cf. [H2, p. 301]. Here $B$ is a large constant, as in [H2, p. 300]. Now for fixed cusps $\eta_j$, $\eta_l$, take $u(z) = E_0^{(j)}(z,s)$, $v(z) = E_0^{(l)}(z,s)$, and do a quick mimic of [H2, pp. 153(bottom)–155(line 5)] along the lines of [H2, pp. 284–285]. We get

$$\begin{aligned} \int_{\mathcal{F}} E_0^{(j)}(z,s)\overline{E_0^{(l)}(z,s)}\, d\mu(z) =& \frac{\delta_{jl} B^{s+\overline{s}-1}}{s+\overline{s}-1} + \varphi_{jl}(s)\frac{B^{\overline{s}-s}}{\overline{s}-s} + \overline{\varphi_{lj}(s)}\frac{B^{s-\overline{s}}}{s-\overline{s}} \\ &+ \left(\sum_k \varphi_{jk}(s)\overline{\varphi_{lk}(s)}\right)\frac{B^{1-s-\overline{s}}}{1-s-\overline{s}}, \end{aligned} \tag{7.3}$$

for $s \in \mathbf{C}$ with $\operatorname{Re} s > 1/2$, $\operatorname{Im} s \neq 0$.

Now mimic [H2, pp. 200(bottom)-201(top)] via [H2, p. 301(d)]. We get

$$\begin{aligned} \int_{\mathcal{F}_Y} E_j(z,s)\overline{E_l(z,s)}\, d\mu(z) =& \frac{\delta_{jl} Y^{2\sigma-1}}{2\sigma-1} - \varphi_{jl}(s)\frac{Y^{-2it}}{2it} + \overline{\varphi_{lj}(s)}\frac{Y^{2it}}{2it} \\ &+ \left(\sum_k \varphi_{jk}(s)\overline{\varphi_{lk}(s)}\right)\frac{Y^{1-2\sigma}}{1-2\sigma} + O\left(\omega(t)\cdot e^{6|t|-4\pi Y}\right), \end{aligned} \tag{7.4}$$

uniformly for $s = \sigma + it$, $\frac{1}{2} < \sigma \leq \frac{3}{2}$, $t \neq 0$, $|s - s_i| \geq \epsilon$ $(i = 0, 1, ..., M_e)$, $Y > B$. Here $\omega(t)$ is an even function of $t \in \mathbf{R}$, $\omega(t) \geq 1$ and

$$\int_{-T}^{T} \omega(t)\, dt = O(T^2) \quad \text{as } T \to \infty. \tag{7.5}$$

Cf. [H2, pp. 161, 299 (line 14), 315 (ii)].

Before continuing, let us note some relations. We have

$$\varphi_{lk}(s) \equiv \overline{\varphi_{lk}(\overline{s})} \equiv \varphi_{kl}(s). \tag{7.6}$$

by [H2, p. 281 (e) and (3.8)]. See also [H2, p. 488 (bottom line)]. One defines

$$\Phi(s) = \left(\varphi_{jk}(s)\right), \tag{7.7}$$

a $\kappa \times \kappa$–matrix (cf. [H2, p. 281]). It satisfies

$$\Phi(s)\Phi(1-s) = I, \qquad \text{i.e. } \sum_k \varphi_{jk}(s)\varphi_{kl}(1-s) = \delta_{jl}, \tag{7.8}$$

for generic $s \in \mathbf{C}$ (cf. [H2, p. 296 (B)]). Differentiating, we get

$$\sum_k \varphi'_{jk}(s)\varphi_{kl}(1-s) = \sum_k \varphi_{jk}(s)\varphi'_{kl}(1-s). \tag{7.9}$$

Also, for $s = \frac{1}{2} + it$, (7.6) and (7.8) gives:

$$\Phi(\frac{1}{2}+it) \text{ is } \textit{unitary}, \text{ i.e. } \sum_k \varphi_{jk}(\frac{1}{2}+it)\overline{\varphi_{lk}(\frac{1}{2}+it)} = \delta_{jl}. \tag{7.10}$$

We now fix $t \neq 0$ and let $\sigma = \frac{1}{2} + h$, $h \to 0^+$ in (7.4). Then

$$\begin{aligned}
&\frac{\delta_{jl}Y^{2\sigma-1}}{2\sigma-1} + \frac{\left(\sum_k \varphi_{jk}(s)\overline{\varphi_{lk}(s)}\right)Y^{1-2\sigma}}{1-2\sigma} \\
&= \frac{\delta_{jl}Y^{2h} - \left(\sum_k \varphi_{jk}(s)\overline{\varphi_{lk}(s)}\right)Y^{-2h}}{2h}. \\
&\left\{\text{Write } Y^{2h} = Y^{2h} - Y^{-2h} + Y^{-2h} \text{ and use (7.10)}\right\} \\
&\longrightarrow 2\delta_{jl} \cdot \ln Y - \frac{1}{2}\sum_k D_h \left(\varphi_{jk}(\frac{1}{2}+h+it)\overline{\varphi_{lk}(\frac{1}{2}+h+it)}\right)_{|h=0}.
\end{aligned}$$

Notice that by (7.6) and (7.9),

$$\begin{aligned}
&\sum_k \varphi'_{jk}(\frac{1}{2}+it)\overline{\varphi_{lk}(\frac{1}{2}+it)} = \sum_k \varphi_{jk}(\frac{1}{2}+it)\overline{\varphi'_{lk}(\frac{1}{2}+it)}, \\
&\text{and if } j = l, \text{ this is real.}
\end{aligned} \tag{7.11}$$

Hence for $t \neq 0$:

$$\begin{aligned}
&\int_{\mathcal{F}_Y} E_j(z,\frac{1}{2}+it)\overline{E_l(z,\frac{1}{2}+it)}\,d\mu(z) \\
&\quad = 2\delta_{jl} \cdot \ln Y - \sum_k \left(\varphi'_{jk}(\frac{1}{2}+it)\overline{\varphi_{lk}(\frac{1}{2}+it)}\right) \\
&\qquad + 2 \cdot \mathrm{Re}\left(\overline{\varphi_{jl}(\frac{1}{2}+it)}\frac{Y^{2it}}{2it}\right) + O\left(\omega(t) \cdot e^{6|t|-4\pi Y}\right),
\end{aligned}$$

and thus:

$$\sum_{j,l} \xi_{lj}(\frac{1}{2}+it) \int_{\mathcal{F}_Y} E_j(z,\frac{1}{2}+it)\overline{E_l(z,\frac{1}{2}+it)}\, d\mu(z)$$

$$\begin{aligned} &= 2\left(\sum_j \xi_{jj}(\frac{1}{2}+it)\right)\cdot \ln Y \\ &\quad - \sum_{j,l,k} \xi_{lj}(\frac{1}{2}+it)\left(\varphi'_{jk}(\frac{1}{2}+it)\overline{\varphi_{lk}(\frac{1}{2}+it)}\right) \\ &\quad + \sum_{j,l} \xi_{lj}(\frac{1}{2}+it)\cdot 2\cdot \mathrm{Re}\,\left(\overline{\varphi_{jl}(\frac{1}{2}+it)}\frac{Y^{2it}}{2it}\right) \\ &\quad + O\left(\sum_{j,l}\left|\xi_{lj}(\frac{1}{2}+it)\right|\cdot\omega(t)\cdot e^{6|t|-4\pi Y}\right). \end{aligned} \tag{7.12}$$

