LECTURE NOTES: RIEMANNIAN GEOMETRY

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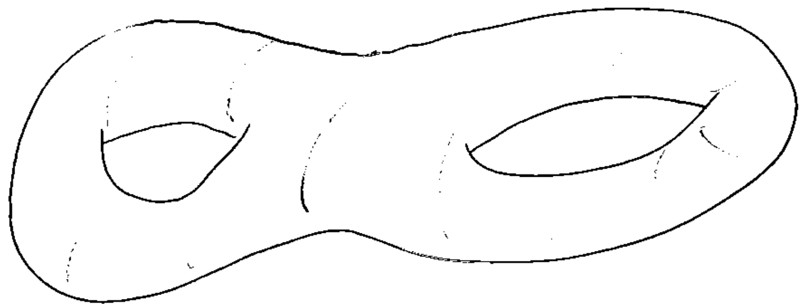
1. Manifolds

#1. Manifolds (topological and C^∞)



Def 1: A <u>(topological) manifold of dimension d</u> is a <u>paracompact</u>, <u>connected</u> <u>Hausdorff space</u> M such that every point $p \in M$ has an open neighborhood, U, which is homeomorphic to an open subset Ω of $\mathbb{R}^d$. {viz., M is <u>locally Euclidean</u>}

Such a homeomorphism $x : U \to \Omega$ is called a <u>(coordinate) chart</u>. An <u>atlas</u> (on M) is a family $\{(U_\alpha, x_\alpha)\}$ of charts such that $M = \bigcup_\alpha U_\alpha$.

Recall: A topological space M is <u>paracompact</u> $\overset{\text{def}}{\Longleftrightarrow}$ "any open cover has a locally finite refinement". However in the above setting, i.e. with M connected, Hausdorff and locally Euclidean, M is paracompact iff M has a countable atlas – see <u>Problem 2</u>.

<u>Remark</u>: The exact def. of "topological manifold" <u>varies</u> in the literature. Often one does <u>not</u> require M connected. Also one sometimes requires M <u>second countable</u>,

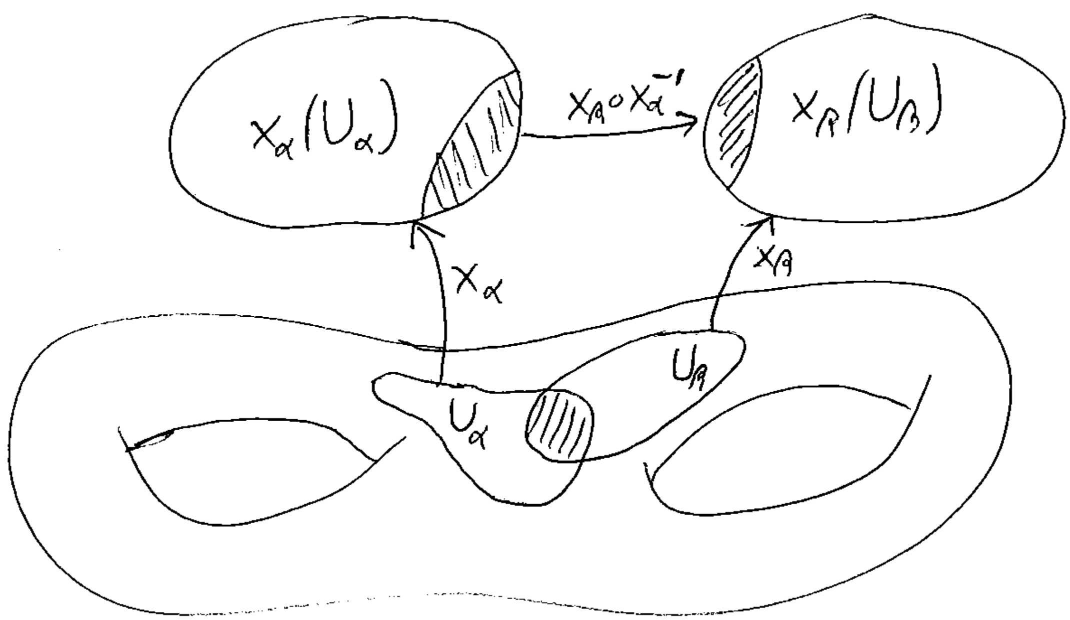
Remark: Does one get more general objects by allowing "varying d" in Def. 1? Answer NO, by Brouwer's Theorem on Invariance of Dimension. — See Problem 3.

Def 2: A C^∞ atlas on a (topological) manifold M is an atlas $\{(U_\alpha, x_\alpha)\}$ such that for any two α, β with $U_\alpha \cap U_\beta \neq \emptyset$, the map

chart transition map

$$x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$$

is C^∞. viz., (U_α, x_α) and (U_β, x_β) are compatible



A C^∞ structure on M is a C^∞ atlas which is maximal (w.r.t. set inclusion).

A C^∞ manifold is a (topological) manifold with a C^∞ structure.

Facts:

In a C^∞ atlas, ~~the~~ any chart transition map $x_\beta \circ x_\alpha^{-1}$ as above must be a diffeomorphism, since $(x_\beta \circ x_\alpha^{-1})^{-1} = x_\alpha \circ x_\beta^{-1}$.

Any C^∞ atlas on M determines a unique C^∞ structure, namely the family of all charts which are compatible with the given atlas. — Problem 4

Two C^∞ atlases determine the same C^∞ structure iff they are compatible (i.e. their union is a C^∞ atlas).

Examples of C^∞ manifolds

- $\mathbb{R}^d$
- S^d {the unit sphere in $\mathbb{R}^{d+1}$; see Jost p. 3.}
- Any connected open subset of a C^∞ manifold M is itself a C^∞ manifold – it is called an <u>open submanifold of M</u>. – <u>Problem 6</u>

  {Hence in particular any connected open subset of $\mathbb{R}^d$ is a C^∞ manifold.}

- T^d – a d-dimensional <u>torus</u>.

  See Jost p. 3 for a detailed construction. See <u>Problem 9</u> regarding more general <u>quotient manifolds</u>.

- If M, N are C^∞ manifolds, then so is <u>$M \times N$</u>. – <u>Problem 8</u>

Def 3: Let M, M' be C^∞ manifolds with C^∞ atlases $\mathcal{A}$ and $\mathcal{A}'$, respectively.
A continuous map $f : M \to M'$ is said to be **C^∞** (or **differentiable**) if all maps

$$\underline{\underline{y \circ f \circ x^{-1}}} \quad (\text{for } (U,x) \in \mathcal{A},\ (V,y) \in \mathcal{A}')$$

are C^∞.

this is a map from $x(U \cap f^{-1}(V))$ to $y(V)$.

If furthermore f is a bijection and f^{-1} is C^∞ then f is called a **diffeomorphism**.

