VISIBILITY AND DIRECTIONS IN QUASICRYSTALS

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Abstract. It is well known that a positive proportion of all points in a d-dimensional lattice is visible from the origin, and that these visible lattice points have constant density in $\mathbb{R}^d$. In the present paper we prove an analogous result for a large class of quasicrystals, including the vertex set of a Penrose tiling. We furthermore establish that the statistical properties of the directions of visible points are described by certain $\mathrm{SL}(d,\mathbb{R})$-invariant point processes. Our results imply in particular existence and continuity of the gap distribution for directions in certain two-dimensional cut-and-project sets. This answers a recent question of Baake et al. [arXiv:1402.2818].

1. Introduction

A point set $\mathcal{P} \subset \mathbb{R}^d$ has constant density in $\mathbb{R}^d$ if there exists $\theta(\mathcal{P}) < \infty$ such that, for any $\mathcal{D} \subset \mathbb{R}^d$ with boundary of Lebesgue measure zero,

$$\lim_{T\to\infty} \frac{\#(\mathcal{P} \cap T\mathcal{D})}{T^d} = \theta(\mathcal{P}) \operatorname{vol}(\mathcal{D}). \tag{1.1}$$

We refer to $\theta(\mathcal{P})$ as the density of $\mathcal{P}$. It is interesting to compare the density of $\mathcal{P}$ with the density of the subset of *visible* points given by

$$\widehat{\mathcal{P}} = \big\{ \boldsymbol{y} \in \mathcal{P} \,:\, t\boldsymbol{y} \notin \mathcal{P} \;\forall t \in (0,1) \big\}. \tag{1.2}$$

This definition assumes that the observer is at the origin $\mathbf{0}$. Note also that, by definition, $\mathbf{0} \notin \widehat{\mathcal{P}}$. A classic example is the set of integer lattice points $\mathcal{P} = \mathbb{Z}^d$. In this case, the set of visible points is given by the primitive lattice points $\widehat{\mathcal{P}} = \{\boldsymbol{m} \in \mathbb{Z}^d \,:\, \gcd(\boldsymbol{m}) = 1\}$. Both sets have constant density with $\theta(\mathcal{P}) = 1$ and $\theta(\widehat{\mathcal{P}}) = 1/\zeta(d)$, where $\zeta(d)$ denotes the Riemann zeta function.

In this paper we are interested in the visible points of a regular cut-and-project set $\mathcal{P} = \mathcal{P}(\mathcal{W},\mathcal{L})$ constructed from a (possibly affine) lattice $\mathcal{L} \subset \mathbb{R}^{d+m}$ and a window set $\mathcal{W} \subset \mathbb{R}^m$ (see Section 2 for detailed definitions). Our first observation is the following.

Theorem 1. *If $\mathcal{P} = \mathcal{P}(\mathcal{W},\mathcal{L})$ is a regular cut-and-project set, then $\mathcal{P}$ and $\widehat{\mathcal{P}}$ have constant density with $0 < \theta(\widehat{\mathcal{P}}) \le \theta(\mathcal{P})$.*

The constant density of $\mathcal{P}$ is a well known fact, cf. Section 2 below. The main point of Theorem 1 is that the *visible* set $\widehat{\mathcal{P}}$ also has a strictly positive constant density. Although for cut-and-project sets $\mathcal{P}$ with generic choices of $\mathcal{L}$ we have $\theta(\widehat{\mathcal{P}}) = \theta(\mathcal{P})$, there are important examples with $\theta(\widehat{\mathcal{P}}) < \theta(\mathcal{P})$. The Penrose tilings and other cut-and-project sets which are based on the construction in [6, Sec. 2.2] fall into this category, cf. [9].

The second result of this paper concerns the distribution of directions in $\mathcal{P}$. Consider a general point set with constant density $\theta(\mathcal{P}) > 0$ ($\mathcal{P}$ may be the visible set itself). We write $\mathcal{P}_T = \mathcal{P} \cap \mathcal{B}_T^d \setminus \{\mathbf{0}\}$ for the subset of points lying in the punctured open ball of radius T, centered at the origin. The number of such points is $\#\mathcal{P}_T \sim v_d\, \theta(\mathcal{P})\, T^d$ as $T \to \infty$, where $v_d = \operatorname{vol}(\mathcal{B}_1^d) = \pi^{d/2}/\Gamma(\frac{d+2}{2})$ is the volume of the unit ball. For each T, we study

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the directions $\|\boldsymbol{y}\|^{-1}\boldsymbol{y} \in \mathrm{S}_1^{d-1}$ with $\boldsymbol{y} \in \mathcal{P}_T$, counted *with* multiplicity (if $\mathcal{P} = \widehat{\mathcal{P}}$ then the multiplicity is naturally one). The asymptotics (1.1) implies that, as $T \to \infty$, the directions become uniformly distributed on S_1^{d-1}. That is, for any set $\mathfrak{U} \subset \mathrm{S}_1^{d-1}$ with boundary of measure zero (with respect to the volume element ω on S_1^{d-1}) we have

$$\lim_{T\to\infty} \frac{\#\{\boldsymbol{y} \in \mathcal{P}_T \,:\, \|\boldsymbol{y}\|^{-1}\boldsymbol{y} \in \mathfrak{U}\}}{\#\mathcal{P}_T} = \frac{\omega(\mathfrak{U})}{\omega(\mathrm{S}_1^{d-1})}. \tag{1.3}$$

Recall that $\omega(\mathrm{S}_1^{d-1}) = d\, v_d$.

To understand the fine-scale distribution of the directions in $\mathcal{P}_T$, we consider the probability of finding r directions in a small open disc $\mathfrak{D}_T(\sigma, \boldsymbol{v}) \subset \mathrm{S}_1^{d-1}$ with random center $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$ and volume $\omega(\mathfrak{D}_T(\sigma, \boldsymbol{v})) = \frac{\sigma d}{\theta(\mathcal{P})T^d}$ with $\sigma > 0$ fixed. Denote by

$$\mathcal{N}_T(\sigma, \boldsymbol{v}, \mathcal{P}) = \#\{\boldsymbol{y} \in \mathcal{P}_T \,:\, \|\boldsymbol{y}\|^{-1}\boldsymbol{y} \in \mathfrak{D}_T(\sigma, \boldsymbol{v})\} \tag{1.4}$$

the number of points in $\mathfrak{D}_T(\sigma, \boldsymbol{v})$. The scaling of the disc size ensures that the expectation value for the counting function is asymptotically equal to σ. That is, for any probability measure λ on S_1^{d-1} with continuous density,

$$\lim_{T\to\infty} \int_{\mathrm{S}_1^{d-1}} \mathcal{N}_T(\sigma, \boldsymbol{v}, \mathcal{P})\, d\lambda(\boldsymbol{v}) = \sigma. \tag{1.5}$$

This fact follows directly from (1.1). In the following, we denote by

$$\kappa_{\mathcal{P}} := \frac{\theta(\widehat{\mathcal{P}})}{\theta(\mathcal{P})} \tag{1.6}$$

the relative density of visible points in $\mathcal{P}$. We will prove:

Theorem 2. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ be a regular cut-and-project set, $\sigma > 0$, $r \in \mathbb{Z}_{\geq 0}$, and let λ be a Borel probability measure on S_1^{d-1} which is absolutely continuous with respect to ω. Then the limits*

$$E(r, \sigma, \mathcal{P}) := \lim_{T\to\infty} \lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \mathcal{N}_T(\sigma, \boldsymbol{v}, \mathcal{P}) = r\}), \tag{1.7}$$

$$E(r, \sigma, \widehat{\mathcal{P}}) := \lim_{T\to\infty} \lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \mathcal{N}_T(\sigma, \boldsymbol{v}, \widehat{\mathcal{P}}) = r\}) \tag{1.8}$$

exist, are continuous in σ and independent of λ. For $\sigma \to 0$ we have

$$E(0, \sigma, \mathcal{P}) = 1 - \kappa_{\mathcal{P}}\, \sigma + o(\sigma), \tag{1.9}$$

$$E(0, \sigma, \widehat{\mathcal{P}}) = 1 - \sigma + o(\sigma). \tag{1.10}$$

This theorem generalizes our previous work on directions in Euclidean lattices [5, Section 2]. The existence of the limit (1.7) has already been established in [6, Thm. A.1]. It is worthwhile noting that, if the set of directions in $\mathcal{P}$ were independent and uniformly distributed random variables in S_1^{d-1}, then (1.7) would converge almost surely to the Poisson distribution

$$E(r, \sigma) = \frac{\sigma^r}{r!}\, \mathrm{e}^{-\sigma}. \tag{1.11}$$

Although (1.10) is consistent with the Poisson distribution, we will see in Section 3 that $E(r, \sigma, \widehat{\mathcal{P}})$ is characterized by a certain point process in $\mathbb{R}^d$ which is determined by a finite-dimensional probability space.

Theorem 2 allows us to answer a recent question of Baake et al. [1] on the existence of the gap distribution for the directions in the class of two-dimensional cut-and-project sets considered here. In dimension $d = 2$, it is convenient to identify the circle S_1^1 with the unit interval mod 1, and represent the set of directions in $\mathcal{P}_T$ as $\frac{1}{2\pi}\arg\big(y_1 + \mathrm{i}y_2\big)$ with $\boldsymbol{y} = (y_1, y_2) \in \mathcal{P}_T$. We label these numbers (with multiplicity) in increasing order by

$$-\tfrac{1}{2} < \xi_{T,1} \leq \xi_{T,2} \leq \cdots \leq \xi_{T,N(T)} \leq \tfrac{1}{2}, \tag{1.12}$$

where $N(T) := \#\mathcal{P}_T$. The analogous construction for the visible set $\widehat{\mathcal{P}}$ yields the multiplicity-free set of directions

$$-\tfrac{1}{2} < \widehat{\xi}_{T,1} < \widehat{\xi}_{T,2} < \cdots < \widehat{\xi}_{T,\widehat{N}(T)} \le \tfrac{1}{2} \tag{1.13}$$

where $\widehat{N}(T) := \#\widehat{\mathcal{P}}_T \le N(T)$. We also set $\xi_{T,0} = \widehat{\xi}_{T,0} = \xi_{T,N(T)} - 1 = \widehat{\xi}_{T,\widehat{N}(T)} - 1$.

Corollary 3. *If $\mathcal{P} = \mathcal{P}(\mathcal{W},\mathcal{L})$ is a regular cut-and-project set in dimension $d = 2$, there exists a continuous decreasing function F on $\mathbb{R}_{\ge 0}$ satisfying $F(0) = 1$ and $\lim_{s\to\infty} F(s) = 0$, such that for every $s \ge 0$,*

$$\lim_{T\to\infty} \frac{\#\{1 \le j \le \widehat{N}(T) \;:\; \widehat{N}(T)(\widehat{\xi}_{T,j} - \widehat{\xi}_{T,j-1}) \ge s\}}{\widehat{N}(T)} = F(s) \tag{1.14}$$

and

$$\lim_{T\to\infty} \frac{\#\{1 \le j \le N(T) \;:\; N(T)(\xi_{T,j} - \xi_{T,j-1}) \ge s\}}{N(T)} = \begin{cases} 1 & \text{if } s = 0 \\ \kappa_{\mathcal{P}} F(\kappa_{\mathcal{P}} s) & \text{if } s > 0. \end{cases} \tag{1.15}$$

It follows from the properties of $F(s)$ that the limit distribution function in (1.15) is continuous at $s = 0$ if and only if $\kappa_{\mathcal{P}} = 1$.

In the special case when $\mathcal{P} = \mathbb{Z}^2$, (1.14) was proved earlier by Boca, Cobeli and Zaharescu [2], who also gave an explicit formula for the limit distribution. More generally for $\mathcal{P}$ any affine lattice in $\mathbb{R}^2$, Corollary 3 was proved in [5, Thm. 1.3, Cor. 2.7].

Baake et al. [1] have observed numerically that the limiting gap distribution in Corollary 3 may vanish near zero. In Section 12 we will explain this hard-core repulsion between visible directions in the case of two-dimensional cut-and-project sets constructed over algebraic number fields, including any $\mathcal{P}$ associated with a Penrose tiling. There is no hard-core repulsion for typical two-dimensional cut-and-project sets. The phenomenon can be completely ruled out in higher dimensions $d \ge 3$, where we show that $E(0, \sigma, \widehat{\mathcal{P}}) > 1 - \sigma$ for all $\sigma > 0$.

The organization of this paper is as follows. In Section 2 we recall the definition of a cut-and-project set of a higher-dimensional lattice. In Section 3 we construct random point processes in $\mathbb{R}^d$ whose realizations yield the visible points in certain $\mathrm{SL}(d,\mathbb{R})$-invariant families of cut-and-project sets. These point processes describe the limit distributions in Theorem 2, cf. Theorem 4 in Section 3. This follows closely the construction in [6] for the full cut-and-project set. An important technical tool in our approach is the Siegel-Veech formula, which is stated and proved in Section 4. In Section 5 we describe the small-σ asymptotics of the void distribution in (1.9) and (1.10). Sections 6–9 are devoted to the proof of Theorem 1, Sections 10 and 11 to the proofs of Theorem 2 and Corollary 3, respectively. Finally in Section 12 we discuss the possible vanishing of the limiting gap distribution near zero.