Now to compute (7.1), we multiply (7.12) with $(4\pi)^{-1}h(t)$ and integrate over $t \in \mathbf{R}$. We get four terms. The first term is

$$\begin{aligned} &\frac{1}{2\pi}\left(\int_{-\infty}^{\infty} h(t)\sum_j \xi_{jj}(\frac{1}{2}+it)\, dt\right)\cdot\, \ln Y \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} h(t)\sum_{j=1}^{\kappa}\sum_{\substack{i=1\\k(i,j)=j}}^{d} \gamma_{ij}^{1/2+it}\, dt\right)\cdot\, \ln Y \qquad \{\text{via } (2.11)\} \\ &= \sum_{\substack{j,i\\k(i,j)=j}} \sqrt{\gamma_{ij}}\cdot g(\ln\gamma_{ij})\;\cdot\; \ln Y. \end{aligned}$$

To compute the *third* term, we look at

$$\begin{aligned} &\frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\xi_{lj}(\frac{1}{2}+it)\cdot 2\cdot \mathrm{Re}\,\left(\overline{\varphi_{jl}(\frac{1}{2}+it)}\frac{Y^{2it}}{2it}\right)\, dt \\ &\qquad\qquad \Big\{\text{cf. } (7.6)\Big\} \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\xi_{lj}(\frac{1}{2}+it)\cdot\left(\frac{\varphi_{jl}(\frac{1}{2}-it)Y^{2it}-\varphi_{jl}(\frac{1}{2}+it)Y^{-2it}}{2it}\right)\, dt \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\xi_{lj}(\frac{1}{2}+it)\cdot\frac{\varphi_{jl}(\frac{1}{2}-it)Y^{2it}-\varphi_{jl}(\frac{1}{2})}{2it}\, dt \\ &\quad - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\xi_{lj}(\frac{1}{2}+it)\cdot\frac{\varphi_{jl}(\frac{1}{2}+it)Y^{-2it}-\varphi_{jl}(\frac{1}{2})}{2it}\, dt. \end{aligned}$$

This is treated as in [H2, p. 202]. We move the contour to $\operatorname{Im} t = \beta - \frac{1}{2}$ and $\operatorname{Im} t = \frac{1}{2} - \beta$, respectively, where $\frac{1}{2} < \beta < s_{M_e}$ as before. Notice that $\varphi_{jl}(s)$ is uniformly bounded for $\frac{1}{2} \leq \operatorname{Re} s \leq \beta$ (cf. [H2, p. 301 (a)]), and $\xi_{lj}(s)$ is bounded in any finite vertical strip. We get

$$\frac{1}{8\pi i}\int_{\operatorname{Im} t=\beta-\frac{1}{2}} h(t)\xi_{lj}(\frac{1}{2}+it)\cdot\frac{\varphi_{jl}(\frac{1}{2}-it)Y^{2it}-\varphi_{jl}(\frac{1}{2})}{t}\,dt$$

$$-\frac{1}{8\pi i}\int_{\operatorname{Im} t=\frac{1}{2}-\beta} h(t)\xi_{lj}(\frac{1}{2}+it)\cdot\frac{\varphi_{jl}(\frac{1}{2}+it)Y^{-2it}-\varphi_{jl}(\frac{1}{2})}{t}\,dt.$$

$$=\frac{\varphi_{jl}(\frac{1}{2})}{8\pi i}\left(\int_{\operatorname{Im} t=\frac{1}{2}-\beta}-\int_{\operatorname{Im} t=\beta-\frac{1}{2}}\right)h(t)\xi_{lj}(\frac{1}{2}+it)\,\frac{dt}{t}+O\left(Y^{1-2\beta}\right).$$

By applying the Cauchy integral formula over $[-\Delta,\Delta]\times[\frac{1}{2}-\beta,\beta-\frac{1}{2}]$ $(\Delta\to\infty)$ we see that this is

$$\frac{1}{4}\cdot\varphi_{jl}(\frac{1}{2})\cdot h(0)\cdot\xi_{lj}(\frac{1}{2})+O\left(Y^{1-2\beta}\right).$$

Finally, the fourth term from (7.12) gives

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}h(t)\cdot O\left(\sum_{j,l}\left|\xi_{lj}(\frac{1}{2}+it)\right|\cdot\omega(t)\cdot e^{6|t|-4\pi Y}\right)\,dt=O\left(e^{-4\pi Y}\right),$$

by (3.3) and (7.5).

Collecting our results, we have

$$\int_{\mathcal{F}_Y}H_{\mathcal{M}}(z,z)\,d\mu(z)\;=\;\sum_{\substack{j,i\\k(i,j)=j}}\sqrt{\gamma_{ij}}\cdot g(\ln\gamma_{ij})\;\cdot\;\ln Y \tag{7.13}$$

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}h(t)\cdot\sum_{j,l,k}\xi_{lj}(\frac{1}{2}+it)\left(\varphi'_{jk}(\frac{1}{2}+it)\overline{\varphi_{lk}(\frac{1}{2}+it)}\right)\,dt$$

$$+\frac{1}{4}h(0)\sum_{j,l}\varphi_{jl}(\frac{1}{2})\xi_{lj}(\frac{1}{2})+O\left(Y^{1-2\beta}\right).$$

## 8. The final form of the trace formula

We substitute (6.3) (modified as in Remark 6.4), (6.5) and (7.13) in (3.28). *Notice that the* $(\ln Y)-$*terms cancel,* and we can compute the limit as $Y\to\infty$ (remember $\beta>\frac{1}{2}$ in (7.13)). Combining this with

(3.22), (3.26) and Theorem 3.6, we then obtain the final form of the trace formula, in the case $\alpha \notin \Gamma$.

It will be convenient to incorporate the case $\alpha \in \Gamma$. In this case, $\mathcal{M}$ is the identity operator (cf. (2.3), (2.4)), and the trace formula is taken from [H2, p. 315]. (Notice that $\alpha \in \Gamma$ implies $d = 1$, $k(1, j) = j$, $\gamma_{1j} = 1$, $r_{1j} \in \mathbf{Z}$ and $\xi_{lj}(s) \equiv \delta_{lj}$.)