Often later, when we have made a choice of a chart (U,x) on M, we identify any point $p \in M$ with its coordinate $x(p) \in \mathbb{R}^d$. In line with this, in the situation in Def. 3, we may sometimes write just "f" to denote $y \circ f \circ x^{-1}$. For example, if f is a C^∞ map $f : M \to \mathbb{R}$ and (U,x) is a chart on M, it is common to write

$$\frac{\partial f}{\partial x^j} \quad \text{in place of} \quad \frac{\partial (f \circ x^{-1})}{\partial x^j} \quad (\text{a function } x(U) \to \mathbb{R}).$$

Lemma 1 (partition of unity):

Let M be a C^∞ manifold and $(U_\alpha)_{\alpha \in A}$ an open covering of M. Then there exists a partition of unity subordinate to (U_α), i.e. a locally finite refinement $(V_\beta)_{\beta \in B}$ of (U_α) and a family $(\varphi_\beta)_{\beta \in B}$ of C_0^∞-functions $\varphi_\beta : M \to \mathbb{R}$ with:

i.e. C^∞-functions with compact support

(i) $\operatorname{supp} \varphi_\beta \subset V_\beta \quad \forall \beta \in B$

(ii) $0 \leq \varphi_\beta(x) \leq 1, \quad \forall \beta \in B,\ x \in M$

(iii) $\underbrace{\sum_{\beta \in B}}_{\text{essentially finite sum}} \varphi_\beta(x) = 1, \quad \forall x \in M$

"Smooth types" (brief survey)

Let M be a topological manifold. If $\dim M \leq 3$ then M has exactly one C^∞ structure ~~up~~ to diffeomorphism (cf. Problem 5)

However for $\dim M = 4$ it may be (even with M compact & simply connected) that M has no C^∞ structure (Donaldson). On the other hand $M = \mathbb{R}^4$ possesses uncountably many non-diffeomorphic C^∞ structures! — "exotic $\mathbb{R}^4$"

Outstanding question: $\boxed{\exists \text{ exotic } S^4 \text{ ??}}$

For $\dim M \geq 5$: M has finitely many C^∞ structures up to diffeomorphism. E.g. S^7 has 28 diffeomorphism classes of C^∞ structures (Milnor, 50's, 60's).

S^8 has 2,
S^9 has 8,
S^{10} has 6.
S^d for $d \leq 6$, $d \neq 4$ has 1.

For more info & refs, see Wikipedia, e.g.
"Differential structure",
"Exotic spheres".

1.1. **Notes.** .

In this lecture we follow Jost, [5, Sec. 1.1].

p. 1: We assume in our lecture that the reader is familiar with basic concepts of point set topology; note that Jost gives a quick summary of most of the pertinent definitions on his [5, p. 1]. One basic concept which Jost does not define, but which appears in his Def. 1.1.1 (= our Def. 1 on p. 1) is the following: A topological space M is said to be *connected* if M cannot be written as a union of two disjoint nonempty open subsets. One can prove that a topological manifold is in fact *path-connected*, meaning that any two points can be joined by a curve;[1] indeed see Problem 1. We will use this fact a lot! (Path-connectedness is apriori a stronger property than connectedness, i.e. for an arbitrary topological space, path-connectedness implies connectedness.)

p. 1: Note that the requirement that a (connected) topological manifold M should be *paracompact* is equivalent to M being *second countable*[2]! Slightly more generally: If M is *any* topological space which is Hausdorff and locally Euclidean[3] then M is second countable iff [M is paracompact and has countably many connected components].
Indeed, cf. math.stackexchange.com/questions/527642.

An example of topological space which is connected, Hausdorff, and locally Euclidean but not paracompact is the so called *"Long Line"*; see wikipedia!

p. 1: In Definition 1, the assumption that M should be *Hausdorff* is certainly not redundant. For a simple example showing this, see the solution to Problem 10.

p. 2: A basic notion appearing here is that of a map in several real variables being C^∞ (=*smooth*). We recall the definition here: If V is an open subset of $\mathbb{R}^d$ then a map $f : V \to \mathbb{R}^n$ is said to be C^m ($m \geq 0$) if, writing $f(x) = (f^1(x), \ldots, f^n(x))$ and $x = (x^1, \ldots, x^d)$, for every $j \in \{1, \ldots, n\}$, $k \in \{0, \ldots, m\}$ and $(\ell_1, \ldots, \ell_k) \in \{1, \ldots, d\}^k$, the partial derivative

$$\frac{\partial^k f^j(x^1, \ldots, x^d)}{\partial x^{\ell_1} \cdots \partial x^{\ell_k}}$$

[1]A "curve" is by definition a continuous function from an interval $I \subset \mathbb{R}$ to a topological space.

[2]Recall that a topological space is said to be second countable if it has a countable *base,* i.e. a countable family $\mathcal{U}$ of open subsets of M such that every open subset of M is a union of some sets in $\mathcal{U}$.

[3]Sometimes one defines a *topological manifold* to be *such* a space, i.e. a topological space which is Hausdorff and locally Euclidean.

exists and is continuous for all $x = (x^1, \ldots, x^d) \in V$. (In particular $f : V \to \mathbb{R}^n$ is C^0 iff f is continuous.) Finally the map $f : V \to \mathbb{R}^n$ is said to be C^∞ if it is C^m for *every* $m \geq 0$.

In the situation above, $f : V \to \mathbb{R}^n$ is said to be a *diffeomorphism* (onto its image) if f is injective, the image $W := f(V)$ is an open subset of $\mathbb{R}^n$, and the inverse map $f^{-1} : f(V) \to V$ is *also* C^∞. In this situation we necessarily have $n = d$.

2. Tangent spaces and the tangent bundle

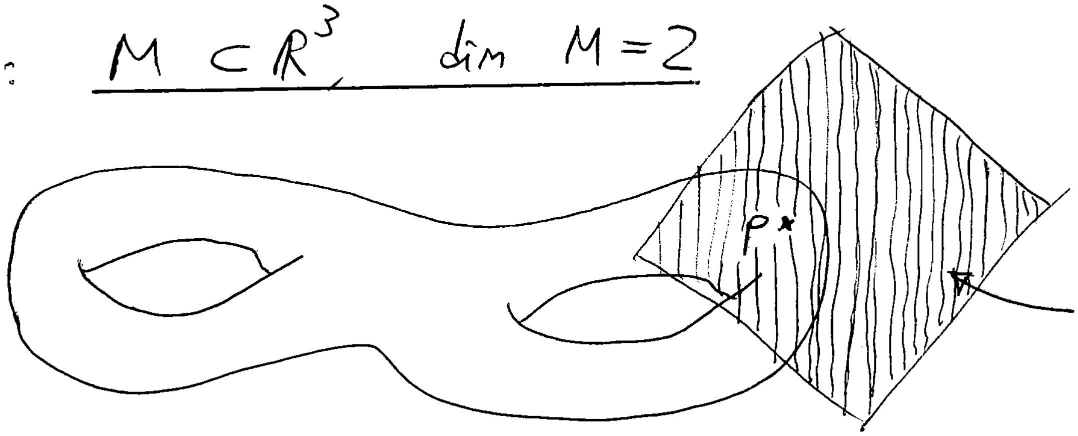
#2. Tangent spaces

Given a C^∞ manifold M and $p \in M$, we want to define the <u>tangent space T_pM</u>.