2. Cut-and-project sets

We start by recalling the definition of a cut-and-project set in $\mathbb{R}^d$ from [6]. Denote by π and π_{int} the orthogonal projection of $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$ onto the first d and last m coordinates. We refer to $\mathbb{R}^d$ and $\mathbb{R}^m$ as the *physical space* and *internal space*, respectively. Let $\mathcal{L} \subset \mathbb{R}^n$ be a lattice of full rank. Then the closure of the set $\pi_{\mathrm{int}}(\mathcal{L})$ is an abelian subgroup $\mathcal{A}$ of $\mathbb{R}^m$. We denote by $\mathcal{A}^\circ$ the connected subgroup of $\mathcal{A}$ containing $\mathbf{0}$; then $\mathcal{A}^\circ$ is a linear subspace of $\mathbb{R}^m$, say of dimension m_1, and there exist $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{m_2} \in \mathcal{L}$ ($m = m_1 + m_2$) such that $\pi_{\mathrm{int}}(\boldsymbol{b}_1), \ldots, \pi_{\mathrm{int}}(\boldsymbol{b}_{m_2})$ are linearly independent in $\mathbb{R}^m/\mathcal{A}^\circ$ and

$$\mathcal{A} = \mathcal{A}^\circ + \mathbb{Z}\pi_{\mathrm{int}}(\boldsymbol{b}_1) + \ldots + \mathbb{Z}\pi_{\mathrm{int}}(\boldsymbol{b}_{m_2}). \tag{2.1}$$

Given $\mathcal{L}$ and a bounded subset $\mathcal{W} \subset \mathcal{A}$ with non-empty interior, we define

$$\mathcal{P}(\mathcal{W},\mathcal{L}) = \{\pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathcal{L},\ \pi_{\mathrm{int}}(\boldsymbol{y}) \in \mathcal{W}\} \subset \mathbb{R}^d. \tag{2.2}$$

We will call $\mathcal{P} = \mathcal{P}(\mathcal{W},\mathcal{L})$ a *cut-and-project set*, and $\mathcal{W}$ the *window*. We denote by $\mu_{\mathcal{A}}$ the Haar measure of $\mathcal{A}$, normalized so that its restriction to $\mathcal{A}^\circ$ is the standard m_1-dimensional

Lebesgue measure. If $\mathcal{W}$ has boundary of measure zero with respect to $\mu_{\mathcal{A}}$, we will say $\mathcal{P}(\mathcal{W},\mathcal{L})$ is *regular*. Set $\mathcal{V}=\mathbb{R}^d\times\mathcal{A}^\circ$; then $\mathcal{L}_{\mathcal{V}}=\mathcal{L}\cap\mathcal{V}$ is a lattice of full rank in $\mathcal{V}$. Let $\mu_{\mathcal{V}}=\mathrm{vol}\times\mu_{\mathcal{A}}$ be the natural volume measure on $\mathbb{R}^d\times\mathcal{A}$ (this restricts to the standard $d+m_1$ dimensional Lebesgue measure on $\mathcal{V}$). It follows from Weyl equidistribution (cf. [6, Prop. 3.2]) that for any regular cut-and-project set $\mathcal{P}$ and any bounded $\mathcal{D}\subset\mathbb{R}^d$ with boundary of measure zero with respect to Lebesgue measure,

$$\lim_{T\to\infty}\frac{\#\{\boldsymbol{b}\in\mathcal{L}\,:\,\pi(\boldsymbol{b})\in\mathcal{P}\cap T\mathcal{D}\}}{T^d}=C_{\mathcal{P}}\,\mathrm{vol}(\mathcal{D}) \tag{2.3}$$

where

$$C_{\mathcal{P}}:=\frac{\mu_{\mathcal{A}}(\mathcal{W})}{\mu_{\mathcal{V}}(\mathcal{V}/\mathcal{L}_{\mathcal{V}})}. \tag{2.4}$$

A further condition often imposed in the quasicrystal literature is that $\pi|_{\mathcal{L}}$ is injective (i.e., the map $\mathcal{L}\to\pi(\mathcal{L})$ is one-to-one); we will not require this here. To avoid coincidences in $\mathcal{P}$, we assume throughout this paper that the window is appropriately chosen so that the map $\pi_{\mathcal{W}}:\{\boldsymbol{y}\in\mathcal{L}:\pi_{\mathrm{int}}(\boldsymbol{y})\in\mathcal{W}\}\to\mathcal{P}$ is bijective. Then (2.3) implies

$$\lim_{T\to\infty}\frac{\#(\mathcal{P}\cap T\mathcal{D})}{T^d}=C_{\mathcal{P}}\,\mathrm{vol}(\mathcal{D}), \tag{2.5}$$

i.e., $\mathcal{P}$ has density $\theta(\mathcal{P})=C_{\mathcal{P}}$. Under the above assumptions $\mathcal{P}(\mathcal{W},\mathcal{L})$ is a Delone set, i.e., uniformly discrete and relatively dense in $\mathbb{R}^d$.

We furthermore extend the definition of cut-and-project sets $\mathcal{P}(\mathcal{W},\mathcal{L})$ to affine lattices $\mathcal{L}=\mathcal{L}_0+\boldsymbol{x}$ with $\boldsymbol{x}\in\mathbb{R}^n$ and $\mathcal{L}_0$ a lattice; note that $\mathcal{P}(\mathcal{W},\mathcal{L}+\boldsymbol{x})=\mathcal{P}(\mathcal{W}-\pi_{\mathrm{int}}(\boldsymbol{x}),\mathcal{L})+\pi(\boldsymbol{x})$.

3. Random cut-and-project sets

Following our approach in [6], we will now, for any given regular cut-and-project set $\mathcal{P}=\mathcal{P}(\mathcal{W},\mathcal{L})$, construct two $\mathrm{SL}(d,\mathbb{R})$-invariant random point processes on $\mathbb{R}^d$ which will describe the limit distributions in Theorem 2. Let $G=\mathrm{ASL}(n,\mathbb{R})=\mathrm{SL}(n,\mathbb{R})\ltimes\mathbb{R}^n$, with multiplication law

$$(M,\boldsymbol{\xi})(M',\boldsymbol{\xi}')=(MM',\boldsymbol{\xi}M'+\boldsymbol{\xi}'). \tag{3.1}$$

Also set $\Gamma=\mathrm{ASL}(n,\mathbb{Z})\subset G$. Choose $g\in G$ and $\delta>0$ so that $\mathcal{L}=\delta^{1/n}(\mathbb{Z}^n g)$, and let φ_g be the embedding of $\mathrm{ASL}(d,\mathbb{R})$ in G given by

$$\varphi_g:\mathrm{ASL}(d,\mathbb{R})\to G,\quad (A,\boldsymbol{x})\mapsto g\left(\begin{pmatrix}A & 0\\ 0 & 1_m\end{pmatrix},(\boldsymbol{x},\boldsymbol{0})\right)g^{-1}. \tag{3.2}$$

It then follows from Ratner's work [10], [11] that there exists a unique closed connected subgroup H_g of G such that $\Gamma\cap H_g$ is a lattice in H_g, $\varphi_g(\mathrm{SL}(d,\mathbb{R}))\subset H_g$, and the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{SL}(d,\mathbb{R}))$ in $\Gamma\backslash G$ is given by

$$X=\Gamma\backslash\Gamma H_g. \tag{3.3}$$

Note that X can be naturally identified with the homogeneous space $(\Gamma\cap H_g)\backslash H_g$. We denote the unique right-H_g invariant probability measure on either of these spaces by μ; sometimes we will also let μ denote the corresponding Haar measure on H_g. For each $x=\Gamma h\in X$ we set

$$\mathcal{P}^x:=\mathcal{P}(\mathcal{W},\delta^{1/n}(\mathbb{Z}^n hg)) \tag{3.4}$$

and denote by $\widehat{\mathcal{P}}^x$ the corresponding set of visible points. Both sets are well defined since $\overline{\pi_{\mathrm{int}}(\delta^{1/n}(\mathbb{Z}^n hg))}\subset\mathcal{A}$ for all $h\in H_g$; in fact $\overline{\pi_{\mathrm{int}}(\delta^{1/n}(\mathbb{Z}^n hg))}=\mathcal{A}$ for μ-almost all $h\in H_g$; cf. [6, Prop. 3.5]. Note that $\mathcal{P}^x$ and $\widehat{\mathcal{P}}^x$ with x random in (X,μ) define random point processes on $\mathbb{R}^d$. The fact that $\varphi_g(\mathrm{SL}(d,\mathbb{R}))\subset H_g$ implies that these processes are $\mathrm{SL}(d,\mathbb{R})$-invariant.

Theorem 4. *The limit distributions in Theorem 2 are given by*

$$E(r,\sigma,\mathcal{P}) = \mu(\{x \in X \,:\, \#(\mathcal{P}^x \cap \mathfrak{C}(\sigma)) = r\}) \tag{3.5}$$

and

$$E(r,\sigma,\widehat{\mathcal{P}}) = \mu(\{x \in X \,:\, \#(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)) = r\}) \tag{3.6}$$

where

$$\mathfrak{C}(\sigma) = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \,:\, 0 < x_1 < 1,\ \|(x_2,\ldots,x_d)\| < \Big(\frac{\sigma d}{C_{\mathcal{P}} v_{d-1}}\Big)^{1/(d-1)} x_1\right\}. \tag{3.7}$$

We note that relation (3.5) is a special case of [6, Thm. A.1]. The new result of the present study is (3.6).

In [6, Section 1.4] we also consider the closed connected subgroup $\widetilde{H}_g$ of G such that $\Gamma\cap\widetilde{H}_g$ is a lattice in $\widetilde{H}_g$, $\varphi_g(\mathrm{ASL}(d,\mathbb{R})) \subset \widetilde{H}_g$, and the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{ASL}(d,\mathbb{R}))$ in $\Gamma\backslash G$ is given by $\widetilde{X} := \Gamma\backslash\Gamma\widetilde{H}_g$. The unique right-$\widetilde{H}_g$ invariant probability measure on $\widetilde{X}$ is denoted by $\widetilde{\mu}$. The point process $\mathcal{P}^x$ in (3.4) with x random in $(\widetilde{X},\widetilde{\mu})$ is now $\mathrm{ASL}(d,\mathbb{R})$-invariant, i.e., in addition to the previous $\mathrm{SL}(d,\mathbb{R})$-invariance we also have translation-invariance. The latter implies that $\mathcal{P}^x = \widehat{\mathcal{P}}^x$ for $\widetilde{\mu}$-almost every $x \in \widetilde{X}$. Proposition 4.5 in [6] shows that for Lebesgue-almost all $\boldsymbol{y} \in \mathbb{R}^d \times \{\boldsymbol{0}\}$ we have $H_{g(1_n,\boldsymbol{y})} = \widetilde{H}_g$. This has the following interesting consequence.

Corollary 5. *Given any regular cut-and-project set $\mathcal{P}$ there is a subset $\mathfrak{S} \subset \mathbb{R}^d$ of Lebesgue measure zero such that for every $\boldsymbol{y} \in \mathbb{R}^d \setminus \mathfrak{S}$*

$$E(r,\sigma,\mathcal{P}+\boldsymbol{y}) = E(r,\sigma,\widehat{\mathcal{P}+\boldsymbol{y}}) = \widetilde{\mu}(\{x \in \widetilde{X} \,:\, \#(\mathcal{P}^x \cap \mathfrak{C}(\sigma)) = r\}). \tag{3.8}$$

That is, all limit distributions are independent of $\boldsymbol{y}$ for Lebesgue-almost every $\boldsymbol{y}$.

4. The Siegel-Veech formula for visible points

Throughout the remaining sections, we let $\mathcal{P} = \mathcal{P}(\mathcal{W},\mathcal{L})$ be a given regular cut-and-project set. We fix $g \in G$ and $\delta > 0$ so that $\mathcal{L} = \delta^{1/n}(\mathbb{Z}^n g)$. In fact, by an appropriate scaling of the length units, we can assume without loss of generality that $\delta = 1$. This assumption will be in force throughout the remaining sections except the last one. Hence we now have $\mathcal{P} = \mathcal{P}(\mathcal{W},\mathbb{Z}^n g)$ and $\mathcal{P}^x = \mathcal{P}(\mathcal{W},\mathbb{Z}^n hg)$ for each $x = \Gamma h \in X$.

The following Siegel-Veech formulas will serve as a crucial technical tool in our proofs of the main theorems.

Theorem 6. *For any $f \in \mathrm{L}^1(\mathbb{R}^d)$,*

$$\int_X \sum_{\boldsymbol{q}\in\mathcal{P}^x} f(\boldsymbol{q})\, d\mu(x) = C_{\mathcal{P}} \int_{\mathbb{R}^d} f(\boldsymbol{x})\, d\boldsymbol{x} \tag{4.1}$$

and

$$\int_X \sum_{\boldsymbol{q}\in\widehat{\mathcal{P}}^x} f(\boldsymbol{q})\, d\mu(x) = \kappa_{\mathcal{P}}\, C_{\mathcal{P}} \int_{\mathbb{R}^d} f(\boldsymbol{x})\, d\boldsymbol{x}. \tag{4.2}$$

Veech has proved formulas of the above type for general $\mathrm{SL}(d,\mathbb{R})$-invariant measures [13, Thm. 0.12]. The proof of Theorem 6 is simpler in the present setting. Relation (4.1) was proved in [6, Theorem 1.5]. In the present section we will prove that *there exists* $0 < \kappa_{\mathcal{P}} \le 1$ *such that relation* (4.2) *holds for all* $f \in \mathrm{L}^1(\mathbb{R}^d)$. We will then later establish that this $\kappa_{\mathcal{P}}$ indeed yields the relative density defined in (1.6).