Our result is:

(8.1)

$$\begin{aligned}
&\sum_{\lambda \geq 0} h(r_\lambda) \cdot \mathrm{Tr}\ \mathcal{M}_{|A(\Gamma\backslash\mathcal{H},\lambda)} \\
&= \begin{Bmatrix} 0 & \text{if} & \alpha \notin \Gamma \\ 1 & \text{if} & \alpha \in \Gamma \end{Bmatrix} \cdot \frac{\mu(\Gamma \setminus \mathcal{H})}{4\pi} \cdot \int_{-\infty}^{\infty} r h(r) \tanh(\pi r)\, dr \\
&+ \sum_{\substack{\{T\} \subset \Gamma\alpha^{-1}\Gamma, \\ \text{hyperbolic,} \\ \text{doesn't fix cusp}}} \frac{\ln N(T_0)}{N(T)^{1/2} - N(T)^{-1/2}} \cdot g(\ln N(T)) \\
&+ \sum_{\substack{\{T\} \subset \Gamma\alpha^{-1}\Gamma, \\ \text{elliptic}}} \frac{1}{2M_T \cdot \sin\theta(T)} \cdot \int_{-\infty}^{\infty} \frac{e^{-2\theta(T) r}}{1 + e^{-2\pi r}} \cdot h(r)\, dr \\
&+ \sum_{i,j}^{(p)} \left[ (C(r_{ij}) - \ln 2) \cdot g(0) + \frac{1}{4} h(0) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\, dr \right] \\
&+ \frac{1}{2} \sum_{i,j}^{(h)} \frac{\sqrt{\gamma_{ij}}}{|x_{ij} - y_{ij}|} \cdot \left( \sum_{\substack{v \\ \bmod |x_{ij} - y_{ij}|}} \ln \mathbf{c} \left( N_j \Gamma N_j^{-1}, \frac{v + y_{ij} r_{ij}}{y_{ij} - x_{ij}} \right) \right) \cdot g(\ln \gamma_{ij}) \\
&+ \frac{1}{2} \sum_{i,j}^{(h)} \sqrt{\gamma_{ij}} \cdot \int_{|\ln \gamma_{ij}|}^{\infty} g(u) \cdot \frac{e^{u/2} + e^{-u/2}}{e^{u/2} - e^{-u/2} + |\gamma_{ij}^{1/2} - \gamma_{ij}^{-1/2}|}\, du \\
&+ \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \cdot \sum_{j,l,k} \xi_{lj}(\tfrac{1}{2} + ir) \varphi'_{jk}(\tfrac{1}{2} + ir) \overline{\varphi_{lk}(\tfrac{1}{2} + ir)}\, dr \\
&- \frac{1}{4} h(0) \cdot \sum_{j,l} \varphi_{jl}(\tfrac{1}{2}) \xi_{lj}(\tfrac{1}{2}).
\end{aligned}$$

We review the notation:

$\mu(\Gamma \backslash \mathcal{H})$ is the hyperbolic area of some (and hence any) fundamental domain for $\Gamma$.

For $T \in \Gamma\alpha^{-1}\Gamma$ hyperbolic, not fixing a cusp, $N(T)$ is the (unique) number $\gamma > 1$ making $T$ conjugate to $\mathbf{a}(\gamma)$ (in $\mathrm{PSL}(2, \mathbf{R})$), and $T_0$ denotes a generator of $Z_\Gamma(T) = \{\sigma \in \Gamma \mid \sigma T = T\sigma\}$. (By Theorem 3.6, $Z_\Gamma(T) \neq \{\mathrm{Id}\}$). For $T \in \Gamma\alpha^{-1}\Gamma$ elliptic, $M_T = |Z_\Gamma(T)|$, and $\theta(T)$ is the unique angle $\theta \in (0, \pi)$ making $T$ conjugate (in $\mathrm{PSL}(2, \mathbf{R})$) to $\mathbf{k}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

For each $i \in \{1, ..., d\}$, $j \in \{1, ..., \kappa\}$ we have a decomposition

$$\alpha_i = A_{ij} N_{k(i,j)}^{-1} S^{r_{ij}} \mathbf{a}(\gamma_{ij}) N_j, \tag{8.2}$$

where

$$A_{ij} \in \Gamma, \quad r_{ij} \in \mathbf{R} \pmod{\mathbf{Z}}$$
$$\gamma_{ij} \in \mathbf{Q}^+; \quad \gamma_{ij} = \frac{x_{ij}}{y_{ij}}, \quad x_{ij}, y_{ij} \in \mathbf{Z}^+, \quad (x_{ij}, y_{ij}) = 1. \tag{8.3}$$

We use $\sum_{i,j}^{(p)}$ and $\sum_{i,j}^{(h)}$ to denote summation over

$$\sum_{i,j}^{(p)}: \quad \{\langle i,j \rangle \mid i \in \{1, ..., d\},\ j \in \{1, ..., \kappa\},\ k(i,j) = j,\ \gamma_{ij} = 1\} \quad \text{and}$$

$$\sum_{i,j}^{(h)}: \quad \{\langle i,j \rangle \mid i \in \{1, ..., d\},\ j \in \{1, ..., \kappa\},\ k(i,j) = j,\ \gamma_{ij} \neq 1\},$$

respectively. For any Fuchsian group $\Gamma'$ with parabolic fixpoint $p \in \mathbf{R}$ we denote (cf. Remark 4.2):

$$\begin{aligned} &\mathbf{c}(\Gamma', p) \\ &= \left[\mathbf{c} > 0 \text{ such that } G = \begin{pmatrix} * & * \\ \mathbf{c} & * \end{pmatrix} \text{ for a generator } G \text{ of } (\Gamma')_p\right] \\ &= \inf\{c > 0 \mid \begin{pmatrix} 1 + cp & -cp^2 \\ c & 1 - cp \end{pmatrix} \in \Gamma'\}. \end{aligned} \tag{8.4}$$

(In (8.1), notice that $\dfrac{v + y_{ij} r_{ij}}{y_{ij} - x_{ij}}$ is simply the *second* fixpoint of $S^{r_{ij} + v/y_{ij}} \mathbf{a}(x_{ij}/y_{ij})$; cf. (6.1), (8.2), and $T_{jiv}^{-1}$.)