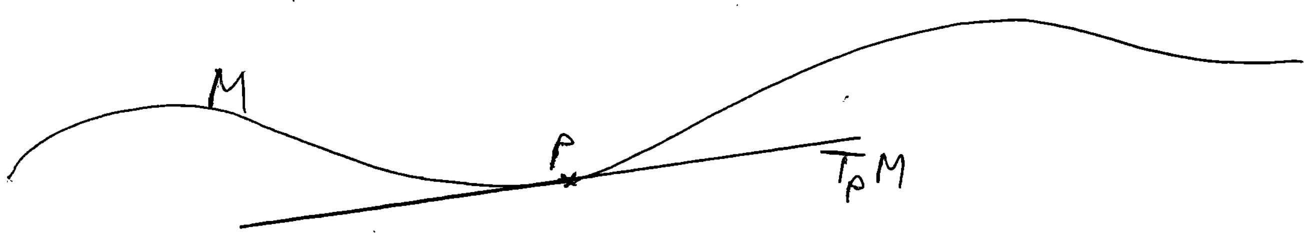
For M a submanifold of $\mathbb{R}^d$ we want T_pM to be the "usual thing"!

"<u>submanifold</u>" — understand intuitively now — we'll define later.

Ex: $M \subset \mathbb{R}^3$, $\dim M = 2$

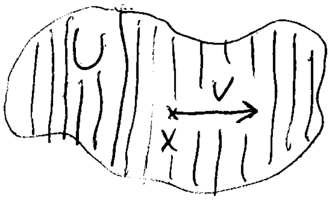


Ex: $M \subset \mathbb{R}^2$, $\dim M = 1$



Actually we will define T_pM as <u>a certain vector space of dimension $= \dim M$</u>. (One should think of $0 \in T_pM$ as lying "at p".) Whenever $M \underset{\text{submanifold}}{\subset} N$, there will exist a natural inclusion $T_pM \hookrightarrow T_pN$.

Def 1: For U open $\subset \mathbb{R}^d$ and $x \in U$, we define $\underline{T_x U := \mathbb{R}^d}$.



View $v \in T_x U = \mathbb{R}^d$ as a vector with "startpoint $= x$"!

Two key uses (meanings) of $v \in T_x U$

① **Tangent vector of a curve** $\quad c : (a, b) \to U$
(a C^∞ map).

For $t \in (a, b)$:

$$\frac{d}{dt} c(t) = \dot{c}(t) := \left(\frac{d}{dt} c^1(t), \dots, \frac{d}{dt} c^d(t)\right) \in T_{c(t)} U$$

Note $\quad \dot{c}(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$ $\quad$ compute in $\mathbb{R}^d$

② **Directional derivative of a function**

For $f : U \to \mathbb{R}$ a C^∞ map and $x \in U$, $v \in T_x U$,

$$\underline{v(f)} := \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h} = v^j \cdot \frac{\partial f}{\partial x^j}$$

Therefore we can write "$\frac{\partial}{\partial x^j}$" for the vector

$(0, \dots, 0, 1, 0, \dots, 0) \in T_x U$, and so $\quad$ for $v^1, \dots, v^d \in \mathbb{R}$

pos. j

"$v^j \cdot \frac{\partial}{\partial x^j}$" stands for $(v^1, \dots, v^d) \in T_x U$.

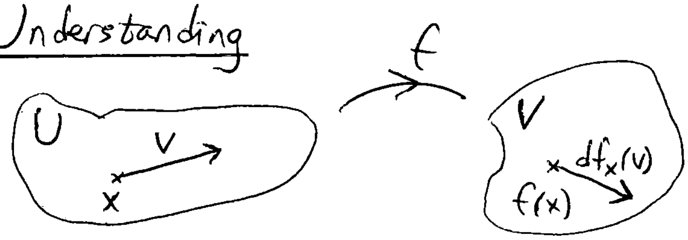
Einstein summation

Def 2: If $f : U \to V$ is a C^∞ map (with U open $\subset \mathbb{R}^d$ and V open $\subset \mathbb{R}^n$), and $x \in U$, we define df_x (or $df(x)$), the differential of f at x as the linear map $df_x : T_x U \to T_{f(x)} V$ with matrix

$$\left(\frac{\partial f^j}{\partial x^k}\right)_{j,k} = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^d} \\ \vdots & & \vdots \\ \frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^d} \end{pmatrix}$$

the Jacobi matrix of f

Understanding



If x is moved with velocity v (an infinitesimal time), this causes $f(x)$ to move with velocity $df_x(v)$

Note: We always represent vectors in $\mathbb{R}^d$ as column matrices; thus

$$\underbrace{df_x(v) = \left(\frac{\partial f^j}{\partial x^k}\right)\begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix}}_{\text{matrix multiplication}} = w \quad \text{with} \quad \underbrace{w^j = \frac{\partial f^j}{\partial x^k} v^k}_{\text{Einstein summation}},$$

$j = 1, \ldots, n.$

Fact: If $U \xrightarrow{f} V \xrightarrow{g} W$, $x \in U$,

then $d(g \circ f)_x = dg_{f(x)} \circ df_x : T_x U \longrightarrow T_{g(f(x))} W$

the "chain rule"

May write as "$d(g \circ f) = dg \circ df$" if the points are clear.

Def 3: For M a d-dimensional C^∞ manifold, and $p \in M$, the tangent space T_pM is defined as

$$\{(U, x, u) \ : \ (U,x) \text{ is a } C^\infty \text{ chart on } M \text{ with } p \in U \text{ and } u \in T_{x(p)}(x(U))\}$$

modulo the equivalence relation $\sim$ given by

$$(U,x,u) \sim (V,y,v) \overset{\text{def}}{\iff} u = d(x \circ y^{-1})_{y(p)}(v)$$

i.e. T_pM is the set of equivalence classes of $\sim$.

this makes sense since $(x \circ y^{-1})(y(p)) = x(p)$.

One should really write "$T^{\text{prel}}_{x(p)}(x(U))$" or similar above, and also in Def. 1; but at the end one can verify that Def 1 can be viewed as a special case of Def 3.

Verify: $\sim$ is indeed an equivalence relation.

Also for each fixed C^∞ chart (U,x) with $p \in U$, $(U,x,u) \sim (U,x,u')$ iff $u = u'$, and in fact we have a bijection

$$T_{x(p)}(x(U)) = \mathbb{R}^d \to T_pM$$
$$u \mapsto [(U,x,u)].$$

Problem 12

When the chart is understood, we'll write simply "u" to denote $[(U,x,u)] \in T_pM$. Finally the vector space operations in each $T_{x(p)}(x(U))$ induce well-defined — " — in T_pM; thus T_pM is a vector space over $\mathbb{R}$, $\dim = d$.