Consider the map

$$B \mapsto \int_X \#(\widehat{\mathcal{P}}^x \cap B)\, d\mu(x) \qquad (B \text{ any Borel subset of } \mathbb{R}^d). \tag{4.3}$$

This map defines a Borel measure on $\mathbb{R}^d$, which is finite on any compact set B (by [6, Theorem 1.5]), invariant under $\mathrm{SL}(d,\mathbb{R})$, and gives zero point mass to $\mathbf{0}\in\mathbb{R}^d$. Hence up to a constant, the measure must equal Lebesgue measure, i.e. there exists a constant $\kappa_{\mathcal{P}}\geq 0$ such that

$$\int_X \#(\widehat{\mathcal{P}}^x\cap B)\,d\mu(x)=\kappa_{\mathcal{P}}C_{\mathcal{P}}\operatorname{vol}(B) \tag{4.4}$$

for every Borel set $B\subset\mathbb{R}^d$. By a standard approximation argument, this implies that (4.2) holds for all $f\in \mathrm{L}^1(\mathbb{R}^d)$. Also $\kappa_{\mathcal{P}}\leq 1$ is immediate from (4.1).

It remains to verify that $\kappa_{\mathcal{P}}>0$. Recall that we are assuming that $\mathcal{W}$ has non-empty interior $\mathcal{W}^\circ$ in $\mathcal{A}=\overline{\pi_{\mathrm{int}}(\mathcal{L})}$. Now take B to be any bounded open set in $\mathbb{R}^d$ which is star-shaped with center $\mathbf{0}$ and such that $(B\setminus\{\mathbf{0}\})\times\mathcal{W}^\circ$ contains some point in the (affine) lattice $\mathcal{L}$. Then the set of $x=\Gamma h$ in X for which $\mathbb{Z}^n hg$ has at least one point in $(B\setminus\{\mathbf{0}\})\times\mathcal{W}^\circ$ is non-empty and open. Note that for any such x, $\mathcal{P}^x=\mathcal{P}(\mathcal{W},\mathbb{Z}^n hg)$ has a point in $B\setminus\{\mathbf{0}\}$, and hence also a *visible* point in $B\setminus\{\mathbf{0}\}$, since B is star-shaped. It follows that the left hand side of (4.4) is positive for our set B. Therefore $\kappa_{\mathcal{P}}>0$, as claimed.

5. The limit distribution for small σ

From now on we take $E(r,\sigma,\mathcal{P})$ and $E(r,\sigma,\widehat{\mathcal{P}})$ to be defined by the relations (3.5), (3.6). Then (1.7) holds by [6, Thm. A.1], and we will prove in Section 10 that also (1.8) holds.

In the present section we will prove that the relation (1.9),

$$E(0,\sigma,\mathcal{P})=1-\kappa_{\mathcal{P}}\,\sigma+o(\sigma), \tag{5.1}$$

holds with the same $\kappa_{\mathcal{P}}\in(0,1]$ as in the Siegel-Veech formula (4.2). Rel. (1.10) is then a simple conseqence of the observation that

$$E(0,\sigma,\widehat{\mathcal{P}})=E(0,\kappa_{\mathcal{P}}^{-1}\sigma,\mathcal{P}). \tag{5.2}$$

To prove (5.1), first note that, for any $\sigma>0$,

$$\begin{aligned}1-E(0,\sigma,\mathcal{P})&=\mu\big(\{x\in X\,:\,\mathcal{P}^x\cap\mathfrak{C}(\sigma)\neq\emptyset\}\big)=\mu\big(\{x\in X\,:\,\widehat{\mathcal{P}}^x\cap\mathfrak{C}(\sigma)\neq\emptyset\}\big)\\&\leq\int_X\#\big(\widehat{\mathcal{P}}^x\cap\mathfrak{C}(\sigma)\big)\,d\mu(x)=\kappa_{\mathcal{P}}C_{\mathcal{P}}\operatorname{vol}(\mathfrak{C}(\sigma))=\kappa_{\mathcal{P}}\sigma,\end{aligned} \tag{5.3}$$

where the integral was evaluated using (4.4).

On the other hand using the fact that the point process $\mathcal{P}^x$ $(x\in(X,\mu))$ is invariant under $\mathrm{SO}(d)$, and $\widehat{\mathcal{P}'k}=\widehat{\mathcal{P}}'k$ for every point set $\mathcal{P}'$ and every $k\in\mathrm{SO}(d)$, we have

$$1-E(0,\sigma,\mathcal{P})=\int_X A(\sigma,\mathcal{P}^x)\,d\mu(x) \tag{5.4}$$

with

$$A(\sigma,\mathcal{P}^x)=\int_{\mathrm{SO}(d)} I\Big(\widehat{\mathcal{P}}^x\cap\mathfrak{C}(\sigma)k\neq\emptyset\Big)\,dk, \tag{5.5}$$

where dk is Haar measure on $\mathrm{SO}(d)$ normalized by $\int_{\mathrm{SO}(d)}dk=1$.

We write $\varphi(\boldsymbol{p},\boldsymbol{q})\in[0,\pi]$ for the angle between any two points $\boldsymbol{p},\boldsymbol{q}\in\mathbb{R}^d\setminus\{\mathbf{0}\}$, as seen from $\mathbf{0}$. Also for any $x\in X$ we set

$$\sigma_0(\mathcal{P}^x)=\frac{C_{\mathcal{P}}v_{d-1}}{d}\Big(\tan\frac{\varphi_0(\mathcal{P}^x)}{2}\Big)^{d-1} \tag{5.6}$$

where

$$\varphi_0(\mathcal{P}^x)=\min\big\{\varphi(\boldsymbol{p},\boldsymbol{q})\,:\,\boldsymbol{p},\boldsymbol{q}\in\widehat{\mathcal{P}}^x\cap\mathcal{B}_1^d,\ \boldsymbol{p}\neq\boldsymbol{q}\big\}, \tag{5.7}$$

with the convention that $\varphi_0(\mathcal{P}^x)=\pi$ and $\sigma_0(\mathcal{P}^x)=+\infty$ whenever $\#(\widehat{\mathcal{P}}^x\cap\mathcal{B}_1^d)\leq 1$. These are measurable functions on X, and $\varphi_0(\mathcal{P}^x)>0$ and $\sigma_0(\mathcal{P}^x)>0$ for all $x\in X$.

Now if $0<\sigma<\sigma_0(\mathcal{P}^x)$ then for any two distinct points $\boldsymbol{p},\boldsymbol{q}\in\widehat{\mathcal{P}}^x\cap\mathcal{B}_1^d$ we have

$$\varphi(\boldsymbol{p},\boldsymbol{q})>2\arctan\Big(\Big(\frac{\sigma d}{C_{\mathcal{P}}v_{d-1}}\Big)^{1/(d-1)}\Big), \tag{5.8}$$

and because of the definition of $\mathfrak{C}(\sigma)$, (3.7), this implies that there does not exist any $k\in\mathrm{SO}(d)$ for which $\mathfrak{C}(\sigma)k$ contains both $\boldsymbol{p}$ and $\boldsymbol{q}$. Hence for $0<\sigma<\sigma_0(\mathcal{P}^x)$ we have (writing $\boldsymbol{e}_1=(1,0,\ldots,0)\in\mathbb{R}^d$)

$$\begin{aligned} A(\sigma,\mathcal{P}^x)\geq\sum_{\boldsymbol{p}\in\widehat{\mathcal{P}}^x\cap\mathcal{B}_1^d}\int_{\mathrm{SO}(d)}I\Big(\boldsymbol{p}\in\mathfrak{C}(\sigma)k\Big)\,dk&=\#\big(\widehat{\mathcal{P}}^x\cap\mathcal{B}_1^d\big)\cdot\int_{\mathrm{SO}(d)}I\Big(\boldsymbol{e}_1\in\mathfrak{C}(\sigma)k\Big)\,dk\\ &=\frac{\mathrm{vol}(\mathfrak{C}(\sigma)\cap\mathcal{B}_1^d)}{\mathrm{vol}(\mathcal{B}_1^d)}\#\big(\widehat{\mathcal{P}}^x\cap\mathcal{B}_1^d\big), \end{aligned} \tag{5.9}$$

and here

$$\frac{\mathrm{vol}(\mathfrak{C}(\sigma)\cap\mathcal{B}_1^d)}{\mathrm{vol}(\mathcal{B}_1^d)}\sim\frac{\mathrm{vol}(\mathfrak{C}(\sigma))}{\mathrm{vol}(\mathcal{B}_1^d)}=\frac{\sigma}{v_dC_{\mathcal{P}}}\qquad\text{as }\ \sigma\to0. \tag{5.10}$$

Hence given any number $K<(v_dC_{\mathcal{P}})^{-1}$, there is some $\sigma(K)>0$ such that for all $0<\sigma<\sigma(K)$ we have

$$1-E(0,\sigma,\mathcal{P})=\int_X A(\sigma,\mathcal{P}^x)\,d\mu(x)\geq K\sigma\int_X I\big(\sigma<\sigma_0(\mathcal{P}^x)\big)\#\big(\widehat{\mathcal{P}}^x\cap\mathcal{B}_1^d\big)\,d\mu(x). \tag{5.11}$$

Furthermore, by the Monotone Convergence Theorem and (4.4),

$$\lim_{\sigma\to0}\int_X I\big(\sigma<\sigma_0(\mathcal{P}^x)\big)\#\big(\widehat{\mathcal{P}}^x\cap\mathcal{B}_1^d\big)\,d\mu(x)=\int_X\#\big(\widehat{\mathcal{P}}^x\cap\mathcal{B}_1^d\big)\,d\mu(\mathcal{P}^x)=\kappa_{\mathcal{P}}C_{\mathcal{P}}v_d. \tag{5.12}$$

We thus conclude

$$\liminf_{\sigma\to0}\frac{1-E(0,\sigma,\mathcal{P})}{\sigma}\geq K\kappa_{\mathcal{P}}C_{\mathcal{P}}v_d. \tag{5.13}$$

The claim (5.1) follows from (5.3) and the fact that (5.13) holds for every $K<(v_dC_{\mathcal{P}})^{-1}$.

6. Lower bound on the density of visible points

Combining (5.1) and (1.7) (recall that the latter was proved in [6, Thm. A.1]), we get the following lower bound on the density $\theta(\widehat{\mathcal{P}})=\kappa_{\mathcal{P}}C_{\mathcal{P}}$ in Theorem 1:

Lemma 7. *Let $\mathfrak{U}$ be any subset of S_1^{d-1} with boundary of measure zero (w.r.t. ω), and let $\mathcal{D}=\{\boldsymbol{v}\in\mathbb{R}^d\,:\,0<\|\boldsymbol{v}\|<1,\ \|\boldsymbol{v}\|^{-1}\boldsymbol{v}\in\mathfrak{U}\}$ be the corresponding sector in $\mathcal{B}_1^d$. Then*

$$\liminf_{T\to\infty}\frac{\#(\widehat{\mathcal{P}}\cap T\mathcal{D})}{T^d}\geq\kappa_{\mathcal{P}}C_{\mathcal{P}}\,\mathrm{vol}(\mathcal{D}). \tag{6.1}$$

Proof. We may assume $\omega(\mathfrak{U})>0$, since otherwise $\mathrm{vol}(\mathcal{D})=0$ and the lemma is trivial. Let $\varepsilon>0$ be given, and let $\mathfrak{U}_\varepsilon^-\subset\mathrm{S}_1^{d-1}$ be the "ε-thinning" of $\mathfrak{U}$, that is

$$\mathfrak{U}_\varepsilon^-=\big\{\boldsymbol{v}\in\mathrm{S}_1^{d-1}\,:\,\big[\varphi(\boldsymbol{w},\boldsymbol{v})<\varepsilon\Rightarrow\boldsymbol{w}\in\mathfrak{U}\big],\ \forall\boldsymbol{w}\in\mathrm{S}_1^{d-1}\big\}. \tag{6.2}$$

(Recall that $\varphi(\boldsymbol{w},\boldsymbol{v})\in[0,\pi]$ is the angle between $\boldsymbol{w}$ and $\boldsymbol{v}$ as seen from $\mathbf{0}$.) Then $\omega(\mathfrak{U}_\varepsilon^-)\to\omega(\mathfrak{U})$ as $\varepsilon\to0$, since $\mathfrak{U}$ by assumption is a Jordan measurable subset of S_1^{d-1}. From now on we assume that ε is so small that $\omega(\mathfrak{U}_\varepsilon^-)>0$. We let λ be ω restricted to $\mathfrak{U}_\varepsilon^-$ and normalized to be a probability measure; thus $\lambda(B)=\omega(\mathfrak{U}_\varepsilon^-)^{-1}\omega(B\cap\mathfrak{U}_\varepsilon^-)$ for any Borel subset $B\subset\mathrm{S}_1^{d-1}$.