For $0 < v \leq 1$ we have defined $C_v$, a generalized Euler constant, by

$$\sum_{0 \leq n \leq x - v} \frac{1}{v + n} = \ln x + C_v + O_v\left(\frac{1}{\sqrt{x}}\right) \qquad \text{for } x > 0. \tag{8.5}$$

In particular, $C = C_1$ is the usual Euler constant. For $r \in \mathbf{R}$ we then define $C(r)$ by

$$C(r) = \begin{cases} \frac{1}{2}(C_v + C_{1-v}) - C & \text{if } r \notin \mathbf{Z},\ v \equiv -r \pmod 1,\ 0 < v < 1 \\ 0 & \text{if } r \in \mathbf{Z} \end{cases},$$

where $r \in \mathbf{Z}$ is included to cover the case $\alpha \in \Gamma$; cf. [H2, p. 315 (line 5)]. Finally (cf. (2.11)):

$$\xi_{lj}(s) = \sum_{\substack{i=1 \\ k(i,j)=l}}^{d} \gamma_{ij}^{s}.$$

Our trace formula (8.1) on p. 42 has been proved for any $h(r)$ satisfying (3.3), and

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iru}\, dr, \quad u \in \mathbf{R}.$$

We want to extend the formula to a larger class of functions. For this we need to give bounds for the sums $\sum_k \varphi'_{jk}(\frac{1}{2} + ir)\overline{\varphi_{lk}(\frac{1}{2} + ir)}$ occurring on p. 42. Recall (7.11). From [H2, p. 315 (i), (iii)] (combined with (7.6) and (7.8)) we know that

$$\sum_j \left( \sum_k \varphi'_{jk}(\frac{1}{2} + ir)\overline{\varphi_{jk}(\frac{1}{2} + ir)} \right) = -\omega(r) + O(1). \tag{8.6}$$

Here $\omega(r)$ is as in (7.4), cf. (7.5), [H2, pp. 161, 299 (line 14), 315 (ii)].

We need to get rid of the $j$-summation. The method we use for this resembles that of Selberg [S2, p. 22].

**Lemma 8.1.** *Recall the definition of $\Phi(s)$; (7.7). Take $B$ large as in (7.3). Then for $s = \sigma + it$, $\sigma > \frac{1}{2}$, $t \neq 0$, and any vector $\mathbf{v} = (v_1, ..., v_\kappa) \in \mathbf{C}^\kappa$,*

(8.7)
$$\mathbf{v}\Phi(s)\overline{\Phi(s)}^T\overline{\mathbf{v}}^T \leq B^{4\sigma-2} \cdot \left( \sqrt{1 + \left(\frac{\sigma - 1/2}{t}\right)^2} + \frac{\sigma - 1/2}{|t|} \right)^2 \mathbf{v}\overline{\mathbf{v}}^T.$$

*(We identify a $1 \times 1-$matrix and its element. Notice that both sides of (8.7) are apriori real.)*

*Proof.* Take $E_0^{(j)} = E_0^{(j)}(z,s)$ as in (7.2). For any $z \in \mathcal{F}$ we have

$$0 \leq \left| \sum_j v_j E_0^{(j)} \right|^2 = \sum_j \sum_l v_j \overline{v_l} E_0^{(j)} \overline{E_0^{(l)}}.$$

Integrating this over $z \in \mathcal{F}$ we get, by (7.3)

$$\begin{aligned} 0 &\leq \frac{B^{2\sigma-1}}{2\sigma-1} \sum_j |v_j|^2 + \frac{B^{-2it}}{-2it} \sum_j \sum_l v_j \overline{v_l} \varphi_{jl} + \frac{B^{2it}}{2it} \sum_j \sum_l v_j \overline{v_l \varphi_{lj}} \\ &\qquad + \frac{B^{1-2\sigma}}{1-2\sigma} \sum_j \sum_l v_j \overline{v_l} \left( \sum_k \varphi_{jk} \overline{\varphi_{lk}} \right) \\ &= \frac{B^{2\sigma-1}}{2\sigma-1} \cdot \mathbf{v}\overline{\mathbf{v}}^T + \frac{1}{2t} \cdot \mathbf{v} \left( B^{-2it} i\Phi - B^{2it} i\overline{\Phi}^T \right) \overline{\mathbf{v}}^T + \frac{B^{1-2\sigma}}{1-2\sigma} \cdot \mathbf{v}\Phi\overline{\Phi}^T \overline{\mathbf{v}}^T. \end{aligned}$$

Now $0 \leq \mathbf{u}\overline{\mathbf{u}}^T$ for any complex row vector $\mathbf{u}$. Applying this to $\mathbf{u} = \mathbf{v} \cdot \left( a^{-1/2} B^{-it} \Phi \pm a^{1/2} B^{it} iE \right)$, where $a$ is an arbitrary positive number, we get

$$\pm \mathbf{v} \left( B^{-2it} i\Phi - B^{2it} i\overline{\Phi}^T \right) \overline{\mathbf{v}}^T \leq \frac{1}{a} \cdot \mathbf{v}\Phi\overline{\Phi}^T \overline{\mathbf{v}}^T + a \cdot \mathbf{v}\overline{\mathbf{v}}^T.$$

Combining this with our first inequality, we get

$$0 \leq \left( \frac{B^{2\sigma-1}}{2\sigma-1} + \frac{a}{2|t|} \right) \mathbf{v}\overline{\mathbf{v}}^T - \left( \frac{B^{1-2\sigma}}{2\sigma-1} - \frac{1}{2a|t|} \right) \mathbf{v}\Phi\overline{\Phi}^T \overline{\mathbf{v}}^T,$$

and thus

$$\left( B^{1-2\sigma} a|t| - (\sigma - \frac{1}{2}) \right) \cdot \mathbf{v}\Phi\overline{\Phi}^T \overline{\mathbf{v}}^T \leq a \left( |t| B^{2\sigma-1} + a(\sigma - \frac{1}{2}) \right) \mathbf{v}\overline{\mathbf{v}}^T.$$

Now let $a = B^{2\sigma-1}(b + \sigma - \frac{1}{2})/|t|$, where $b > 0$ :

$$\begin{aligned} b \cdot \mathbf{v}\Phi\overline{\Phi}^T \overline{\mathbf{v}}^T &\leq \frac{B^{2\sigma-1}(b+\sigma-1/2)}{|t|} \\ &\qquad \times \left( |t| B^{2\sigma-1} + B^{2\sigma-1} \frac{b+\sigma-1/2}{|t|} (\sigma - \frac{1}{2}) \right) \mathbf{v}\overline{\mathbf{v}}^T; \\ \mathbf{v}\Phi\overline{\Phi}^T \overline{\mathbf{v}}^T &\leq \frac{B^{4\sigma-2}}{t^2} (b + \sigma - \frac{1}{2}) \left( \frac{t^2 + (\sigma-1/2)^2}{b} + \sigma - \frac{1}{2} \right) \mathbf{v}\overline{\mathbf{v}}^T. \end{aligned}$$

Choosing $b$ optimally, $b = \sqrt{t^2 + (\sigma - \frac{1}{2})^2}$, we get (8.7). $\square$

**Lemma 8.2.** *Take $j, l \in \{1, ..., \kappa\}$. Then*

$$\left| \sum_k \varphi'_{jk}(\frac{1}{2} + it)\overline{\varphi_{lk}(\frac{1}{2} + it)} \right| \leq \omega(t) + O(1).$$