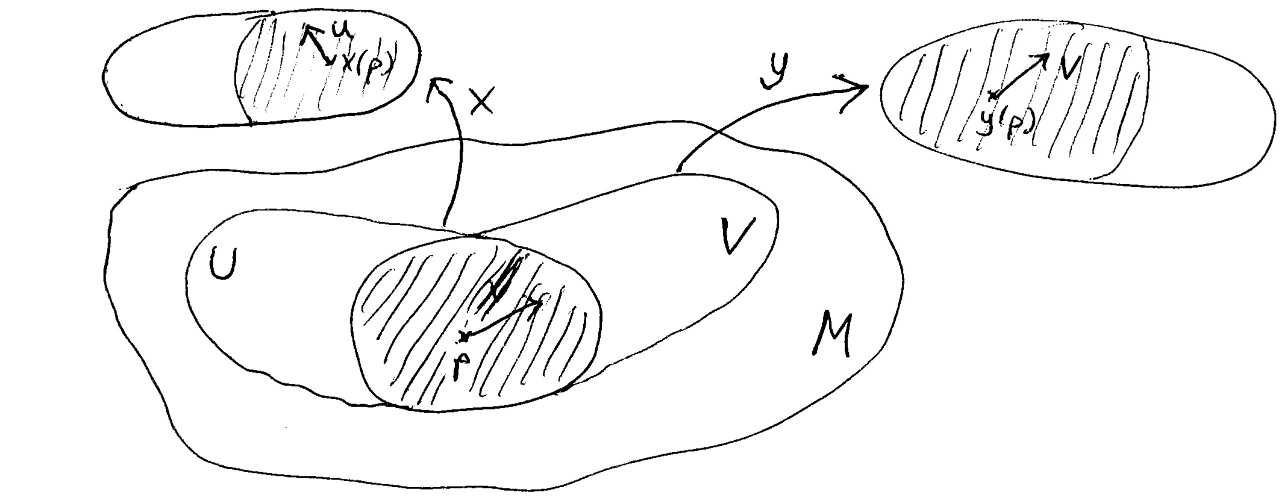
Example / fact: If M is a (finite dimensional) vector space over $\mathbb{R}$, or more generally if M is an open subset of a vector space V over $\mathbb{R}$, then we identify each T_pM with V!

See Problem 13(c) regarding the fact that this identification is possible.

A common special case:

For any C^∞ manifold M, $p \in M$, $v \in T_pM$, we identify $T_v(T_pM) = T_pM$!

Explicit transformation formula



Suppose a certain fixed vector in T_pM is represented by $u \in \mathbb{R}^d$ wrt (U, x), and represented by $v \in \mathbb{R}^d$ wrt (V, y).

Then $u = \underbrace{d(x \circ y^{-1})_{y(p)}}_{\left(\frac{\partial x^j}{\partial y^k}\right) \text{ at } y = y(p)} (v)$ ie $u^j = \frac{\partial x^j}{\partial y^k} v^k$.

Now "$v = u$" (shorthand for $[(U, x, u)] = [(V, y, v)]$)

$$\Rightarrow \quad \underbrace{v^j \frac{\partial}{\partial y^j} = u^j \frac{\partial}{\partial x^j}}$$

Notation from p. 2; we here skip the quotation marks; the two sides are obviously (in general) different vectors in $\mathbb{R}^d$, and the equality means this.

$$\Rightarrow \quad v^j \frac{\partial}{\partial y^j} = \left(v^k \frac{\partial x^j}{\partial y^k} \right) \frac{\partial}{\partial x^j}$$

and in particular, taking

$v = (0, \dots, 0, 1, 0, \dots, 0)$ gives

pos k

$$\boxed{\frac{\partial}{\partial y^k} = \frac{\partial x^j}{\partial y^k} \frac{\partial}{\partial x^j}}$$

$\in T_pM$ $\quad \in \mathbb{R}$ $\quad \in T_pM$

Def 4: Given a C^∞ map $f: M \to N$ (where M, N are C^∞ manifolds), and $p \in M$, the ***differential of f at p***, $\underline{df_p}$ (or $\underline{df(p)}$) is the linear map $df_p : T_pM \to T_{f(p)}N$ which w.r.t. any charts (U, x) on M with $p \in U$ and (V, y) on N with $f(p) \in V$, is given by $\underline{df_p = d(y \circ f \circ x^{-1})_{x(p)} : T_{x(p)}\big(x(U)\big) \to T_{y(f(p))}\big(y(V)\big)}$

This is well-defined; see Problem 13(d)!

Home assignment

Fact: If $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ (C^∞ maps between C^∞ manifolds M_1, M_2, M_3), then $\underline{d(g \circ f) = dg \circ df}$, i.e.

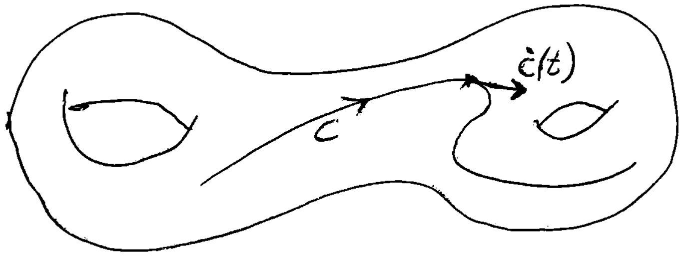
$$d(g \circ f)_p = dg_{f(p)} \circ df_p : T_pM_1 \to T_{g(f(p))}, \quad \forall p \in M_1.$$

the chain rule; it is proved using the chain rule for "$\mathbb{R}^d$-maps", see p. 2. See Problem 13(e).

Def 5: A C^∞ map $f: M \to N$ is called a(n) $\left\{\begin{matrix}\text{submersion}\\ \text{immersion}\end{matrix}\right\}$ if df_p is $\left\{\begin{matrix}\text{surjective}\\ \text{injective}\end{matrix}\right\}$ $\forall p \in M$.

The two key uses of $v \in T_pM$

① Tangent vector of a curve $\quad c : (a, b) \to M$ (a C^∞ map)



For $t \in (a,b)$, DEF: $\dot{c}(t) = \frac{d}{dt} c(t) := dc_t(1) \in T_{c(t)}M$

Explanation: $dc_t : T_t(a,b) = \mathbb{R} \to T_{c(t)}M$

See also Problem 14; in any chart, $\dot{c}(t)$ is computed "as one would expect"!

② Directional derivative of a function

For $f : M \to \mathbb{R}$ a C^∞ map, $p \in M$, $v \in T_pM$,

DEF: $v(f) := df_p(v) \in T_{f(p)}\mathbb{R} = \mathbb{R}$.

The following facts just say that "everything works as expected"...

FACT 1: For any chart (U, x) on M, if $p \in U$ and $v = v^j \frac{\partial}{\partial x^j} \in T_pM$ then $v(f) = \left(v^j \frac{\partial}{\partial x^j}\right) f = v^j \frac{\partial f}{\partial x^j}$

pedantically: $v^j \frac{\partial (f \circ x^{-1})}{\partial x^j} \Big|_{x = x(p)}$

Problem 13(f)

FACT: If $c:(a,b) \to M$ is a C^∞ curve and $f: M \to \mathbb{R}$ then $\dot{c}(t)(f) = \frac{d}{dt} f(c(t))$.