Now note that, by the definitions of $\mathcal{N}_T(\sigma,\boldsymbol{v},\mathcal{P})$ and $\widehat{\mathcal{P}}$, for any $\sigma>0$, $T>0$ and $\boldsymbol{v}\in\mathrm{S}_1^{d-1}$ we have $\mathcal{N}_T(\sigma,\boldsymbol{v},\mathcal{P})>0$ if and only if there is some $\boldsymbol{y}\in\widehat{\mathcal{P}}\cap\mathcal{B}_T^d$ such that $\|\boldsymbol{y}\|^{-1}\boldsymbol{y}\in\mathfrak{D}_T(\sigma,\boldsymbol{v})$. Furthermore, if T is larger than a certain constant depending on $\sigma,\mathcal{P},\varepsilon$, then $\mathfrak{D}_T(\sigma,\boldsymbol{v})\subset\mathfrak{U}$

for every $\boldsymbol{v} \in \mathfrak{U}_\varepsilon^-$, meaning that $\|\boldsymbol{y}\|^{-1}\boldsymbol{y} \in \mathfrak{D}_T(\sigma, \boldsymbol{v})$ implies $\boldsymbol{y} \in \mathbb{R}_{>0}\mathcal{D}$. Hence for such T and σ we have

$$\lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \mathcal{N}_T(\sigma, \boldsymbol{v}, \mathcal{P}) > 0\}) = \lambda\big(\big\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \big[\exists \boldsymbol{y} \in \widehat{\mathcal{P}} \cap \mathcal{B}_T^d : \|\boldsymbol{y}\|^{-1}\boldsymbol{y} \in \mathfrak{D}_T(\sigma, \boldsymbol{v})\big]\big\}\big)$$
$$\leq \sum_{\boldsymbol{y} \in \widehat{\mathcal{P}} \cap T\mathcal{D}} \lambda\big(\big\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \|\boldsymbol{y}\|^{-1}\boldsymbol{y} \in \mathfrak{D}_T(\sigma, \boldsymbol{v})\big\}\big) \leq \frac{\omega(\mathfrak{D}_T(\sigma, \boldsymbol{e}_1))}{\omega(\mathfrak{U}_\varepsilon^-)} \cdot \#\big(\widehat{\mathcal{P}} \cap T\mathcal{D}\big)$$
$$= \frac{\sigma d}{\omega(\mathfrak{U}_\varepsilon^-) C_{\mathcal{P}} T^d} \cdot \#\big(\widehat{\mathcal{P}} \cap T\mathcal{D}\big). \tag{6.3}$$

Hence, letting $T \to \infty$ and applying (1.7) we have, for any fixed $\sigma > 0$,

$$\liminf_{T \to \infty} \frac{\#\widehat{\mathcal{P}} \cap T\mathcal{D}}{T^d} \geq \frac{\omega(\mathfrak{U}_\varepsilon^-) C_{\mathcal{P}}}{d} \cdot \frac{1 - E(0, \sigma, \mathcal{P})}{\sigma}. \tag{6.4}$$

Letting $\sigma \to 0$ in the right hand side and using (5.1), this gives

$$\liminf_{T \to \infty} \frac{\#\widehat{\mathcal{P}} \cap T\mathcal{D}}{T^d} \geq \kappa_{\mathcal{P}} C_{\mathcal{P}} \frac{\omega(\mathfrak{U}_\varepsilon^-)}{d}. \tag{6.5}$$

Finally letting $\varepsilon \to 0$ and using $\omega(\mathfrak{U})/d = \mathrm{vol}(\mathcal{D})$ we obtain the statement of the lemma. □

7. Continuity in the space of cut-and-project sets

Next, in Lemma 9 and Lemma 10, we will prove that for almost all $x \in X$, both $\mathcal{P}^x$ and $\widehat{\mathcal{P}}^x$ *vary continuously* as we perturb x.

Lemma 8. *For any $\boldsymbol{m} \in \mathbb{R}^n$, if $\pi(\boldsymbol{m}hg) \neq \boldsymbol{0}$ for some $h \in H_g$ then $\pi(\boldsymbol{m}hg) \neq \boldsymbol{0}$ for μ-almost all $h \in H_g$. Similarly, for any $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{R}^n$, if* $\dim \mathrm{Span}\{\pi(\boldsymbol{n}hg), \pi(\boldsymbol{m}hg)\} = 2$ *for some $h \in H_g$ then* $\dim \mathrm{Span}\{\pi(\boldsymbol{n}hg), \pi(\boldsymbol{m}hg)\} = 2$ *for μ-almost all $h \in H_g$.*

Proof. H_g is a connected, real-analytic manifold; hence any real-analytic function on H_g which does not vanish identically is non-zero almost everywhere. The first part of the lemma follows by applying this principle to the coordinate functions $h \mapsto \pi(\boldsymbol{m}hg) \cdot \boldsymbol{e}_j$ for $j = 1, \ldots, d$. The second part of the lemma follows by applying the same principle to the functions

$$h \mapsto (\pi(\boldsymbol{m}hg) \cdot \boldsymbol{e}_i)(\pi(\boldsymbol{n}hg) \cdot \boldsymbol{e}_j) - (\pi(\boldsymbol{m}hg) \cdot \boldsymbol{e}_j)(\pi(\boldsymbol{n}hg) \cdot \boldsymbol{e}_i), \tag{7.1}$$

for $1 \leq i < j \leq d$. □

Lemma 9. *For μ-almost every $x \in X$, and for every bounded open set $U \subset \mathbb{R}^d$ with $\mathcal{P}^x \cap \partial U = \emptyset$, there is an open set $\Omega \subset X$ with $x \in \Omega$ such that $\#(\mathcal{P}^{x'} \cap U) = \#(\mathcal{P}^x \cap U)$ for all $x' \in \Omega$.*

Proof. For each $\boldsymbol{m} \in \mathbb{Z}^n$, by an argument as in Lemma 8 we either have $\boldsymbol{m}hg \neq \boldsymbol{0}$ for almost all $h \in H_g$ or else $\boldsymbol{m}hg = \boldsymbol{0}$ for all $h \in H_g$. By taking $h = 1$ we see that the latter property can hold for at most one $\boldsymbol{m} \in \mathbb{Z}^n$, and if it holds then we necessarily have $\boldsymbol{m} = \boldsymbol{0}g^{-1}$, and $H_g \subset g\,\mathrm{SL}(n, \mathbb{R})g^{-1}$. If such an exceptional $\boldsymbol{m}$ exists we call it $\boldsymbol{m}_E$, and we set $(\mathbb{Z}^n)' := \mathbb{Z}^n \setminus \{\boldsymbol{m}_E\}$; otherwise we set $(\mathbb{Z}^n)' := \mathbb{Z}^n$.

Now consider the following two subsets of H_g:

$$S_1 = \big\{h \in H_g : (\mathbb{Z}^n)'hg \cap (\mathbb{R}^d \times \partial\mathcal{W}) \neq \emptyset\big\}; \tag{7.2}$$

$$S_2 = \big\{h \in H_g : \exists \boldsymbol{\ell}_1 \neq \boldsymbol{\ell}_2 \in \mathbb{Z}^n hg \cap \pi_{\mathrm{int}}^{-1}(\mathcal{W}) \text{ satisfying } \pi(\boldsymbol{\ell}_1) = \pi(\boldsymbol{\ell}_2)\big\}. \tag{7.3}$$

We have $\mu(S_1) = 0$, by [6, Theorem 5.1]. Also $\mu(S_2) = 0$, by [6, Prop. 3.7] applied to $\mathcal{W}^\circ$. We will prove the lemma by showing that for every $h \in H_g \setminus (S_1 \cup S_2)$, the point $x = \Gamma h \in X$ has the property described in the lemma.

Thus let $h \in H_g \setminus (S_1 \cup S_2)$ be given, set $x = \Gamma h \in X$, and let U be an arbitrary bounded open subset of $\mathbb{R}^d$ with boundary disjoint from $\mathcal{P}^x = \mathcal{P}(\mathcal{W}, \mathbb{Z}^n hg)$. Assume that the desired property does *not* hold. Then there is a sequence $h_1, h_2, \ldots$ in H_g tending to h such that

$$\#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U) \neq \#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n hg) \cap U), \qquad \forall j. \tag{7.4}$$

Let F be the (finite) set

$$F = \{\boldsymbol{m} \in \mathbb{Z}^n \,:\, \boldsymbol{m}hg \in U \times \mathcal{W}\}. \tag{7.5}$$

Note that $\boldsymbol{m}hg \in U \times \mathcal{W}^\circ$ for every $\boldsymbol{m} \in F \cap (\mathbb{Z}^n)'$, since $h \notin S_1$. But $U \times \mathcal{W}^\circ$ is open; hence by continuity we also have $\boldsymbol{m}h'g \in U \times \mathcal{W}^\circ$ for every $h' \in H_g$ sufficiently near h and all $\boldsymbol{m} \in F \cap (\mathbb{Z}^n)'$. Note also that if the exceptional point $\boldsymbol{m}_E$ exists and belongs to F then $\boldsymbol{0} = \boldsymbol{m}_E h'g \in U \times \mathcal{W}$ for *all* $h' \in H_g$. Hence, for every $h' \in H_g$ near h we have

$$\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h'g) \supset \{\pi(\boldsymbol{m}h'g) \,:\, \boldsymbol{m} \in F\}. \tag{7.6}$$

Because of $h \notin S_2$, the points $\pi(\boldsymbol{m}hg)$ for $\boldsymbol{m} \in F$ are pairwise distinct. By continuity it then also follows that for any $h' \in H_g$ sufficiently near h, the points $\pi(\boldsymbol{m}h'g)$ for $\boldsymbol{m} \in F$ are pairwise distinct. Hence $\#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n hg) \cap U) = \#F$ and $\#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h'g) \cap U) \geq \#F$ for every h' near h. Therefore in (7.4), the left hand side must be *larger* than $\#F$, for all large j. Hence for each large j there is some $\boldsymbol{m} \in \mathbb{Z}^n \setminus F$ such that $\boldsymbol{m}h_j g \in U \times \mathcal{W}$. But for any compact $C \subset H_g$ the set $\cup_{h' \in C}(U \times \mathcal{W})g^{-1}h'^{-1}$ is bounded and hence has finite intersection with $\mathbb{Z}^n$. Therefore there is a bounded number of possibilities for $\boldsymbol{m}$ as j varies, and by passing to a subsequence we may assume that $\boldsymbol{m}$ is independent of j.

Now for our fixed $\boldsymbol{m} \in \mathbb{Z}^n \setminus F$ we have $\boldsymbol{m}h_j g \in U \times \mathcal{W}$ for all j, but $\boldsymbol{m}h_j g \to \boldsymbol{m}hg \notin U \times \mathcal{W}$ as $j \to \infty$; this forces $\boldsymbol{m}hg \in \partial(U \times \mathcal{W})$, and it also implies that we cannot have $\boldsymbol{m} = \boldsymbol{m}_E$. But $\pi_{\mathrm{int}}(\boldsymbol{m}hg) \notin \partial\mathcal{W}$ since $h \notin S_1$, and thus we must have $\pi(\boldsymbol{m}hg) \in \partial U$. Note also that $\pi_{\mathrm{int}}(\boldsymbol{m}hg)$ cannot belong to the exterior of $\mathcal{W}$, since then the same would hold for $\pi_{\mathrm{int}}(\boldsymbol{m}h_j g)$ for j large, contradicting $\boldsymbol{m}h_j g \in U \times \mathcal{W}$. Hence $\pi_{\mathrm{int}}(\boldsymbol{m}hg)$ must belong to the interior of $\mathcal{W}$; therefore $\pi(\boldsymbol{m}hg) \in \mathcal{P}^x = \mathcal{P}(\mathcal{W}, \mathbb{Z}^n hg)$. This contradicts our assumption that $\mathcal{P}^x$ is disjoint from ∂U, and so the lemma is proved. □

Lemma 10. *For μ-almost every $x \in X$, and for every bounded open set $U \subset \mathbb{R}^d$ with $\widehat{\mathcal{P}}^x \cap \partial U = \emptyset$, there is an open set $\Omega \subset X$ with $x \in \Omega$ such that $\#(\widehat{\mathcal{P}}^{x'} \cap U) = \#(\widehat{\mathcal{P}}^x \cap U)$ for all $x' \in \Omega$.*

Proof. Let $\boldsymbol{m}_E$, $(\mathbb{Z}^n)'$, S_1 and S_2 be as in the proof of Lemma 9. Also set

$$S_3 = \{h \in H_g \,:\, \exists \boldsymbol{m} \in \mathbb{Z}^n,\ h' \in H_g \text{ satisfying } \pi(\boldsymbol{m}hg) = \boldsymbol{0},\ \pi(\boldsymbol{m}h'g) \neq \boldsymbol{0}\}$$

$$S_4 = \{h \in H_g \,:\, \exists \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^n,\ h' \in H_g \text{ satisfying } \dim \mathrm{Span}\{\pi(\boldsymbol{n}hg), \pi(\boldsymbol{m}hg)\} \leq 1$$
$$\text{and } \dim \mathrm{Span}\{\pi(\boldsymbol{n}h'g), \pi(\boldsymbol{m}h'g)\} = 2\}.$$

Using Lemma 8 and the fact that $\mathbb{Z}^n$ is countable, we have $\mu(S_3) = \mu(S_4) = 0$.

Now let $h \in H_g \setminus (S_1 \cup S_2 \cup S_3 \cup S_4)$ be given, set $x = \Gamma h \in X$, and let U be an arbitrary bounded open subset of $\mathbb{R}^d$ with boundary disjoint from $\widehat{\mathcal{P}}^x = \widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n hg)$. Assume that there is a sequence $h_1, h_2, \ldots$ in H_g tending to h such that

$$\#(\widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U) \neq \#(\widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n hg) \cap U), \qquad \forall j. \tag{7.7}$$

We will show that this leads to a contradiction, and this will complete the proof of the lemma (cf. the proof of Lemma 9).