*(The implied constant may depend on $\Gamma$.)*

*Proof.* Without loss of generality, $|t| \geq 1$. We freeze such a $t$. We first deal with the case $j = l$. Apply (8.7) with $\mathbf{v} = (v_1, ..., v_\kappa)$, $v_i = \delta_{ij}$ :

$$\sum_k |\varphi_{jk}(\sigma + it)|^2 \leq B^{4\sigma-2} \cdot \left( \sqrt{1 + \left(\frac{\sigma - 1/2}{t}\right)^2} + \frac{\sigma - 1/2}{|t|} \right)^2$$

for $\sigma > 1/2$. From (7.10) we know that $\sum_k |\varphi_{jk}(\frac{1}{2} + it)|^2 = 1$, i.e. our inequality reduces to an equality for $\sigma = 1/2$. Hence also

$$\frac{d}{d\sigma} \left( \sum_k |\varphi_{jk}(\sigma + it)|^2 \right)_{\sigma = \frac{1}{2}}$$
$$\leq \frac{d}{d\sigma} \left( B^{4\sigma-2} \left( \sqrt{1 + \left(\frac{\sigma - 1/2}{t}\right)^2} + \frac{\sigma - 1/2}{|t|} \right)^2 \right)_{\sigma = \frac{1}{2}}$$

Using (7.11), we see that this means

$$\sum_k \varphi'_{jk}(\frac{1}{2} + it)\overline{\varphi_{jk}(\frac{1}{2} + it)} \leq \frac{1}{|t|} + 2 \ln B, \quad \text{for each } j.$$

Let $C = 1 + 2 \ln B$. By applying (8.6), we get:

$$\sum_k \varphi'_{jk}(\frac{1}{2} + it)\overline{\varphi_{jk}(\frac{1}{2} + it)} \geq -\omega(t) + O(1) - (\kappa - 1)C.$$

Hence:

$$-\omega(t) + O(1) \leq \sum_k \varphi'_{jk}(\frac{1}{2} + it)\overline{\varphi_{jk}(\frac{1}{2} + it)} \leq O(1). \tag{8.8}$$

We now assume $j \neq l$. Take $u, w \in \mathbf{C}$ and apply (8.7) with $v_j = u$, $v_l = w$ and $v_k = 0$ for $k \notin \{j, l\}$ :

$$|u|^2 \sum_k |\varphi_{jk}(s)|^2 + |w|^2 \sum_k |\varphi_{lk}(s)|^2 + 2\mathrm{Re}\left(u\overline{w} \sum_k \varphi_{jk}(s)\overline{\varphi_{lk}(s)}\right)$$

$$\leq \left(|u|^2 + |w|^2\right) \cdot B^{4\sigma-2} \cdot \left(\sqrt{1 + \left(\frac{\sigma - 1/2}{t}\right)^2} + \frac{\sigma - 1/2}{|t|}\right)^2$$

for $\sigma > 1/2$. As before, we have equality for $\sigma = 1/2$, and we may differentiate with respect to $\sigma$. We get

$$|u|^2 \sum_k \varphi'_{jk}\overline{\varphi_{jk}} + |w|^2 \sum_k \varphi'_{lk}\overline{\varphi_{lk}} + 2\mathrm{Re}\left(u\overline{w} \sum_k \varphi'_{jk}\overline{\varphi_{lk}}\right)$$

$$\leq \left(|u|^2 + |w|^2\right) \cdot \left(\frac{1}{|t|} + 2\ln B\right)$$

at $s = 1/2 + it$. Now choose $u, w$ such that $|u| = |w| = 1$ and $\mathrm{Re}\left(u\overline{w}\sum_k \varphi'_{jk}\overline{\varphi_{lk}}\right) = \left|\sum_k \varphi'_{jk}\overline{\varphi_{lk}}\right|$. Using (8.8) we then get the result. □

**Theorem 8.3.** *Suppose that $h(r)$ satisfies the following hypotheses:*

*(a) $h(r)$ is analytic on $|\mathrm{Im}\, r| \leq \frac{1}{2} + \delta$ for some $\delta > 0$;*

*(b) $h(-r) = h(r)$;*

*(c) $|h(r)| \leq A\,(1 + |\mathrm{Re}\, r|)^{-2-\delta}$.*

*Suppose further that*

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-iru}\,dr, \quad u \in \mathbf{R}.$$

*Then the formula (8.1) on p. 42 holds, where the various sums and integrals are all absolutely convergent. In the next to last term, this is to be interpreted as*

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} |h(r)| \cdot \sum_{j,l} \left|\xi_{lj}(\tfrac{1}{2} + ir)\right| \cdot \left|\sum_k \varphi'_{jk}(\tfrac{1}{2} + ir)\overline{\varphi_{lk}(\tfrac{1}{2} + ir)}\right|\, dr < \infty.$$

*Proof.* We first check that for $h(r)$ as in the theorem, the sums and integrals on p. 42 are indeed absolutely convergent, with good majorants.

For the left hand side, this is seen from [H2, p. 315 (ii)], and

$$\left|\mathrm{Tr}\ \mathcal{M}_{|A(\Gamma\setminus\mathcal{H},\lambda)}\right| \leq \|\mathcal{M}\|_{L_2} \cdot \dim A(\Gamma\setminus\mathcal{H},\lambda),$$

cf. Lemma 2.1, Corollary 2.3 and Lemma 2.5.

Concerning the hyperbolic sum, recall that we have shown in our proof of Theorem 3.6 that [H1, p. 486, Prop 3.1] extends to our case; i.e. there is a majorant corresponding to $\Phi(t) = (t+4)^{-1-\delta}$, $g(u) = c_1(\delta)\left[\mathrm{sech}(u/2)\right]^{1+2\delta}$, $h(r) = c_2(\delta)\left|\Gamma(\frac{1}{2}+\delta+ir)\right|^2$. Cf. [H1, pp. 411(6.7), 412(6.9)]. In view of [H1, p. 30, Prop 7.3], this proves the absolute convergence.

The elliptic sum is finite, by Theorem 3.6.

For the next to last term on p. 42, we use $\xi_{lj}(\frac{1}{2}+ir) = O(1)$, Lemma 8.2, (7.5) and partial integration. Cf. [H2, p. 210].

Now to prove equality on p. 42, we use approximation. Without loss of generality, $h(\overline{r}) = \overline{h(r)}$. Take $h_\epsilon(r) = h(r)e^{-\epsilon r^2}$ and consider the $[h_\epsilon, g_\epsilon]$ analog of the formula on p. 42. The desired argument is now quite similar to [H1, p. 34, step II]. □

*Remark 8.4.* We have taken $\eta_1, ..., \eta_\kappa$ to be the vertices at infinity of some canonical fundamental domain for $\Gamma\setminus\mathcal{H}$, and we have related the $N_j$:s to this fundamental domain by (1.2). However, the only assumptions on $\eta_j$ and $N_j$ that we have used in an essential way is that

(a) $\{\eta_1, ...\eta_\kappa\}$ is a full system of inequivalent cusps;
(b) $N_j \in \mathrm{PSL}(2,\mathbf{R})$ is such that $N_j\eta_j = \infty$ and $\Gamma_{\eta_j} = [N_j^{-1}S^{-1}N_j]$.