FACT: If $v \in T_pM$ and $f: M \to \mathbb{R}$ and $g: M \to \mathbb{R}$ are C^∞ maps then

$$\underline{v(fg) = f(p)\cdot v(g) + g(p)\cdot v(f)} \in \mathbb{R}$$

Leibniz' rule

Closely related fact:

$$\underbrace{d(fg)_p = g(p)\cdot df_p + f(p)\cdot dg_p}_{\text{equality between linear maps } T_pM \to \mathbb{R}.}$$

Def 6: TM, the tangent bundle of a C^∞ manifold M

As a set, $TM := \bigsqcup_{p \in M} T_p M$.

Define $\pi : TM \to M$; $\pi(w) = p$ for $w \in T_p M$.

Topology on TM; see (Problem 16).

For $U \subset M$, write $TU := \pi^{-1}(U) = \bigsqcup_{p \in U} T_p M$.

For (U, x) any C^∞ chart on M, define

$$\varphi_x : TU \to \mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$$

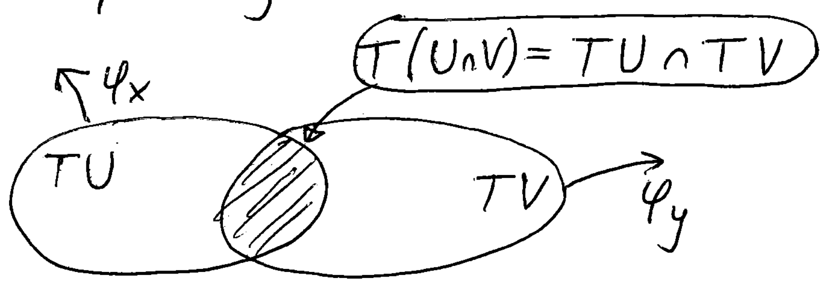
$$\varphi_x(w) = \left(x(\pi(w)),\ dx_{\pi(w)}(w) \right)$$

Then $\{ (TU, \varphi_x) \ : \ (U, x) \in [\text{a fixed } C^\infty \text{ atlas for } M] \}$

is a C^∞ atlas for TM, so that

TM is a $2d$-dimensional C^∞ manifold.

Check of C^∞ compatibility: Let (U,x), (V,y) be C^∞ charts on M.



Then $\varphi_y \circ \varphi_x^{-1} : \varphi_x(T(U \cap V)) \to \varphi_y(T(U \cap V))$

$$\varphi_y \circ \varphi_x^{-1}(z, v) = \left((y \circ x^{-1})(z),\ d(y \circ x^{-1})_z(v) \right) \in \mathbb{R}^d \times \mathbb{R}^d$$

(z, v in $\mathbb{R}^d$)

The last expression is clearly a C^∞ function of z, v. Done!

Later we'll see: TM is a vector bundle over M.

Fact: $\pi : TM \to M$ is a C^∞ map.

Def 7: If $f : M \to N$ is a C^∞ map, then df (the differential of f) is the map

$df : TM \to TN$, $\quad df(w) := df_{\pi(w)}(w) \in T_{f(\pi(w))} N$

Fact: $df : TM \to TN$ is a C^∞ map, and $\pi' \circ df = f \circ \pi$

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{f} & N \end{array}$$

See Problem 17

With the above notation, if (U, x) is a chart for M, $x : U \to x(U)$ is a C^∞ map and thus

$$dx : TU \to T(x(U)) = \coprod_{p \in x(U)} T_p(x(U)) = \underbrace{\coprod_{p \in x(U)} \mathbb{R}^d}$$

With this, "$\varphi_x = dx$" $\longrightarrow$ Naturally identified with $x(U) \times \mathbb{R}^d$

2.1. **Notes.** .

In this lecture we follow Jost, [5, Sec. 1.2].

p. 7, Def. 4: I most often prefer to use the notation "df_p", whereas Jost writes "$df(p)$". Example: Later we will work a lot with a certain map "$\exp_p$" which is a C^∞ map from an open subset $\mathcal{D}_p$ of T_pM to M. We will often consider the differential of this map at a point $v \in \mathcal{D}_p$. Note that this is a map $T_v(\mathcal{D}_p) \to T_{\exp_p(v)}(M)$, but as mentioned on p. 5 we identify $T_v(\mathcal{D}_p) = T_pM$. Thus I most often write

$$\text{"}(d\exp_p)_v : T_p(M) \to T_{\exp_p(v)}(M)\text{"}$$

for this map, whereas Jost writes

$$\text{"}(d\exp_p)(v) : T_p(M) \to T_{\exp_p(v)}(M)\text{"}.$$

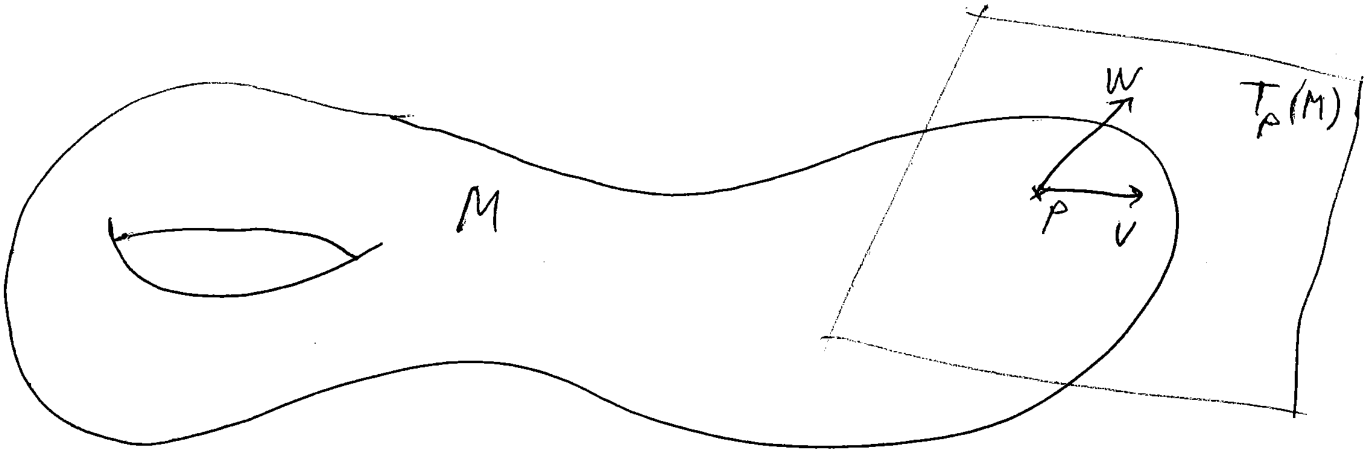
p. 7, Def. 5: Here we also wish to mention the concept of **submanifolds** (cf. [5, Sec. 1.3]). We will not have time in the course to develop the basic facts about submanifolds in any detail; however we state here the most important facts. Cf. also, e.g., [2, Ch. III.4-5].

Let N be a C^∞ manifold. There exist some different notions of "submanifolds" which are often considered. An *immersed submanifold* of N is defined to be a subset $M \subset N$ which is endowed with a structure of a C^∞ manifold such that the inclusion map $i : M \to N$ is a C^∞ immersion. Note that in general the topology of M does not agree with the relative topology of M as a subset of N! (Of course the topology of M must be at least as strong as the relative topology, since i is continuous.) *If* the topology of M agrees with the relative topology, then Jost calls M a *differentiable submanifold* of N (this is often also called an *embedded submanifold* or a *regular submanifold*; cf. wikipedia).