As an initial reduction, let us note that we may assume $\mathcal{P}^x \cap \partial U = \emptyset$. Indeed, recall that $\mathcal{P}^x$ is locally finite (cf. [6, Prop. 3.1]); hence the set $A = \mathcal{P}^x \cap \partial U$ is certainly finite. Also every point in A is invisible in $\mathcal{P}^x$, since we are assuming $\widehat{\mathcal{P}}^x \cap \partial U = \emptyset$. If $A \neq \emptyset$ then fix $r > 0$ so small that $(\boldsymbol{p} + \mathcal{B}_{2r}^d) \cap \mathcal{P}^x = \{\boldsymbol{p}\}$ for each $\boldsymbol{p} \in A$, and set $U' = U \cup (\cup_{\boldsymbol{p} \in A}(\boldsymbol{p} + \mathcal{B}_r^d))$ and $U'' = U \setminus (\cup_{\boldsymbol{p} \in A}(\boldsymbol{p} + \overline{\mathcal{B}_r^d}))$. These are bounded open sets satisfying $\#(\widehat{\mathcal{P}}^x \cap U') = \#(\widehat{\mathcal{P}}^x \cap U'') = \#(\widehat{\mathcal{P}}^x \cap U)$ and $\mathcal{P}^x \cap \partial U' = \mathcal{P}^x \cap \partial U'' = \emptyset$. For each j we must have either $\#(\widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U') > \#(\widehat{\mathcal{P}}^x \cap U)$ or $\#(\widehat{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U'') < \#(\widehat{\mathcal{P}}^x \cap U)$, because of $U'' \subset U \subset U'$ and (7.7). Hence after replacing U by U' or U'', and passing to a subsequence, we are in a situation where (7.7) holds, and also $\mathcal{P}^x \cap \partial U = \emptyset$.

Now take F as in (7.5); it then follows from the proof of Lemma 9 that $\#(\mathcal{P}^x \cap U) = \#F$ and also $\#(\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g) \cap U) = \#F$ for every large j. Hence (7.7) implies that for every large j there is some $\boldsymbol{m} \in F$ such that either $\pi(\boldsymbol{m}h_j g)$ is visible in $\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$ but $\pi(\boldsymbol{m}hg)$ is invisible in $\mathcal{P}^x$, or the other way around. Since F is finite we may assume, by passing to a subsequence, that $\boldsymbol{m}$ is independent of j.

First assume that $\pi(\boldsymbol{m}hg)$ is invisible in $\mathcal{P}^x$ but $\pi(\boldsymbol{m}h_j g)$ is visible in $\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$ for every large j. In particular then $\pi(\boldsymbol{m}h_j g) \neq \mathbf{0}$ for large j, and since $h \notin S_3$ this implies $\pi(\boldsymbol{m}hg) \neq \mathbf{0}$. The invisibility of $\pi(\boldsymbol{m}hg)$ means that there exist $\boldsymbol{n} \in \mathbb{Z}^n$ and $0 < t < 1$ such that $\pi_{\text{int}}(\boldsymbol{n}hg) \in \mathcal{W}$ and $\pi(\boldsymbol{n}hg) = t\pi(\boldsymbol{m}hg)$. Now $\pi_{\text{int}}(\boldsymbol{n}hg) \in \mathcal{W}$ and $h \notin S_1$ force $\pi_{\text{int}}(\boldsymbol{n}hg) \in \mathcal{W}^\circ$; hence $\pi_{\text{int}}(\boldsymbol{n}h_j g) \in \mathcal{W}^\circ$ for all large j and so $\pi(\boldsymbol{n}h_j g) \in \mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$. On the other hand $\dim \operatorname{Span}\{\pi(\boldsymbol{n}hg), \pi(\boldsymbol{m}hg)\} = 1$ together with $h \notin S_4$ imply $\dim \operatorname{Span}\{\pi(\boldsymbol{n}h'g), \pi(\boldsymbol{m}h'g)\} \leq 1$ for *all* $h' \in H_g$. Using also $h_j \to h$, $\pi(\boldsymbol{m}hg) \neq 0$ and $0 < t < 1$, this implies that for every large j there is $0 < t_j < 1$ such that $\pi(\boldsymbol{n}h_j g) = t_j \pi(\boldsymbol{m}h_j g)$. Hence $\pi(\boldsymbol{m}h_j g)$ is invisible in $\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$ for every large j, contradicting our earlier assumption.

It remains to treat the case when $\pi(\boldsymbol{m}hg)$ is visible in $\mathcal{P}^x$ but $\pi(\boldsymbol{m}h_j g)$ is invisible in $\mathcal{P}(\mathcal{W}, \mathbb{Z}^n h_j g)$ for every large j. Then for every large j there exist $\boldsymbol{n} \in \mathbb{Z}^n$ and $0 < t_j < 1$ such that $\pi_{\text{int}}(\boldsymbol{n}h_j g) \in \mathcal{W}$ and $\pi(\boldsymbol{n}h_j g) = t_j \pi(\boldsymbol{m}h_j g)$. It is easily seen that there are only a finite number of possibilities for $\boldsymbol{n}$, and hence by passing to a subsequence we may assume that $\boldsymbol{n}$ is independent of j. Since $\pi(\boldsymbol{m}hg)$ is visible in $\mathcal{P}^x$ we have $\pi(\boldsymbol{m}hg) \neq \mathbf{0}$; hence also $\pi(\boldsymbol{m}h_j g) \neq \mathbf{0}$ for all large j, and this forces $\boldsymbol{n} \neq \boldsymbol{m}$. Also $\pi(\boldsymbol{m}h_j g) \to \pi(\boldsymbol{m}hg) \neq \mathbf{0}$ and $t_j \pi(\boldsymbol{m}h_j g) = \pi(\boldsymbol{n}h_j g) \to \pi(\boldsymbol{n}hg)$ imply that $t = \lim_{j\to\infty} t_j \in [0,1]$ exists, and $\pi(\boldsymbol{n}hg) = t\pi(\boldsymbol{m}hg)$. Using $h \notin S_1$ and $\pi_{\text{int}}(\boldsymbol{n}h_j g) \in \mathcal{W}$ it follows that also $\pi_{\text{int}}(\boldsymbol{n}hg) \in \mathcal{W}$ and so $\pi(\boldsymbol{n}hg) \in \mathcal{P}^x$. Using $h \notin S_3$ and $\pi(\boldsymbol{n}h_j g) \neq \mathbf{0}$ for j large, it follows that $\pi(\boldsymbol{n}hg) \neq \mathbf{0}$; furthermore using $h \notin S_2$ we have $\pi(\boldsymbol{n}hg) \neq \pi(\boldsymbol{m}hg)$. Hence $0 < t < 1$, and so $\pi(\boldsymbol{m}hg)$ is invisible in $\mathcal{P}^x$, contradicting our earlier assumption. □

8. Upper bound on the density of visible points

We are now in position to prove an upper bound complementing Lemma 7.

Lemma 11. *We have* $\lim_{T\to\infty} \frac{\#(\widehat{\mathcal{P}} \cap \mathcal{B}_T^d)}{T^d} = \kappa_{\mathcal{P}} C_{\mathcal{P}} v_d$.

Proof. For any $\mathcal{P}' \subset \mathbb{R}^d$, let us write $\widetilde{\mathcal{P}'} = \mathcal{P}' \setminus \widehat{\mathcal{P}'}$ for the set of *in*visible points in $\mathcal{P}'$. Define $F : X \to \mathbb{Z}_{\geq 0}$ through

$$F(x) = \liminf_{x' \to x} \#(\widetilde{\mathcal{P}}^{x'} \cap \mathcal{B}_1^d). \tag{8.1}$$

Then F is lower semicontinuous by construction. Hence by [6, Thm. 4.1] and the Portmanteau theorem (cf., e.g., [15, Thm. 1.3.4(iv)]),

$$\liminf_{R\to\infty} \int_{\mathrm{SO}(d)} F(\Gamma\varphi_g(k\Phi^{\log R}))\, dk \geq \int_X F\, d\mu, \tag{8.2}$$

with

$$\Phi^t = \begin{pmatrix} e^{-(d-1)t} & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{0} & e^t 1_{d-1} \end{pmatrix} \in \mathrm{SL}(d, \mathbb{R}). \tag{8.3}$$

Now in the left hand side of (8.2), we use the fact that for any $x = \Gamma\varphi_g(T)$, $T \in \mathrm{SL}(d, \mathbb{R})$, we have

$$F(x) \leq \#(\widetilde{\mathcal{P}}^x \cap \mathcal{B}_1^d) = \#(\widetilde{\mathcal{P}}(\mathcal{W}, \mathbb{Z}^n \varphi_g(T) g) \cap \mathcal{B}_1^d) = \#(\widetilde{\mathcal{P}} \cap \mathcal{B}_1^d T^{-1}). \tag{8.4}$$

In the right hand side of (8.2) we note that if $x = \Gamma h$ has both the continuity properties described in Lemmata 9 and 10, and if furthermore $\mathcal{P}^x \cap \mathrm{S}_1^{d-1} = \emptyset$, then in fact $F(x) = \#(\widetilde{\mathcal{P}}^x \cap \mathcal{B}_1^d)$. But these conditions are fulfilled for μ-almost all $x \in X$ (concerning $\mathcal{P}^x \cap \mathrm{S}_1^{d-1} = \emptyset$, use [6, Thm. 1.5]). Hence it follows from (8.2) that

$$\liminf_{R\to\infty} \int_{\mathrm{SO}(d)} \#\big(\widetilde{\mathcal{P}} \cap \mathcal{B}_1^d \Phi^{-\log R} k^{-1}\big)\, dk \geq \int_X \#\big(\widetilde{\mathcal{P}}^x \cap \mathcal{B}_1^d\big)\, d\mu(x) = (1 - \kappa_{\mathcal{P}}) C_{\mathcal{P}} v_d, \tag{8.5}$$

where the last equality holds by Theorem 6.

But exactly as in the proof of Theorem 5.1 in [6], we have for any $R > 1$

$$\int_{\mathrm{SO}(d)} \#(\widetilde{\mathcal{P}} \cap \mathcal{B}_1 \Phi^{-\log R} k^{-1})\, dk = \sum_{\boldsymbol{p} \in \widetilde{\mathcal{P}}} A_R(\|\boldsymbol{p}\|) = \int_0^\infty A_R(\tau)\, d\widetilde{N}(\tau) = -\int_0^\infty \widetilde{N}(\tau)\, dA_R(\tau), \tag{8.6}$$

where

$$\widetilde{N}(T) = \#(\widetilde{\mathcal{P}} \cap \mathcal{B}_T^d), \tag{8.7}$$

and A_R is the continuous and decreasing function from $\mathbb{R}_{\geq 0}$ to $[0,1]$ given by $A_R(0) = 1$ and

$$A_R(\tau) = \frac{\omega\big(\mathrm{S}_1^{d-1} \cap \tau^{-1} \mathcal{B}_1^d \Phi^{-\log R}\big)}{\omega(\mathrm{S}_1^{d-1})} \qquad \text{for } \tau > 0. \tag{8.8}$$

(Thus $A_R(\tau) = 1$ for $0 \leq \tau \leq R^{-1}$ and $A_R(\tau) = 0$ for $\tau \geq R^{d-1}$.) Hence (8.5) says that

$$\liminf_{R \to \infty} \int_0^\infty \widetilde{N}(\tau)\, \big(-dA_R(\tau)\big) \geq C' := (1 - \kappa_{\mathcal{P}}) C_{\mathcal{P}} v_d. \tag{8.9}$$

In view of (2.5) and Lemma 7 (with $\mathcal{D} = \mathcal{B}_1^d$), the statement of the present lemma is equivalent with $\liminf_{\tau \to \infty} \tau^{-d} \widetilde{N}(\tau) \geq C'$. Assume that this is *false*. Then there is some $\eta > 0$ and a sequence $1 < \tau_1 < \tau_2 < \cdots$ with $\tau_j \to \infty$ such that $\widetilde{N}(\tau_j) < (1-\eta) C' \tau_j^d$ for all j. Using the fact that $\widetilde{N}(\tau)$ is an increasing function of τ we see that by shrinking $\eta > 0$ if necessary, we may actually assume that $\widetilde{N}(\tau) < (1-\eta) C' \tau^d$ for all $\tau \in [(1-\eta)\tau_j, \tau_j]$ and all j. By Lemma 7 and (2.5) we have $\limsup_{\tau \to \infty} \tau^{-d} \widetilde{N}(\tau) \leq C'$; thus for any given $\varepsilon > 0$ there is some $\tau_0 > 0$ such that $\widetilde{N}(\tau) \leq (1+\varepsilon) C' \tau^d$ for all $\tau \geq \tau_0$. Now for any j with $(1-\eta)\tau_j > \tau_0$, and any $R > \tau_j^{1/(d-1)}$:

$$\begin{aligned} \int_0^\infty \widetilde{N}(\tau)\, \big(-dA_R(\tau)\big) \leq \int_0^{\tau_0} \widetilde{N}(\tau)(-dA_R(\tau)) + (1+\varepsilon) C' \int_{\tau_0}^{R^{d-1}} \tau^d\, (-dA_R(\tau)) \\ - (\varepsilon + \eta) C' \int_{(1-\eta)\tau_j}^{\tau_j} \tau^d\, (-dA_R(\tau)). \end{aligned} \tag{8.10}$$

Here the sum of the first two terms tends to $(1+\varepsilon)C'$ as $R \to \infty$, as in [6, (5.11)-(5.13)]. Furthermore, if we choose $R = (2\tau_j)^{1/(d-1)}$ and let $j \to \infty$ then

$$\begin{aligned} \int_{(1-\eta)\tau_j}^{\tau_j} \tau^d\, (-dA_R(\tau)) = \frac{d}{\omega(\mathrm{S}_1^{d-1})} \operatorname{vol}\Big(\mathcal{B}_1^d \Phi^{-\log R} \cap \mathcal{B}^d_{\frac{1}{2} R^{d-1}} \setminus \mathcal{B}^d_{\frac{1}{2}(1-\eta) R^{d-1}}\Big) \\ \to \frac{2 v_{d-1}}{v_d} \int_{(1-\eta)/2}^{1/2} (1 - x^2)^{(d-1)/2}\, dx. \end{aligned} \tag{8.11}$$