Hence the trace formula remains true under these milder assumptions. This would also be easy to check directly at this point.

*Remark 8.5.* The trace formula is trivially generalized to the following situation. We recall that

$$\mathrm{PGL}^+(2,\mathbf{R}) = \{\sigma \in \mathrm{GL}(2,\mathbf{R}) \mid \det\sigma > 0\} / \sim,$$

where $\sigma \sim \sigma'$ if and only if $\sigma = \pm\sigma'$. We will always let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (where $a,b,c,d \in \mathbf{R}$, $ad-bc > 0$) denote an element of $\mathrm{PGL}^+(2,\mathbf{R})$, *not* $\mathrm{GL}^+(2,\mathbf{R})$. Cf. our convention on p. 4 on notation of elements from $\mathrm{PSL}(2,\mathbf{R})$.

Let $H$ be any subset of $\mathrm{PGL}^+(2, \mathbf{R})$ satisfying

(8.9) $\quad H\Gamma = \Gamma H = H$

(8.10) $\quad [H : \Gamma] < \infty,$ i.e. $H = \bigsqcup_{i=1}^{d} \Gamma\alpha_i$ for some $\alpha_i \in H \quad (d < \infty).$

(cf. [G, p. 57]). We recall that the action of any $\alpha \in \mathrm{PGL}^+(2, \mathbf{R})$ on $\mathcal{H}$ is defined to be the same as the action of $(\det \alpha)^{-1/2}\alpha \in \mathrm{PSL}(2, \mathbf{R})$ on $\mathcal{H}$. Thus

$$\alpha(z) = \frac{az + b}{cz + d} \quad \text{for } z \in \mathcal{H},\ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now for $f \in L_2(\Gamma \setminus \mathcal{H})$ or $f \in C(\Gamma \setminus \mathcal{H})$ we define $\mathcal{M}f$ by

$$(\mathcal{M}f)(z) = \sum_{i=1}^{d} f(\alpha_i z),$$

and the obvious analog of Lemma 2.1 holds.

Each $\alpha_i$ has a decomposition $(\det \alpha_i)^{-1/2}\alpha_i = A_{ij} N_{k(i,j)}^{-1} S^{r_{ij}} \mathbf{a}(\gamma_{ij}) N_j$ for each $j$ as in (8.2), and the trace formula for $\mathcal{M}$ is again given by the trace formula (8.1) on p. 42, with only a few changes: The first factor in the identity term (which is 0 or 1 on p. 42) is to be replaced by

(8.11) $$\operatorname{card} \left\{ \mu > 0 \mid \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \in H \right\}$$

In the second and third terms, the condition $\{T\} \subset \Gamma\alpha^{-1}\Gamma$ should be replaced by $\{T\} \subset H^{-1}$. Also, the definition of $N(T)$ and $\theta(T)$ has to be extended in the obvious way.

All of this follows from the case that we have already treated, since by (8.9) and (8.10), $H$ is a disjoint union of finitely many double cosets $\Gamma\alpha^{[m]}\Gamma$ $(m = 1, 2, .., N)$, the operator $\mathcal{M}$ is just the sum of the modular correspondences coming from $(\det \alpha^{[m]})^{-1/2}\alpha^{[m]}$ $(m = 1, 2, .., N)$, and the same is true for the trace formula. More general $\mathcal{M}$ of this sort are important for the theory of Hecke operators.

## 9. Hecke Operators on the Modular Group

In this section, we investigate further the trace formula on p. 42 in the case of an arbitrary Hecke operator on the modular group; $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$. Our result will be *an extension* of the formula in [H3, (11.10)], which concerns Hecke operators of *prime* order. Cf. also

[B, p. 268], wherein the corresponding formula for the subspace of *odd* eigenfunctions is derived. (Note, however, that the "odd" subspace has no contribution from the Eisenstein series in its spectral decomposition)

The usual choice of fundamental region for $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$ is

$$\mathcal{F} = \{z \in \mathcal{H} \mid -\frac{1}{2} < \operatorname{Re} z < \frac{1}{2}, \quad |z| > 1\}.$$

The elliptic fixpoints correspond to $i$ and $e^{2\pi i/3}$. $\Gamma$ has signature $(0,3;\ 2,3,\infty)$. It follows that $\mu(\Gamma \setminus \mathcal{H}) = \pi/3$.

There is only one cusp up to equivalence, i.e. $\kappa = 1$. Skipping the superfluous index $j = 1$, we can take $\eta = \infty$, $N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ on p. 4.

Concerning $\varphi(s)$ (i.e. $\varphi_{11}(s)$), it is well know that (cf. [H2, pp. 76, 508])

$$\varphi(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} = \pi^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}\frac{\zeta(2-2s)}{\zeta(2s)}. \tag{9.1}$$

We now fix some $m \in \mathbf{Z}^+$ and define $H_m \subset \mathrm{PGL}^+(2, \mathbf{R})$ by

$$H_m = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbf{Z}, \quad ad - bc = m\}.$$

It is clear that $\Gamma H_m = H_m \Gamma = H_m$. One checks by a straightforward computation that

$$H_m = \bigsqcup_{\substack{ad=m \\ a,d \in \mathbf{Z}^+}} \bigsqcup_{b \bmod d} \Gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \text{(disjoint union)}. \tag{9.2}$$

Hence $H_m$ satisfies the conditions (8.9), (8.10), and we may define the *Hecke operator* $T_m$ as

$$T_m = \frac{1}{\sqrt{m}} \cdot [\text{the modular correspondence coming from } H_m]. \tag{9.3}$$

This is familiar from [G, pp. 58–59] and [M, §2.7, §4.5]. The extra factor $\frac{1}{\sqrt{m}}$ is customary when working with nonholomorphic cusp forms; cf. [Iw, p. 127]. Using $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \cdot (H_m)^{-1} = H_m$ and Lemma 2.2, one proves that $T_m$ is a *self-adjoint* operator on $L_2(\Gamma \setminus \mathcal{H})$. Thus we can choose our eigenfunctions $\phi_0, \phi_1, ...$ to be eigenfunctions also of $T_m$, i.e.

$$T_m \phi_n = \omega_n^{(m)} \cdot \phi_n \qquad \text{for some } \omega_n^{(m)} \in \mathbf{R}. \tag{9.4}$$

We are going to evaluate the trace formula for the operator $T_m$. The left hand side will now be $\sum_{n=0}^{\infty} \omega_n^{(m)} h(r_n)$.