Let $n = \dim N$ and take $1 \leq m \leq n$. It turns out that an arbitrary *subset* $M \subset N$ has a structure as a differentiable submanifold of N of dimension m if and only if for every $p \in M$ there is a C^∞ chart (U, φ) of N such that $p \in U$, $\varphi(p) = 0$, $\varphi(U)$ is an open cube $(-\varepsilon, \varepsilon)^n$, and $\varphi(U \cap M) = (-\varepsilon, \varepsilon)^m \times \{0\}^{n-m}$; furthermore the C^∞ manifold structure of M is then uniquely determined; indeed a C^∞ atlas is made out of all charts of the form $(U \cap M, \mathrm{pr} \circ \varphi)$ with (U, φ) as above, where pr is the projection map $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$. (Cf., e.g., [2, Sec. III.5].)

Note that Jost's [5, Lemma 1.3.2] gives an often convenient way to prove that a subset of a manifold is a submanifold. We here repeat that result, with a somewhat more precise statement of the conclusion: *Let M and N be C^∞ manifolds, and assume $m = \dim M \geq \dim N = n$. Let $p \in N$, and let $f : M \to N$ be a C^∞ map such that df_x has rank n for all $x \in M$ with $f(x) = p$. Then each connected component of the subset $f^{-1}(p) \subset M$ is a closed differentiable submanifold of M of dimension $m - n$.*

#3. Riemannian manifolds; (geodesics)



$\langle v, w\rangle$

$\|v\| = \sqrt{\langle v, v\rangle}$

Def 1: A <u>Riemannian metric</u> on a C^∞ manifold M is given by a scalar product on each tangent space T_pM which depends smoothly on $p \in M$. {a bit vague...}

A <u>Riemannian manifold</u> is a C^∞ manifold equipped with a Riemannian metric.

<u>Understanding</u>: If (U, x) C^∞ chart on M then at each $p \in U$, the scalar product on T_pM is represented by a matrix $(g_{ij})_{i,j=1,\dots,d}$ (symmetric and positive definite), namely

$$\left\langle \xi^i \frac{\partial}{\partial x^i}, \eta^j \frac{\partial}{\partial x^j} \right\rangle = (\xi^1, \dots, \xi^d)\,(g_{ij}) \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^d \end{pmatrix} = \xi^i \eta^j g_{ij}$$

(g_{ij}) depends on p; thus $\underline{\underline{(g_{ij}(x))}}$, and the requirement "depends smoothly" means: ~~that~~ each $g_{ij}(x)$ is a C^∞-function on $x(U) \subset \mathbb{R}^d$.

If (V, y) is another C^∞ chart on M, on which the Riemannian metric is represented by (h_{ij}) then on $U \cap V$:

$$\xi^i \frac{\partial}{\partial x^i} = \left(\xi^i \frac{\partial y^k}{\partial x^i}\right) \frac{\partial}{\partial y^k}, \qquad \eta^j \frac{\partial}{\partial x^j} = \left(\eta^j \frac{\partial y^\ell}{\partial x^j}\right) \frac{\partial}{\partial y^\ell}$$

thus

$$\xi^i \frac{\partial y^k}{\partial x^i} \eta^j \frac{\partial y^\ell}{\partial x^j} h_{k\ell} = \xi^i \eta^j g_{ij}, \quad \forall (\xi^i), (\eta^j) \in \mathbb{R}^d.$$

$$\therefore \quad \boxed{g_{ij} = \frac{\partial y^k}{\partial x^i} \frac{\partial y^\ell}{\partial x^j} h_{k\ell}}$$

$\because$ "covariant tensor": indices at bottom...

Hence in Def 1 it suffices to check smoothness wrt any fixed C^∞-atlas.

Examples :

Kronecker symbol

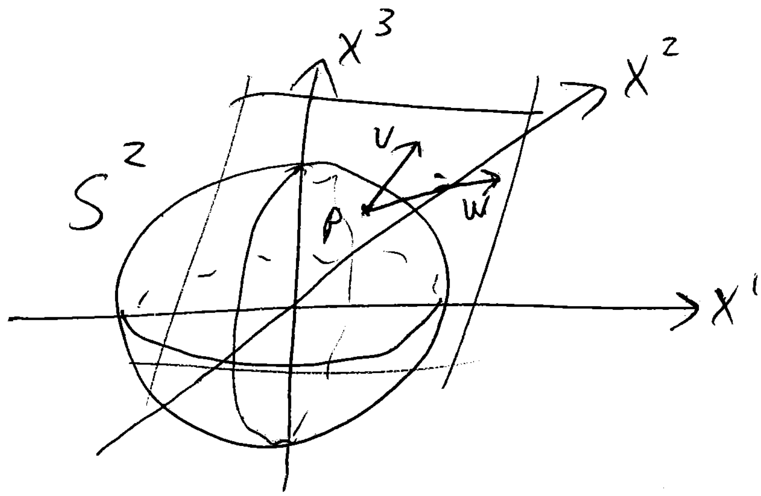
- Euclidean $\mathbb{R}^d$; here $(g_{ij}) = (\delta_{ij})$

{ Of course this is wrt the standard chart, $(\mathbb{R}^d, 1_{\mathbb{R}^d})$ }

- If $f: N \to M$ immersion and M Riemannian $\rightsquigarrow$ N Riemannian.

See Problem 18

- E.g. $S^{d-1} \hookrightarrow \mathbb{R}^d \Rightarrow S^{d-1}$ gets a standard Riemannian metric.



- $\underline{H^d}$ - Hyperbolic d-space.

See Problem 20.

- $\underline{\text{Any}}$ C^∞ manifold can (in $\underline{\text{many}}$ ways) be equipped with a Riemannian metric!

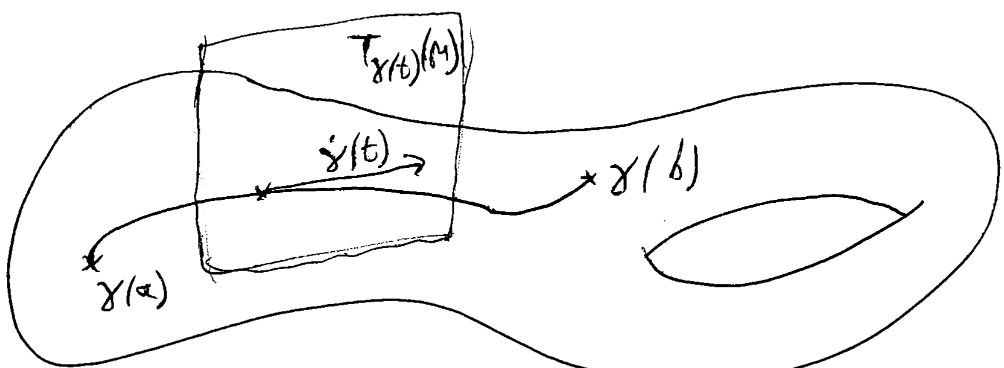
Jost Thm 1.4.1 (Nice (standard) application of $\underline{\text{partition}}$ of unity.