Hence we conclude that there is a constant $c(\eta) > 0$, independent of ε, such that

$$\liminf_{R \to \infty} \int_0^\infty \widetilde{N}(\tau)\, \big(-dA_R(\tau)\big) \leq (1 + \varepsilon - c(\eta)) C'. \tag{8.12}$$

Letting now $\varepsilon \to 0$ we run into a contradiction against (8.9). This concludes the proof of the lemma. $\square$

9. Proof of Theorem 1

Combining Lemma 7 and Lemma 11 we can now complete the proof of Theorem 1. First let $\mathfrak{U}$, $\mathcal{D}$ be as in Lemma 7. Then by Lemma 7 applied to $\mathrm{S}_1^{d-1} \setminus \mathfrak{U}$,

$$\liminf_{T \to \infty} \frac{\#(\widehat{\mathcal{P}} \cap \mathcal{B}_T^d \setminus T\mathcal{D})}{T^d} \geq \kappa_{\mathcal{P}} C_{\mathcal{P}} \big(v_d - \operatorname{vol}(\mathcal{D})\big). \tag{9.1}$$

Combining this with Lemma 11 we get

$$\limsup_{T\to\infty} \frac{\#(\widehat{\mathcal{P}}\cap T\mathcal{D})}{T^d} = \limsup_{T\to\infty}\left(\frac{\#(\widehat{\mathcal{P}}\cap \mathcal{B}_T^d)}{T^d} - \frac{\#(\widehat{\mathcal{P}}\cap \mathcal{B}_T^d\setminus T\mathcal{D})}{T^d}\right) \leq \kappa_{\mathcal{P}} C_{\mathcal{P}}\,\mathrm{vol}(\mathcal{D}). \tag{9.2}$$

Combining this with Lemma 7 (applied to $\mathfrak{U}$ itself) we conclude

$$\lim_{T\to\infty} \frac{\#(\widehat{\mathcal{P}}\cap T\mathcal{D})}{T^d} = \kappa_{\mathcal{P}} C_{\mathcal{P}}\,\mathrm{vol}(\mathcal{D}). \tag{9.3}$$

By a scaling and subtraction argument it follows that (9.3) is true more generally for any $\mathcal{D}\in\mathcal{F}$, where $\mathcal{F}$ is the family of sets of the form $\mathcal{D}=\{\boldsymbol{v}\in\mathbb{R}^d\,:\,r_1\leq\|\boldsymbol{v}\|<r_2,\ \boldsymbol{v}\in\|\boldsymbol{v}\|\mathfrak{U}\}$, for any $0\leq r_1<r_2$ and any $\mathfrak{U}\subset \mathrm{S}_1^{d-1}$ with $\omega(\partial\mathfrak{U})=0$.

Now let $\mathcal{D}$ be an arbitrary subset of $\mathbb{R}^d$ with boundary of measure zero. Note that the validity of (9.3) does not change if we replace $\mathcal{D}$ by $\mathcal{D}\cup\{\boldsymbol{0}\}$ or by $\mathcal{D}\setminus\{\boldsymbol{0}\}$. The proof of Theorem 6 is now completed by approximating $\mathcal{D}\cup\{\boldsymbol{0}\}$ from above and $\mathcal{D}\setminus\{\boldsymbol{0}\}$ from below by finite unions of sets in $\mathcal{F}$.

10. Proof of Theorem 2

Recall that (1.7) was proved in [6, Thm. A.1] and we have proved (1.9) and (1.10) in Section 5. Also the continuity of $E(r,\sigma,\mathcal{P})$ and $E(r,\sigma,\widehat{\mathcal{P}})$ with respect to σ is immediate from (3.5), (3.6) combined with Theorem 6. Hence it remains to prove (1.8).

Thus let λ be a Borel probability measure on S_1^{d-1} which is absolutely continuous with respect to ω, and let $\sigma>0$ and $r\in\mathbb{Z}_{\geq 0}$. Let us fix, once and for all, a map $K:\mathrm{S}_1^{d-1}\to\mathrm{SO}(d)$ such that $\boldsymbol{v}K(\boldsymbol{v})=\boldsymbol{e}_1=(1,0,\ldots,0)$ for all $\boldsymbol{v}\in\mathrm{S}_1^{d-1}$; we assume that K is smooth when restricted to S_1^{d-1} minus one point, cf. [5, Footnote 3, p. 1968]. Recall the definitions of $\mathfrak{C}(\sigma)$ and Φ^t in (3.7) and (8.3).

On verifies that if σ',σ'',α are any fixed numbers satisfying $0<\sigma'<\sigma<\sigma''$ and $\sigma'/\sigma<\alpha<1$, then for any $\boldsymbol{v}\in\mathrm{S}_1^{d-1}$ and all sufficiently large T, the set of $\boldsymbol{y}\in\mathcal{B}_T^d\setminus\{\boldsymbol{0}\}$ satisfying $\|\boldsymbol{y}\|^{-1}\boldsymbol{y}\in\mathfrak{D}_T(\kappa_{\mathcal{P}}^{-1}\sigma,\boldsymbol{v})$ is *contained* in $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'')\Phi^{-(\log T)/(d-1)}K(\boldsymbol{v})^{-1}$, and *contains* $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma')\Phi^{-(\log(\alpha T))/(d-1)}K(\boldsymbol{v})^{-1}$. It follows that

$$\begin{aligned}
&\lambda\big(\big\{\boldsymbol{v}\in\mathrm{S}_1^{d-1}\,:\,\#\big(\widehat{\mathcal{P}}\cap\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'')\Phi^{-(\log T)/(d-1)}K(\boldsymbol{v})^{-1}\big)\leq r\big\}\big)\\
&\quad\leq\lambda\big(\big\{\boldsymbol{v}\in\mathrm{S}_1^{d-1}\,:\,\mathcal{N}_T(\sigma,\boldsymbol{v},\widehat{\mathcal{P}})\leq r\big\}\big)\\
&\qquad\leq\lambda\big(\big\{\boldsymbol{v}\in\mathrm{S}_1^{d-1}\,:\,\#\big(\widehat{\mathcal{P}}\cap\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma')\Phi^{-(\log(\alpha T))/(d-1)}K(\boldsymbol{v})^{-1}\big)\leq r\big\}\big).
\end{aligned}\tag{10.1}$$

Recalling the definition of $\mathcal{P}=\mathcal{P}(\mathcal{W},\mathbb{Z}^n g)$ we see that $\widehat{\mathcal{P}}A=\widehat{\mathcal{P}}(\mathcal{W},\mathbb{Z}^n\varphi_g(A)g)$ for any $A\in\mathrm{SL}(d,\mathbb{R})$. Hence if we define

$$\mathcal{E}(\sigma,r)=\big\{x\in X\,:\,\#\big(\widehat{\mathcal{P}}^x\cap\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)\big)\leq r\big\}, \tag{10.2}$$

then the left hand side in (10.1) equals

$$\lambda\big(\big\{\boldsymbol{v}\in\mathrm{S}_1^{d-1}\,:\,\Gamma\varphi_g\big(K(\boldsymbol{v})\Phi^{(\log T)/(d-1)}\big)\in\mathcal{E}(\sigma'',r)\big\}\big) \tag{10.3}$$

Hence by [6, Thm. 4.1] and the Portmanteau theorem:

$$\liminf_{T\to\infty}\lambda\big(\big\{\boldsymbol{v}\in\mathrm{S}_1^{d-1}\,:\,\mathcal{N}_T(\sigma,\boldsymbol{v},\widehat{\mathcal{P}})\leq r\big\}\big)\geq\mu\big(\mathcal{E}(\sigma'',r)^\circ\big)=\mu\big(\mathcal{E}(\sigma'',r)\big). \tag{10.4}$$

Here the last equality is proved by using Lemma 10 with $U=\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'')$, and noticing that Theorem 6 implies that $\widehat{\mathcal{P}}^x\cap\partial U=\emptyset$ for μ-almost all $x\in X$. Similarly, using the right relation in (10.1), we obtain

$$\limsup_{T\to\infty}\lambda\big(\big\{\boldsymbol{v}\in\mathrm{S}_1^{d-1}\,:\,\mathcal{N}_T(\sigma,\boldsymbol{v},\widehat{\mathcal{P}})\leq r\big\}\big)\leq\mu\big(\overline{\mathcal{E}(\sigma',r)}\big)=\mu\big(\mathcal{E}(\sigma',r)\big). \tag{10.5}$$

Note that $\mathcal{E}(\sigma'',r)\subset\mathcal{E}(\sigma,r)\subset\mathcal{E}(\sigma',r)$, since $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'')\supset\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)\supset\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma')$. Also, if x lies in $\mathcal{E}(\sigma,r)$ but not in $\mathcal{E}(\sigma'',r)$, then $\widehat{\mathcal{P}}^x$ must have some point in $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'')\setminus\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)$, and so by Theorem 6,

$$\mu\big(\mathcal{E}(\sigma,r)\big)-\mu\big(\mathcal{E}(\sigma'',r)\big)\leq\kappa_{\mathcal{P}}C_{\mathcal{P}}\operatorname{vol}\big(\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma'')\setminus\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)\big).\tag{10.6}$$

Similarly

$$\mu\big(\mathcal{E}(\sigma',r)\big)-\mu\big(\mathcal{E}(\sigma,r)\big)\leq\kappa_{\mathcal{P}}C_{\mathcal{P}}\operatorname{vol}\big(\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)\setminus\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma')\big).\tag{10.7}$$

Now by taking σ',σ'' sufficiently near σ, the right hand sides of (10.6) and (10.7) can be made as small as we like. Hence from (10.4) and (10.5) we obtain in fact

$$\lim_{T\to\infty}\lambda\big(\{\boldsymbol{v}\in\mathrm{S}_1^{d-1}\;:\;\mathcal{N}_T(\sigma,\boldsymbol{v},\widehat{\mathcal{P}})\leq r\}\big)=\mu\big(\mathcal{E}(\sigma,r)\big)=\mu\big(\{x\in X\;:\;\#\big(\widehat{\mathcal{P}}^x\cap\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)\big)\leq r\}\big).\tag{10.8}$$

Note here that the right hand side is the same as $\sum_{r'=0}^{r}E(r,\sigma,\widehat{\mathcal{P}})$; cf. (3.6). Hence since (10.8) has been proved for arbitrary $r\geq 0$, also (1.8) holds for arbitrary $r\geq0$, and we are done.

11. Proof of Corollary 3

It follows from Theorem 2 and a general statistical argument (cf. e.g. [4]) that if we define $F(0)=0$ and

$$F(s)=-\frac{d}{ds}E(0,s,\widehat{\mathcal{P}}),\tag{11.1}$$

then the limit relation (1.14) holds at each point $s\geq0$ where $F(s)$ is defined. In fact the function $s\mapsto E(0,s,\widehat{\mathcal{P}})$ is convex; hence $F(s)$ exists for all $s>0$ except at most a countable number of points, and is continuous at each point where it exists. Also $F(s)$ is decreasing, and satisfies $\lim_{s\to0^+}F(s)=1=F(0)$ (cf. (1.10)) and $\lim_{s\to\infty}F(s)=0$. Note also that (1.15) is an immediate consequence of (1.14), the definition of $\widehat{\xi}_{T,j}$ and the fact that $N(T)\sim\kappa_{\mathcal{P}}^{-1}\widehat{N}(T)$ as $T\to\infty$ (cf. Theorem 1 and (1.6)).