The decomposition (8.2), generalized as in Remark 8.5, is trivial. We see that, for $\alpha_i = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$,

$$\gamma_{\langle a,d,b\rangle} = \frac{a}{d}, \quad x_{\langle a,d,b\rangle} = \frac{a}{(a,d)}, \quad y_{\langle a,d,b\rangle} = \frac{d}{(a,d)}, \quad \text{and}$$

$$r_{\langle a,d,b\rangle} \equiv \frac{b}{d} \bmod \mathbf{Z},$$

where the notation is adapted to (9.2) in an obvious way.

The $\sum^{(p)}$ –term on p. 42 is thus seen to be 0 if $\sqrt{m} \notin \mathbf{Z}$, and if $\sqrt{m} \in \mathbf{Z}$ (remember the factor $1/\sqrt{m}$ from (9.3)):

$$\frac{1}{\sqrt{m}} \sum_{b \bmod \sqrt{m}} \left[ (C(\frac{b}{\sqrt{m}}) - \ln 2) \cdot g(0) + \frac{1}{4} h(0) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\, dr \right]$$

$$= \frac{1}{\sqrt{m}} \sum_{b=1}^{\sqrt{m}-1} \left( \frac{1}{2} \left( C_{1-b/\sqrt{m}} + C_{b/\sqrt{m}} \right) - C \right) \cdot g(0)$$

$$- (\ln 2) \cdot g(0) + \frac{1}{4} h(0) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\, dr$$

To compute the factor in front of the first $g(0)$, we use its symmetry and (8.5). We get

$$\frac{1}{\sqrt{m}} \lim_{x\to\infty} \sum_{b=1}^{\sqrt{m}-1} \left( \sum_{0\le n\le x-b/\sqrt{m}} \frac{1}{b/\sqrt{m}+n} - \ln x - \sum_{0\le n\le x-1} \frac{1}{1+n} + \ln x \right)$$

$$= \lim_{x\to\infty} \sum_{b=1}^{\sqrt{m}} \left( \sum_{0\le n\le x-b/\sqrt{m}} \frac{1}{b+n\sqrt{m}} - \sum_{0\le n\le x-1} \frac{1}{(1+n)\sqrt{m}} \right)$$

$$= \lim_{x\to\infty} \left( \sum_{1\le n\le x\sqrt{m}} \frac{1}{n} - \sum_{1\le n\le x} \frac{1}{n} \right) \quad = \quad \ln\sqrt{m}.$$

Hence the whole expression is (when $\sqrt{m} \in \mathbf{Z}$):

$$\ln\left(\frac{\sqrt{m}}{2}\right) \cdot g(0) + \frac{1}{4}h(0) - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\,dr.$$

To continue, notice that $\mathbf{c}(\Gamma, p) = \inf\{c \in \mathbf{Z}^+ \mid cp^2 \in \mathbf{Z}\}$ for each $p \in \mathbf{Q}$ (see (8.4)).

Hence the first $\sum^{(h)}$ −term on p. 42 is (we keep $a, d \in \mathbf{Z}^+$)

$$\frac{1}{2\sqrt{m}} \sum_{\substack{ad=m\\ a\neq d}} \sum_{b \bmod d} \frac{\sqrt{a/d}\cdot(a,d)}{|a-d|}\left(\sum_{v \bmod \frac{|a-d|}{(a,d)}} \ln \mathbf{c}(\Gamma, \frac{v(a,d)+b}{d-a})\right)\cdot g(\ln(\frac{a}{d}))$$

$$= \frac{1}{2}\sum_{\substack{ad=m\\ a\neq d}} \sum_{b \bmod d} \frac{(a,d)}{d\;|a-d|}\left(\sum_{v \bmod \frac{|a-d|}{(a,d)}} 2\ln\left(\frac{|a-d|}{\gcd(a-d, v(a,d)+b)}\right)\right)\cdot g(\ln(\frac{a}{d})).$$

To avoid any misconception: Notice that $\gcd(a-d, v(a,d)+b)$ only depends on the congruence class of $v \bmod (|a-d|/(a,d))$. (This is in agreement with Remark 6.4.) On the other hand, it is *not* necessarily invariant under changes of $b$ by multiples of $d$. The $v-$sum as a whole, however, does possess this invariance. It actually only depends on the congruence class of $b \bmod (a,d)$, since $(v(a,d)+b)$ then runs through a system of representatives of all residues $\bmod |a-d|$ that are $\equiv b \bmod (a,d)$. We get

$$\frac{1}{2}\sum_{\substack{ad=m\\ a\neq d}} \frac{(a,d)}{d|a-d|}\cdot\left(\frac{d}{(a,d)}\cdot\sum_{w \bmod |a-d|} 2\ln\left(\frac{|a-d|}{(a-d,w)}\right)\right)\cdot g(\ln(\frac{a}{d}))$$

$$= \sum_{\substack{ad=m\\ a\neq d}}\left(\ln|a-d| - \frac{\ln X(|a-d|)}{|a-d|}\right)\cdot g(\ln(\frac{a}{d})),$$

where

$$X(n) = \prod_{k \bmod n} (k,n) \qquad \text{for } n \in \mathbf{Z}^+. \tag{9.5}$$

(Then note that $\frac{\ln X(n)}{n} = \sum_{p|n}\left(\frac{1}{p} + \frac{1}{p^2} + ... + \frac{1}{(p^\infty, n)}\right)\cdot \ln p$)

Similarly, but easier, the second $\sum^{(h)}-$ term on p. 42 is seen to be

$$\frac{1}{2}\sum_{\substack{ad=m\\ a\neq d}}\int_{|\ln(\frac{a}{d})|}^{\infty} g(u)\cdot\frac{e^{u/2}+e^{-u/2}}{e^{u/2}-e^{-u/2}+|\sqrt{a/d}-\sqrt{d/a}|}\,du$$

For the next to last term on p. 42 we first notice that (again skipping unnecessary indices)

$$\xi(s)=\sum_{ad=m}\ \sum_{b \bmod d}\left(\frac{a}{d}\right)^{s}=\sqrt{m}\cdot\sum_{\substack{d|m\\ a=m/d}}\left(\frac{a}{d}\right)^{s-1/2}.$$

Hence by Lemma 2.6 and (9.3), the action of $T_m$ on the Eisenstein series $E(z,s)$ associated with the only cusp of $\Gamma$ is given by

$$T_mE(z,s)=\left(\sum_{ad=m}\left(\frac{a}{d}\right)^{s-1/2}\right)\cdot E(z,s).$$

Cf. [Iw, (8.33)]. Also, from (9.1) we see that

$$\begin{aligned}\varphi'(s)\varphi(1-s)&=\frac{\varphi'(s)}{\varphi(s)}\\ &=2\ln\pi-\frac{\Gamma'(s)}{\Gamma(s)}-\frac{\Gamma'(1-s)}{\Gamma(1-s)}-2\frac{\zeta'(2s)}{\zeta(2s)}-2\frac{\zeta'(2-2s)}{\zeta(2-2s)}.\end{aligned}$$