Def 2: For $\gamma:[a,b]\to M$ a C^∞ curve,

$$\underline{L(\gamma)} := \int_a^b \|\dot\gamma(t)\|\,dt \longleftarrow \text{the \underline{length} of } \gamma$$

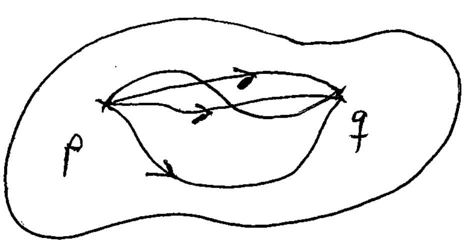
$$\underline{E(\gamma)} := \frac{1}{2}\int_a^b \|\dot\gamma(t)\|^2\,dt \longleftarrow \text{the \underline{energy} of } \gamma$$

– Also for <u>piecewise C^∞</u> ("pw C^∞") curve.



Def 3: For $p,q\in M$, the <u>distance between p and q</u> is

$$\underline{d(p,q)} := \inf\{L(\gamma) : \gamma:[a,b]\to M \text{ pw } C^\infty \text{ curve with } \gamma(a)=p,\ \gamma(b)=q\}$$



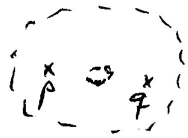
<u>Isometries</u> – say def!

<u>Ex</u>: On <u>$\mathbb{R}^d$</u>, <u>S^{d-1}</u> – nice exercise to do directly from the definition! See also Jost p. 26–27 for S^{d-1}, using symmetry.
Also <u>H^d</u> – Problem 20.

– Is the infimum in Def 3 always attained?

Answer: NO

Ex open subset of $\mathbb{R}^d$ with "holes"



Lemma 1: {= Jost Lemma 1.4.1}

d is a metric on M, i.e.

(1) $d(p,q) \geq 0$, $\forall p, q \in M$ and $d(p,q) > 0$ when $p \neq q$.

{This also includes: $d(p,q) < \infty$!}

(2) $d(p,q) = d(q,p)$

(3) $d(p,q) \leq d(p,r) + d(r,q)$.

proof: (2) & (3) obvious.

(1) – the issue is to show $d(p,q) > 0$ when $p \neq q$;
Jost gives a detailed proof using local coordinates.
But one must also prove $d(p,q) < \infty$! – (Problem 18)

Lemma 2: {= Jost Cor. 1.4.1}

The topology on M induced by the metric d
= the original topology.

{Again, Jost gives a detailed proof.}

Fix notation for open ball

In any metric space (X,d) we write,

for $p \in X$, $r>0$: $\underline{B_r(p) := \{q \in X : d(p,q) < r\}}$.

"The open ball of radius r, center p"

{ Jost uses "$B(p,\rho)$", "$d_p(0)$" (p. 24), "$D_\varepsilon(x)$" (p. 16)...

In particular,

For $p \in M$; $B_r(p) \subset M$

For $0 \in \mathbb{R}^d$; $B_r(0) = \{v \in \mathbb{R}^d : \|v\| < r\}$

Euclidean norm

For $0 \in T_pM$: $B_r(0) = \{v \in T_pM : \|v\| < r\}$

Given by the Riemannian metric

We now prove two basic facts about L and E. They basically show that "minimizing L and E are equivalent but L is independent of the parametrization of the curve".

Lemma 3: Jost Lemma 1.4.2

For each pw C^∞ curve $\gamma : [a,b] \to M$,

$$\underline{L(\gamma)^2 \leq 2(b-a)\,E(\gamma)}$$

with equality iff $\|\dot\gamma(t)\| = \text{const.}$

proof: By Cauchy–Schwarz,

$$L(\gamma) = \int_a^b \|\dot\gamma(t)\|\,dt \leq \sqrt{\int_a^b \|\dot\gamma(t)\|^2 dt}\ \sqrt{\int_a^b 1^2\,dt}$$

$$= \sqrt{b-a}\cdot\sqrt{2E(\gamma)}.$$ $\square$

Lemma 4: Jost Lemma 1.4.3

A "change of parameter"

For γ as above and $\psi : [\alpha,\beta] \to [a,b]$ pw C^∞ and strictly increasing or decreasing, $\underline{L(\gamma\circ\psi) = L(\gamma)}$.

proof: $$(\gamma\circ\psi)'(s) = \underbrace{\psi'(s)}_{\in\mathbb{R}}\cdot\underbrace{\gamma'(\psi(s))}_{\in T_{\gamma(\psi(s))}(M)}$$ chain rule

$$\therefore\ L(\gamma\circ\psi) = \int_\alpha^\beta \|(\gamma\circ\psi)'(s)\|\,ds = \int_\alpha^\beta |\psi'(s)|\cdot\|\gamma'(\psi(s))\|\,ds$$

$$\overset{t=\psi(s);\ dt=\psi'(s)ds}{=} \int_a^b \|\gamma'(t)\|\,dt = L(\gamma).$$ $\square$

Now write $\underline{\underline{x(t) := x(\gamma(t))}}$ as short-hand!

{For γ a curve & (U, x) a chart.}

Lemma 5: {Jost Lemma 1.4.4}

The Euler-Lagrange (E-L) equations for $E(\gamma)$

are: $\underline{\ddot{x}^i(t) + \Gamma^i_{jk}(x(t)) \cdot \dot{x}^j(t)\, \dot{x}^k(t) = 0}, \quad i = 1, \dots, d,$

where $\underline{\underline{\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(g_{jl,k} + g_{kl,j} - g_{jk,l} \right)}}$

{"$\underline{\text{Christoffel symbols}}$"}

with $(g^{ij}) = (g_{ij})^{-1}$ (thus $g^{ij} g_{jk} = \delta_{ik}$)

and $g_{jl,k} := \frac{\partial}{\partial x^k} g_{jl}$

$\underline{\underline{\text{Explanation}}}$: (1) We pretend $\underline{\gamma([a,b]) \subset U}$, i.e. the same local coordinates work on all γ!

{This assumption can be removed - see notes!}

(2) The E-L equations are

$\overset{(\text{def})}{\Longleftrightarrow}$ For any $\underline{\text{proper variation}}$ $\gamma(t,s)$ of $\gamma(t)$ (i.e. $\gamma : [a,b] \times (-\varepsilon, \varepsilon) \to M$, C^∞, and $\gamma(t, 0) \equiv \gamma(t)$, $\gamma(a, s) \equiv \gamma(a)$, $\gamma(b, s) \equiv \gamma(b)$)

We have $\frac{d}{ds} E(\gamma(\cdot, s))\big|_{s=0} = 0$.

Proper variation of γ:

Immediate consequence of Lemma 5:

If γ is a C^∞ curve and γ gives a "local minimum" of $E(\gamma)$ among curves with fixed endpoints

meaning: any curve $\tilde{\gamma}$ near γ and with same endpoints as γ, has $E(\tilde{\gamma}) \geqslant E(\gamma)$.

then γ is a geodesic (p. 10), i.e. satisfies the E-L equations.