It now only remains to prove that $F(s)$ is continuous, or equivalently that the derivative in (11.1) exists for every $s>0$. Assume the contrary, and let $s_0>0$ be a point where the derivative does *not* exist. By convexity, both the left and right derivative exist at s_0; thus

$$-\lim_{s\to s_0^-}\frac{E(0,s_0,\widehat{\mathcal{P}})-E(0,s,\widehat{\mathcal{P}})}{s_0-s}>-\lim_{s\to s_0^+}\frac{E(0,s,\widehat{\mathcal{P}})-E(0,s_0,\widehat{\mathcal{P}})}{s-s_0}\geq0.\tag{11.2}$$

In dimension $d=2$, using the fact that the point process $x\mapsto\widehat{\mathcal{P}}^x$ is invariant under $\left(\begin{smallmatrix}1&r\\0&1\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{R})$ for any $r\in\mathbb{R}$, it follows that the formula (3.5) holds with $\mathfrak{C}(\sigma)$ replaced by $\mathfrak{C}(a,a+\sigma)$ for any $a\in\mathbb{R}$, where

$$\mathfrak{C}(a_1,a_2)=\Big\{\boldsymbol{y}=(y_1,y_2)\in\mathbb{R}^2\;:\;0<y_1<1,\;\frac{2}{\kappa_{\mathcal{P}}C_{\mathcal{P}}}a_1y_1<y_2<\frac{2}{\kappa_{\mathcal{P}}C_{\mathcal{P}}}a_2y_1\Big\}$$

In particular, for any $0<s<s'$ and $a\in\mathbb{R}$,

$$E(0,s,\widehat{\mathcal{P}})-E(0,s',\widehat{\mathcal{P}})=\mu\big(\{x\in X\;:\;\widehat{\mathcal{P}}^x\cap\mathfrak{C}(a,a+s)=\emptyset,\;\widehat{\mathcal{P}}^x\cap\mathfrak{C}(a,a+s')\neq\emptyset\}\big).\tag{11.3}$$

For given $x\in X$, we order the numbers

$$\frac{\kappa_{\mathcal{P}}C_{\mathcal{P}}}{2}\cdot\frac{y_2}{y_1}\qquad\text{for }\boldsymbol{y}=(y_1,y_2)\in\widehat{\mathcal{P}}^x\cap((0,1)\times\mathbb{R}_{>0})$$

as $0<\lambda_{x,1}<\lambda_{x,2}<\ldots$. We also set $\lambda_{x,0}=0$. Taking $s'=s_0>s$ in (11.3), integrating over $a\in(0,a_0)$ for some fixed $a_0>0$, and using Fubini's Theorem, we obtain

$$a_0\big(E(0,s,\widehat{\mathcal{P}})-E(0,s_0,\widehat{\mathcal{P}})\big)\leq\int_X(s_0-s)\#\big\{j\geq0\;:\;\lambda_{x,j+1}-\lambda_{x,j}>s,\;\lambda_{x,j+1}<a_0+s_0\big\}\,d\mu(x).$$

Hence

(11.4)

$$-a_0 \lim_{s\to s_0^-} \frac{E(0,s_0,\widehat{\mathcal{P}})-E(0,s,\widehat{\mathcal{P}})}{s_0-s} \leq \int_X \#\big\{j\geq 0 \,:\, \lambda_{x,j+1}-\lambda_{x,j}\geq s_0,\ \lambda_{x,j+1}<a_0+s_0\big\}\,d\mu(x).$$

Similarly, replacing s by s_0 and s' by s in (11.3), we obtain

(11.5)

$$-a_0 \lim_{s\to s_0^+} \frac{E(0,s,\widehat{\mathcal{P}})-E(0,s_0,\widehat{\mathcal{P}})}{s-s_0} \geq \int_X \#\big\{j\geq 0 \,:\, \lambda_{x,j+1}-\lambda_{x,j}> s_0,\ \lambda_{x,j+1}<a_0+s_0\big\}\,d\mu(x).$$

It follows from (11.2), (11.4) and (11.5) that there is a set $A\subset X$ with $\mu(A)>0$ such that for every $x\in A$, there is some $j\geq 0$ such that $\lambda_{x,j+1}-\lambda_{x,j}=s_0$ and $\lambda_{x,j}<a_0$. Note that $\lambda_{x,1}\neq s_0$ for μ-almost all $x\in X$, by Theorem 6 applied with f as the characteristic function of the line $y_2=s_0\frac{2}{\kappa_{\mathcal{P}}C_{\mathcal{P}}}y_1$ in $\mathbb{R}^2$. Hence after removing a null set from A, we have for each $x\in A$ that $\widehat{\mathcal{P}}^x$ contains a pair of points $\boldsymbol{y}=(y_1,y_2)$ and $\boldsymbol{y}'=(y_1',y_2')$ satisfying

$$0<y_1,y_1'<1, \qquad \frac{y_2'}{y_1'}-\frac{y_2}{y_1}=\frac{2}{\kappa_{\mathcal{P}}C_{\mathcal{P}}}s_0, \qquad 0<\frac{y_2}{y_1}<\frac{2}{\kappa_{\mathcal{P}}C_{\mathcal{P}}}a_0.$$

However this is easily seen to violate the $\mathrm{SL}(2,\mathbb{R})$-invariance of the point process $x\mapsto\widehat{\mathcal{P}}^x$. For example, for each $\frac12\leq\lambda\leq 1$, because of the invariance under $\left(\begin{smallmatrix}\sqrt{\lambda} & 0\\ 0 & 1/\sqrt{\lambda}\end{smallmatrix}\right)$, there is a subset $A_\lambda\subset X$ with $\mu(A_\lambda)=\mu(A)>0$ such that for each $x\in A_\lambda$, $\widehat{\mathcal{P}}^x$ contains a pair of points $\boldsymbol{y}=(y_1,y_2)$ and $\boldsymbol{y}'=(y_1',y_2')$ satisfying

$$0<y_1,y_1'<\sqrt{\lambda}, \qquad \frac{y_2'}{y_1'}-\frac{y_2}{y_1}=\frac{2}{\kappa_{\mathcal{P}}C_{\mathcal{P}}}\frac{s_0}{\lambda}, \qquad 0<\frac{y_2}{y_1}<\frac{2}{\kappa_{\mathcal{P}}C_{\mathcal{P}}}\frac{a_0}{\lambda}.$$

Let R be the rectangle $(0,1)\times(0,\frac{4}{\kappa_{\mathcal{P}}C_{\mathcal{P}}}(a_0+s_0))$ in $\mathbb{R}^2$. By taking N sufficiently large we can ensure that the set $X_{R,N}:=\{x\in X \,:\, \#(\widehat{\mathcal{P}}^x\cap R)\leq N\}$ has measure $\mu(X_{R,N})\geq 1-\frac12\mu(A)$. It follows that $\mu(A_\lambda\cap X_{R,N})\geq\frac12\mu(A)$ for each $\frac12\leq\lambda\leq 1$, and so if Λ is any infinite subset of $[\frac12,1]$ then the integral $\int_{X_{R,N}}\sum_{\lambda\in\Lambda}I(x\in A_\lambda)\,d\mu(x)$ is infinite. On the other hand the definition of $X_{R,N}$ implies that $\sum_{\lambda\in\Lambda}I(x\in A_\lambda)\leq\binom{N}{2}$ for each $x\in X_{R,N}$.

We have thus reached a contradiction, and we conclude that (11.2) cannot hold, i.e. $F(s)$ is continuous for all $s\geq 0$.

12. Vanishing near zero of the gap distribution

The gap distribution obtained in Corollary 3 may sometimes vanish near zero. This phenomenon was noted numerically in [1] in several examples. In the case when $\mathcal{P}$ is a *lattice*, this vanishing is well understood; cf. [2], [5].

Let $\mathcal{P}=\mathcal{P}(\mathcal{W},\mathcal{L})$ be a regular cut-and-project set. We define $m_{\widehat{\mathcal{P}}}\geq 0$ to be the supremum of all $\sigma\geq 0$ with the property that $\#(\widehat{\mathcal{P}}^x\cap\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma))\leq 1$ for (μ-)almost all $x\in X$. Then the computation in (5.3) (together with (5.2)) shows that

$$E(0,\sigma,\widehat{\mathcal{P}}) \quad \begin{cases} =1-\sigma & \text{when } 0\leq\sigma\leq m_{\widehat{\mathcal{P}}}\\ >1-\sigma & \text{when } \sigma>m_{\widehat{\mathcal{P}}}.\end{cases} \tag{12.1}$$

We note that if $d\geq 3$ then $m_{\widehat{\mathcal{P}}}=0$, because of the $\mathrm{SL}(d,\mathbb{R})$-invariance and the fact that $\mathrm{SL}(d,\mathbb{R})$ acts transitively on pairs of non-proportional vectors in $\mathbb{R}^d\setminus\{\mathbf{0}\}$ when $d\geq 3$.

Let us now assume $d=2$. Note that by (12.1) and the discussion at the beginning of Sec. 11, the function F in Corollary 3 satisfies

$$F(s) \quad \begin{cases} =1 & \text{if } 0\leq s\leq m_{\widehat{\mathcal{P}}}\\ <1 & \text{if } s>m_{\widehat{\mathcal{P}}}.\end{cases}$$

In other words, $m_{\widehat{\mathcal{P}}}$ is the largest number with the property that the limiting gap distribution obtained in Corollary 3 is supported on the interval $[m_{\widehat{\mathcal{P}}}, \infty)$. In particular, the support of the limiting gap distribution is separated from 0 if and only if $m_{\widehat{\mathcal{P}}} > 0$.

Let us also note that if $d = 2$, $m \geq 1$, and $\mathcal{L}$ is a "generic" lattice or affine lattice, so that either $H_g = \mathrm{SL}(n, \mathbb{R})$ or $H_g = G = \mathrm{ASL}(n, \mathbb{R})$, then we have $m_{\widehat{\mathcal{P}}} = 0$, again using the transitivity of the action of $\mathrm{SL}(n, \mathbb{R})$ on pairs of non-proportional vectors in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ for $n \geq 3$.

On the other hand, we will now recall (for general d) a standard construction of cut-and-project sets using algebraic number theory, which can be used to produce several of the most well-known quasicrystals (cf., e.g., [9]; see also [7, Ch. II, Prop. 6] and [8, Thm. 6]). We will see that in special cases with $d = 2$, this construction leads to quasicrystals for which $m_{\widehat{\mathcal{P}}} > 0$.

We follow [6, Sec. 2.2]. Let K be a totally real number field of degree $N \geq 2$ over $\mathbb{Q}$, let $\mathcal{O}_K$ be its subring of algebraic integers, and let $\pi_1, \ldots, \pi_N$ be the distinct embeddings of K into $\mathbb{R}$. We will always view K as a subset of $\mathbb{R}$ via π_1; in other words we agree that π_1 is the identity map. Fix $d \geq 1$ and set $n = dN$. By abuse of notation we write π_j also for the coordinate-wise embedding of K^d into $\mathbb{R}^d$, and for the entry-wise embedding of $M_d(K)$ (the algebra of $d \times d$ matrices with entries in K) into $M_d(\mathbb{R})$. Let $\mathcal{L}$ be the lattice in $\mathbb{R}^n = (\mathbb{R}^d)^N$ given by

$$\mathcal{L} = \mathcal{L}_K^d := \Big\{ (\boldsymbol{x}, \pi_2(\boldsymbol{x}), \ldots, \pi_N(\boldsymbol{x})) \ : \ \boldsymbol{x} \in \mathcal{O}_K^d \Big\}. \tag{12.2}$$

As usual we set $m = n - d = (N-1)d$, let π and π_{int} be the projections of $\mathbb{R}^n = (\mathbb{R}^d)^N = \mathbb{R}^d \times \mathbb{R}^m$ onto the first d and last m coordinates. It follows from [14, Cor. 2 in Ch. IV-2] that $\pi_{\mathrm{int}}(\mathcal{L})$ is dense in $\mathbb{R}^m$, i.e. we have $\mathcal{A} = \mathbb{R}^m$ and $\mathcal{V} = \mathbb{R}^n$ in the present situation. Hence the window $\mathcal{W}$ should be taken as a subset of $\mathbb{R}^m$, and we consider the cut-and-project set $\mathcal{P}(\mathcal{W}, \mathcal{L}) \subset \mathbb{R}^d$.

Choose $\delta > 0$ and $g \in \mathrm{SL}(n, \mathbb{R})$ such that

$$\mathcal{L} = \delta^{1/n} \mathbb{Z}^n g. \tag{12.3}$$

In fact

$$\delta = |D_K|^{d/2}, \tag{12.4}$$

where D_K is the discriminant of K; cf., e.g., [3, Ch. V.2, Lemma 2]. As proved in [6, Sec. 2.2.1], in this situation we have

$$H_g = g \, \mathrm{SL}(d, \mathbb{R})^N g^{-1}; \tag{12.5}$$

where $\mathrm{SL}(d, \mathbb{R})^N$ is embedded as a subgroup of $G = \mathrm{ASL}(n, \mathbb{R})$ through

$$(A_1, \ldots, A_N) \mapsto \Big(\mathrm{diag}[A_1, \ldots, A_N], \mathbf{0} \Big), \tag{12.6}$$

where $\mathrm{diag}[A_1, \ldots, A_N]$ is the block matrix whose diagonal blocks are $A_1, \ldots, A_N$ in this order, and all other blocks vanish.