Hence the next to last term on p. 42 equals

$$\begin{aligned}\frac{1}{4\pi}\int_{-\infty}^{\infty}h(r)\cdot\left(\sum_{ad=m}\left(\frac{a}{d}\right)^{ir}\right)\cdot\Bigg(&2\ln\pi-\frac{\Gamma'(\frac12+ir)}{\Gamma(\frac12+ir)}-\frac{\Gamma'(\frac12-ir)}{\Gamma(\frac12-ir)}\\ &-2\frac{\zeta'(1+2ir)}{\zeta(1+2ir)}-2\frac{\zeta'(1-2ir)}{\zeta(1-2ir)}\Bigg)\,dr\end{aligned}\tag{9.6}$$

To simplify, we let (cf. [H2, p. 508 (bottom line)])

$$A(r)=\frac{\zeta'(1+2ir)}{\zeta(1+2ir)}+\frac{1}{2ir}.$$

This combination is holomorphic on $\operatorname{Im} r\leq 0$, and

$$\begin{aligned}A(r)&=-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+2ir}}+\frac{1}{2ir} &&\text{for } \operatorname{Im} r<0\\ A(r)&=O\big(\ln(2+|r|)\big) &&\text{for } \operatorname{Im} r\leq 0.\end{aligned}$$

(cf. [T, p. 60 (3.11.7)])

The computation of (9.6) is now very similar to [H2, p. 509]. We get

$$
\begin{aligned}
&(\ln \pi) \sum_{ad=m} g(\ln(\frac{a}{d})) \\
&- \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \cdot \sum_{ad=m} \left(\frac{a}{d}\right)^{ir} \left( \frac{\Gamma'(\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}+ir)} + \frac{\Gamma'(\frac{1}{2}-ir)}{\Gamma(\frac{1}{2}-ir)} \right) dr \\
&- \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \cdot \sum_{ad=m} \left(\frac{a}{d}\right)^{ir} \cdot (A(r) + A(-r))\, dr
\end{aligned}
$$

$$
\left\{ h(r) \cdot \sum_{ad=m} (a/d)^{ir} \text{ is even} \right\}
$$

$$
\begin{aligned}
&= (\ln \pi) \sum_{ad=m} g(\ln(\frac{a}{d})) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \cdot \sum_{ad=m} \left(\frac{a}{d}\right)^{ir} \cdot \frac{\Gamma'(\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}+ir)}\, dr \\
&\qquad - \frac{1}{\pi} \int_{-\infty}^{\infty} h(r) \cdot \sum_{ad=m} \left(\frac{a}{d}\right)^{ir} \cdot A(r)\, dr
\end{aligned}
$$

$$
= \ldots.. - \frac{1}{\pi} \int_{\operatorname{Im} r = -\epsilon} h(r) \cdot \sum_{ad=m} \left(\frac{a}{d}\right)^{ir} \cdot \left( - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+2ir}} + \frac{1}{2ir} \right) dr
$$

$$
\begin{aligned}
&= (\ln \pi) \sum_{ad=m} g(\ln(\frac{a}{d})) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \cdot \sum_{ad=m} \left(\frac{a}{d}\right)^{ir} \cdot \frac{\Gamma'(\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}+ir)}\, dr \\
&\qquad - \frac{1}{2} \left( \sum_{ad=m} 1 \right) \cdot h(0) + 2 \cdot \sum_{n=1}^{\infty} \sum_{ad=m} \frac{\Lambda(n)}{n} \cdot g\left( \ln(\frac{a}{d}) - 2 \ln n \right).
\end{aligned}
$$

Finally, $\varphi(\frac{1}{2}) = -1$ and $\xi(\frac{1}{2}) = \sqrt{m}\,(\sum_{ad=m} 1)$; hence the *last term* on p. 42 gives $\frac{1}{4}h(0) \cdot (\sum_{ad=m} 1)$. We also notice that the identity term on p. 42, modified as in (8.11), occurs only if $\sqrt{m} \in \mathbf{Z}$, and then it occurs once. Adding, we finally obtain the trace formula for the operator $T_m$ on $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$ :

(9.7)

$$\sum_{n=0}^{\infty} \omega_n^{(m)} h(r_n) = \left[\text{if } \sqrt{m} \in \mathbf{Z}: \quad \frac{1}{12\sqrt{m}} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r)\, dr\right]$$

$$+ \frac{1}{\sqrt{m}} \cdot \sum_{\substack{\{T\} \subset H_m^{-1}, \\ \text{hyperbolic,} \\ \text{doesn't fix cusp}}} \frac{\ln N(T_0)}{N(T)^{1/2} - N(T)^{-1/2}} \cdot g(\ln N(T))$$

$$+ \frac{1}{\sqrt{m}} \cdot \sum_{\substack{\{T\} \subset H_m^{-1}, \\ \text{elliptic}}} \frac{1}{2M_T \cdot \sin\theta(T)} \cdot \int_{-\infty}^{\infty} \frac{e^{-2\theta(T)r}}{1 + e^{-2\pi r}} \cdot h(r)\, dr$$

$$+ \left[\text{if } \sqrt{m} \in \mathbf{Z}: \left(\ln \pi + \ln\left(\frac{\sqrt{m}}{2}\right)\right) \cdot g(0) + \frac{1}{4} h(0) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\, dr\right]$$

$$+ \sum_{\substack{ad=m \\ a \neq d}} \left(\ln \pi + \ln |a-d| - \frac{\ln X(|a-d|)}{|a-d|}\right) \cdot g(\ln(\frac{a}{d}))$$

$$+ \frac{1}{2} \sum_{\substack{ad=m \\ a \neq d}} \int_{|\ln(\frac{a}{d})|}^{\infty} g(u) \cdot \frac{e^{u/2} + e^{-u/2}}{e^{u/2} - e^{-u/2} + |\sqrt{a/d} - \sqrt{d/a}|}\, du$$

$$- \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \cdot \sum_{ad=m} \left(\frac{a}{d}\right)^{ir} \cdot \frac{\Gamma'(\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}+ir)}\, dr - \frac{1}{4}\left(\sum_{ad=m} 1\right) \cdot h(0)$$

$$+ 2 \cdot \sum_{n=1}^{\infty} \sum_{ad=m} \frac{\Lambda(n)}{n} \cdot g\left(\ln(\frac{a}{d}) - 2\ln n\right).$$

Here $a, d \in \mathbf{Z}^+$. $\omega_n^{(m)}$ is given by (9.4). For $X(n)$, see (9.5). Notice that this formula is a generalization of both [H2, p. 510 (2.6)] and [H3, (11.10)], to which it reduces when $m$ is 1 and when $m$ is a prime, respectively.

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