Lemma 12. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ be a regular cut-and-project set with $\mathcal{L}$ as in* (12.2)*, and with $d = N = 2$ (thus K is a real quadratic number field). Let $\varepsilon > 1$ be the fundamental unit of $\mathcal{O}_K$, and set $R = \sup\{\|\boldsymbol{w}\| \ : \ \boldsymbol{w} \in \mathcal{W}\}$. Then*

$$m_{\widehat{\mathcal{P}}} \geq \frac{\kappa_{\mathcal{P}} C_{\mathcal{P}} \delta}{(\varepsilon^2 + \varepsilon^{-2})^2 R^2}. \tag{12.7}$$

Proof. Let $\sigma > 0$ and $x \in X$ be given and assume that $\#(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)) \geq 2$. It suffices to prove that we must then have $\sigma \geq \frac{\kappa_{\mathcal{P}} C_{\mathcal{P}} \delta}{(\varepsilon^2+\varepsilon^{-2})^2 R^2}$. The area of $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)$ equals r^2 where $r := \sqrt{\frac{\sigma}{\kappa_{\mathcal{P}} C_{\mathcal{P}}}}$; hence there is some $A \in \mathrm{SL}(2, \mathbb{R})$ which maps $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma)$ to the open triangle $\mathfrak{C}_r := \{\boldsymbol{x} \in \mathbb{R}^2 \ : \ 0 < x_1 < r, \ |x_2| < x_1\}$. Take $(A_1, A_2) \in \mathrm{SL}(2, \mathbb{R})^2$ (embedded in G as in (12.6)) so that $x = \Gamma g (A_1, A_2) g^{-1}$. Set $\widetilde{A} = (A_1 A, A_2)$; then $\mathcal{P}^x A = \mathcal{P}(\mathcal{W}, \mathcal{L}\widetilde{A})$. We set

$\gamma = \mathrm{diag}[\varepsilon^{-k}, \varepsilon^{-k}, \varepsilon^k, \varepsilon^k] \in \mathrm{SL}(4,\mathbb{R})$, where k is an integer which we will choose below. Then $\mathcal{L}\widetilde{A} = \mathcal{L}\gamma\widetilde{A} = \mathcal{L}\widetilde{A}\gamma$, by (12.2) and since $\widetilde{A}$ is block diagonal. Hence

$$\mathcal{P}^x A = \mathcal{P}(\mathcal{W}, \mathcal{L}\widetilde{A}) = \mathcal{P}(\mathcal{W}, \mathcal{L}\widetilde{A}\gamma) = \varepsilon^{-k}\mathcal{P}(\varepsilon^{-k}\mathcal{W}, \mathcal{L}\widetilde{A}).$$

Now $\#(\widehat{\mathcal{P}}^x A \cap \mathfrak{C}_r) \geq 2$ and thus $\mathcal{L}\widetilde{A}$ contains two points in $(\varepsilon^k\mathfrak{C}_r) \times (\varepsilon^{-k}\mathcal{W})$ which have non-proportional images under π (the projection onto the physical space $\mathbb{R}^2$). In other words, there exist $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{O}_K^2 \subset \mathbb{R}^2$ which are linearly independent over $\mathbb{R}$ (thus also over K) such that $\boldsymbol{b}_1 = (\boldsymbol{x}, \overline{\boldsymbol{x}})\widetilde{A}$ and $\boldsymbol{b}_2 = (\boldsymbol{x}', \overline{\boldsymbol{x}'})\widetilde{A}$ lie in $(\varepsilon^k\mathfrak{C}_r) \times (\varepsilon^{-k}\mathcal{W})$. Here we write $\boldsymbol{x} \mapsto \overline{\boldsymbol{x}}$ for the nontrivial automorphism of K. It follows that also $\boldsymbol{b}_3 = (\varepsilon\boldsymbol{x}, \overline{\varepsilon\boldsymbol{x}})\widetilde{A}$ and $\boldsymbol{b}_4 = (\varepsilon\boldsymbol{x}', \overline{\varepsilon\boldsymbol{x}'})\widetilde{A}$ lie in $(\varepsilon^{k+1}\mathfrak{C}_r) \times (\varepsilon^{-k-1}\mathcal{W})$. However the four vectors $(\boldsymbol{x}, \overline{\boldsymbol{x}})$, $(\boldsymbol{x}', \overline{\boldsymbol{x}'})$, $(\varepsilon\boldsymbol{x}, \overline{\varepsilon\boldsymbol{x}})$, $(\varepsilon\boldsymbol{x}', \overline{\varepsilon\boldsymbol{x}'})$ lie in $\mathcal{L}$ and form a K-linear basis of K^4. Hence $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4$ lie in $\mathcal{L}\widetilde{A}$ and are linearly independent over $\mathbb{R}$. However $\|\boldsymbol{b}_j\| < r'$ for $j = 1, 2, 3, 4$, where

$$r' = \max\left(\sqrt{(\varepsilon^k r)^2 + (\varepsilon^{-k}R)^2}, \sqrt{(\varepsilon^{k+1}r)^2 + (\varepsilon^{-k-1}R)^2}\right),$$

and thus δ, the covolume of $\mathcal{L}\widetilde{A}$, must be less than r'^4. Now choose k so as to minimize r'. Then $r' \leq \sqrt{\varepsilon^2 + \varepsilon^{-2}}\sqrt{Rr}$, and combining this with $\delta < r'^4$ and $r = \sqrt{\frac{\sigma}{\kappa_{\mathcal{P}} C_{\mathcal{P}}}}$ we obtain $\sigma > \frac{\kappa_{\mathcal{P}} C_{\mathcal{P}} \delta}{(\varepsilon^2 + \varepsilon^{-2})^2 R^2}$, as desired. □

Let us make some further observations in this vein. First, note the general relation

$$\mathcal{P}(\mathcal{W}, q^{-1}\mathcal{L}) = q^{-1}\mathcal{P}(q\mathcal{W}, \mathcal{L}), \qquad \forall\ q > 0 \text{ (real)}.$$

Using this relation with q an appropriate positive integer, it is clear that if $\mathcal{L}$ is any lattice in $\mathbb{R}^n$ such that the cut-and-project set $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ satisfies $m_{\widehat{\mathcal{P}}} > 0$ for every admissible window set $\mathcal{W}$ (for example this holds when $\mathcal{L}$ is as in Lemma 12), then $m_{\widehat{\mathcal{P}}} > 0$ *also* holds for any cut-and-project set obtained from $\mathcal{P}(\mathcal{W}, \mathcal{L})$ by the "union of rational translates" construction in [6, Sec. 2.3.1]. Furthermore, the property of having $m_{\widehat{\mathcal{P}}} > 0$ is also, obviously, preserved under "passing to a sublattice" as in [6, Sec. 2.4]. In particular, by [6, Sec. 2.5], we have $m_{\widehat{\mathcal{P}}} > 0$ for any $\mathcal{P}$ associated with a Penrose tiling.

Remark 12.1. We do not expect the lower bound in Lemma 12 to be sharp, and the argument which we gave regarding the construction in [6, Sec. 2.3.1] certainly does not lead to a sharp bound. It would be interesting to try to determine the *exact* value of $m_{\widehat{\mathcal{P}}}$ for the Penrose tiling, and also for some of the cases discussed in [1].

It is interesting to note that for a large class of regular cut-and-project sets with $m_{\widehat{\mathcal{P}}} > 0$, a corresponding lower bound on the gap length is present in the set of directions (1.13) not only in the limit $T \to \infty$, but for *any fixed* T:

Lemma 13. *Let $\mathcal{P} = \mathcal{P}(\mathcal{W}, \mathcal{L})$ be a regular cut-and-project set in dimension $d = 2$ such that either $\mathbf{0} \notin \mathcal{P}$ or $\mathbf{0} \in \mathcal{P}^x$ for all $x \in X$, and furthermore $\pi_{\mathrm{int}}(\boldsymbol{y}) \notin \partial\mathcal{W}$ for all $\boldsymbol{y} \in \mathcal{L}$ (viz., there are no "singular vertices"; cf.* [1, p. 6]*). Then for any non-proportional vectors $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \widehat{\mathcal{P}}$, the triangle with vertices $\mathbf{0}, \boldsymbol{p}_1, \boldsymbol{p}_2$ has area $\geq (\kappa_{\mathcal{P}} C_{\mathcal{P}})^{-1} m_{\widehat{\mathcal{P}}}$. In particular, for any $T > 0$ and $1 \leq j \leq \widehat{N}(T)$ we have $\widehat{\xi}_{T,j} - \widehat{\xi}_{T,j-1} \geq \min(\frac{1}{2}, (\pi\kappa_{\mathcal{P}} C_{\mathcal{P}})^{-1} m_{\widehat{\mathcal{P}}} T^{-2})$.*

(Using the last bound of Lemma 13 together with $\widehat{N}(T) \sim \pi\kappa_{\mathcal{P}} C_{\mathcal{P}} T^2$ as $T \to \infty$ in the limit relation (1.14) in Corollary 3, we immediately recover the fact that $F(s) = 1$ for $0 \leq s \leq m_{\widehat{\mathcal{P}}}$. We also remark that the condition $\mathbf{0} \in \mathcal{P}^x$ for all $x \in X$ is fulfilled whenever $\mathbf{0} \in \mathcal{W}$ and $\mathcal{L}$ is a lattice, since then $H_g \subset \mathrm{SL}(n, \mathbb{R})$.)

Proof. Assume that $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \widehat{\mathcal{P}}$ are non-proportial vectors and that the triangle $\triangle(\mathbf{0}, \boldsymbol{p}_1, \boldsymbol{p}_2)$ has area less than $(\kappa_{\mathcal{P}} C_{\mathcal{P}})^{-1} m_{\widehat{\mathcal{P}}}$. Note that for any $\boldsymbol{p}_1', \boldsymbol{p}_2' \in \mathbb{R}^2$ such that $\triangle(\mathbf{0}, \boldsymbol{p}_1', \boldsymbol{p}_2')$ has the same area and orientation as $\triangle(\mathbf{0}, \boldsymbol{p}_1, \boldsymbol{p}_2)$, there exists $A \in \mathrm{SL}(2, \mathbb{R})$ with $\boldsymbol{p}_1' = \boldsymbol{p}_1 A$ and $\boldsymbol{p}_2' = \boldsymbol{p}_2 A$. In particular there are some $A \in \mathrm{SL}(2, \mathbb{R})$ and $\sigma_0 \in (0, m_{\widehat{\mathcal{P}}})$ such that $\boldsymbol{p}_1 A, \boldsymbol{p}_2 A \in$

$\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma_0)$. Now there are $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathcal{L}$ such that $\pi(\boldsymbol{y}_j) = \boldsymbol{p}_j$ and $\pi_{\text{int}}(\boldsymbol{y}_j) \in \mathcal{W}$ for $j = 1, 2$, and by assumption neither $\pi_{\text{int}}(\boldsymbol{y}_1)$ nor $\pi_{\text{int}}(\boldsymbol{y}_2)$ lie in $\partial\mathcal{W}$; hence $\boldsymbol{y}_j \left(\begin{smallmatrix} A & 0 \\ 0 & 1_m \end{smallmatrix}\right) \in \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma_0) \times \mathcal{W}^\circ$ for $j = 1, 2$. It follows that $\#(\widehat{\mathcal{P}}^x \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma_0)) \geq 2$ for $x = \Gamma\varphi_g(A) \in X$. In fact, using our assumptions on $\mathcal{P}$ and the fact that $\mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma_0) \times \mathcal{W}^\circ$ is open, we have $\#(\widehat{\mathcal{P}}^{x'} \cap \mathfrak{C}(\kappa_{\mathcal{P}}^{-1}\sigma_0)) \geq 2$ for all x' in some open neighbourhood of $x = \Gamma\varphi_g(A)$ (cf. the proof of Lemma 10). However this violates our definition of $m_{\widehat{\mathcal{P}}}$. We have thus proved the first part of the lemma.

To prove the second statement we merely have to note that $\widehat{\xi}_{T,j} - \widehat{\xi}_{T,j-1} = (2\pi)^{-1}\varphi(\boldsymbol{p}_1, \boldsymbol{p}_2)$ for some $\boldsymbol{p}_1 \neq \boldsymbol{p}_2 \in \widehat{\mathcal{P}}_T$. If $\boldsymbol{p}_1, \boldsymbol{p}_2$ are not proportional then since $\triangle(\boldsymbol{0}, \boldsymbol{p}_1, \boldsymbol{p}_2)$ has area $\frac{1}{2}\|\boldsymbol{p}_1\|\|\boldsymbol{p}_2\| \sin\varphi(\boldsymbol{p}_1, \boldsymbol{p}_2) < \frac{1}{2}T^2 \sin\varphi(\boldsymbol{p}_1, \boldsymbol{p}_2)$, the first part of the lemma implies $\varphi(\boldsymbol{p}_1, \boldsymbol{p}_2) > \sin\varphi(\boldsymbol{p}_1, \boldsymbol{p}_2) > 2(\kappa_{\mathcal{P}} C_{\mathcal{P}})^{-1} m_{\widehat{\mathcal{P}}} T^{-2}$; on the other hand if $\boldsymbol{p}_1, \boldsymbol{p}_2$ are proportional then necessarily $\varphi(\boldsymbol{p}_1, \boldsymbol{p}_2) = \pi$. $\square$

References

[1] M. Baake, F. Götze, C. Huck, and T. Jakobi, Radial spacing distributions from planar point sets, arXiv:1402.2818.

[2] F.P. Boca, C. Cobeli and A. Zaharescu, Distribution of lattice points visible from the origin. Comm. Math. Phys. **213** (2000), 433–470.

[3] S. Lang, *Algebraic Number Theory*, Springer-Verlag, New York, 1994.

[4] J. Marklof, Distribution modulo one and Ratner's theorem, *Equidistribution in Number Theory, An Introduction,* eds. A. Granville and Z. Rudnick, Springer 2007, pp. 217-244.

[5] J. Marklof and A. Strömbergsson, The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems, Annals of Math. **172** (2010), 1949–2033.

[6] J. Marklof and A. Strömbergsson, Free path lengths in quasicrystals, Comm. Math. Phys. DOI 10.1007/s00220-014-2011-3, arXiv:1304.2044.

[7] Y. Meyer, *Algebraic numbers and harmonic analysis*, North-Holland Publishing Co., Amsterdam, 1972.

[8] Y. Meyer, Quasicrystals, Diophantine approximation and algebraic numbers, in *Beyond quasicrystals (Les Houches, 1994)*, pp. 3–16, Springer, Berlin, 1995.

[9] P.A.B. Pleasants, Lines and planes in 2- and 3-dimensional quasicrystals, in *Coverings of discrete quasiperiodic sets*, Springer Tracts Modern Phys., **180**, pp. 185–225, Springer, Berlin, 2003.

[10] M. Ratner, On Raghunathan's measure conjecture, Ann. of Math. **134** (1991) 545–607.

[11] M. Ratner, Raghunathan's topological conjecture and distributions of unipotent flows, Duke Math. J. **63** (1991), 235–280.

[12] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972.

[13] W. A. Veech, Siegel measures, Ann. of Math. **148** (1998), 895–944.

[14] A. Weil, *Basic Number Theory,* 3rd ed., Springer-Verlag, New York, 1974.

[15] Aad W. van der Vaart and Jon A. Wellner, *Weak convergence and empirical processes*, Springer-Verlag, New York, 1996.

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