Using also Lemma 3, 4 $\longrightarrow$ same conclusion if γ is a local minimum of $L(\gamma)$. and γ is parametrized by arc length

In particular, if γ is a C^∞ curve realizing the infimum defining $d(p,q)$, then γ is a geodesic!

Later we'll see: True also for γ pw C^∞

— see Lecture #5, Thm 2

(=Problem 24)

Def 4: A geodesic is a C^∞ curve $\gamma : [a,b] \to M$ which satisfies

$$\ddot{x}^j(t) + \Gamma^j_{ik}\, \dot{x}^i(t)\, \dot{x}^k(t) \equiv 0 \qquad (\forall j)$$

{for every chart (U,x) and all $t \in [a,b] \cap \gamma^{-1}(U)$}

{Equivalently: For some set of charts covering γ. Indeed the E-L equation "transforms correctly", as is clear from Lemma 5.}

Remark: If γ is a geodesic then $\|\dot{\gamma}(t)\| = \text{const.}$

{This can actually be deduced as a consequence of Lemma 5 using Lemma 4; however here we give a direct proof.}

proof:

$$\frac{d}{dt}\langle \dot{x}, \dot{x}\rangle = \frac{d}{dt}\left(g_{ij}(x(t))\, \dot{x}^i(t)\, \dot{x}^j(t)\right) =$$

$$= \underbrace{g_{ij}\, \ddot{x}^i(t)\, \dot{x}^j(t)}_{} + \underbrace{g_{ij} \cdot \dot{x}^i(t)\, \ddot{x}^j(t)}_{} + \left(\frac{\partial}{\partial x^k} g_{ij}\right) \dot{x}^k \dot{x}^i \dot{x}^j$$

same!

See proof of Lemma 5 below; $2 g_{ij}\, \ddot{x}^i = \ldots$

$$= -\left(g_{jk,l} + g_{jl,k} - g_{lk,i}\right) \dot{x}^k \dot{x}^l \dot{x}^j + g_{ij,k}\, \dot{x}^k \dot{x}^i \dot{x}^j$$

all four terms are same $\quad = 0. \qquad \square$

proof of Lemma 5:

Let us first derive (review) the general E-L equation for proper variations of a C^∞ curve $x:[a,b]\to\mathbb{R}^d$, functional $\int_a^b L(t,x(t),\dot{x}(t))\,dt$:

Consider a variation $x(t,s)$. Then

$$\frac{d}{ds}\int_a^b L(t,x(t,s),\dot{x}(t,s))\,dt\Big|_{s=0} =$$

$$= \int_a^b \left(\frac{\partial L}{\partial x^j}\cdot\frac{\partial x^j}{\partial s} + \frac{\partial L}{\partial \dot{x}^j}\cdot\frac{\partial^2 x^j}{\partial s\,\partial t}\right)\Big|_{s=0} dt$$

integrate by parts, use $x(a,s)\equiv x(a)$, $x(b,s)\equiv x(b)$.
$\Rightarrow \frac{\partial x^j}{\partial s}(t,s)=0$ for $s=a,b$.

$$= \int_a^b \left(\frac{\partial L}{\partial x^j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^j}\right)\underbrace{\frac{\partial x^j}{\partial s}(t,0)}_{\text{can choose "arbitrarily"!}}\,dt$$

$\therefore$ Euler-Lagrange equation:

$$\boxed{\frac{\partial L}{\partial x^j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^j} = 0}$$

at all points along the curve

proof of Lemma 5, cont'd

Our functional is $E(\gamma) = \frac{1}{2} \int_a^b g_{jk}(x(t))\, \dot{x}^j(t)\, \dot{x}^k(t)\, dt,$

thus E-L eq is:

$$\cancel{\tfrac{1}{2}} \cdot \frac{d}{dt}\underbrace{\left(g_{ik}\, \dot{x}^k(t) + g_{ji}\, \dot{x}^j(t)\right)} - g_{jk,i} \cdot \dot{x}^j(t)\, \dot{x}^k(t) = 0 \qquad \text{for } i = 1, \dots, d$$

think re. $\dot{x}^i(t)$-contr!

Carry out $\frac{d}{dt}$

Note: $g_{ji} \cdot \dot{x}^j(t) = g_{ik}\, \dot{x}^k(t)$

$$\Leftrightarrow 2\left(g_{ik} \cdot \ddot{x}^k(t) + g_{ik,j} \cdot \dot{x}^j(t)\, \dot{x}^k(t)\right) - g_{jk,i}\, \dot{x}^j(t)\, \dot{x}^k(t) = 0$$

$$\Leftrightarrow 2 g_{ik}\, \ddot{x}^k(t) + \left(g_{ik,j} + g_{ij,k} - g_{jk,i}\right) \dot{x}^j(t)\, \dot{x}^k(t) = 0$$

using the $j \leftrightarrow k$ symmetry of $\dot{x}^j(t)\, \dot{x}^k(t)$

{True for $\forall i$. Now multiply with g^{li}, and add over i!}

(any $l \in \{1, \dots, d\}$)

$$\Leftrightarrow \qquad 2\ddot{x}^l(t) + \underbrace{g^{li}\left(g_{ik,j} + g_{ij,k} - g_{jk,i}\right)}_{2 \cdot \Gamma^l_{jk}} \dot{x}^j(t)\, \dot{x}^k(t) = 0$$

since $(g_{ij})(g^{jk}) = I$

□

Note: This ODE is invariant under change of coord since it is the E-L eq. for the ("invariant") ~~field~~ functional $E(\gamma)$!

3.1. **Notes.** .

In this lecture we follow (in some sense) Jost [5, start of Sec. 1.4].

p. 4 (in Def. 2, and many times later): Here we consider a "C^∞ curve" defined on the closed interval $[a, b]$. This raises a **technical point:** Recall that a *curve* on a manifold M is simply a continuous function from an interval (in $\mathbb{R}$) to M. Similarly, a C^∞ *curve* on a C^∞ manifold is a C^∞ function from an interval to M. If the interval is *open* then this is a well-defined concept since Lecture #1, since an open interval is itself a (1-dim) C^∞ manifold. However if the interval is *closed* (or half-closed) then the interval is no longer a C^∞ manifold[4] and so we need to define the concept here. Thus: A function $f : [a, b] \to M$ is said to be C^∞ if f is continuous, and the derivatives of all orders exist (at the endpoints a and b the appropriate one-sided derivatives exist)[5] and are continuous on all $[a, b]$. It turns out that this is *equivalent* to requiring that f can be extended to a C^∞ function from the *open* interval $(a - \varepsilon, b + \varepsilon)$ to M, for some $\varepsilon > 0$. For the proof of this equivalence, one immediately reduces to the case of $M = \mathbb{R}^d$, and there it follows from a lemma of Borel (cf., e.g., wikipedia).

We also need to extend the above definition to higher dimension, since later in the lecture we consider "C^∞ variations" of a curve (and in later lectures we may also consider multi-parameter variations). Thus we define: A map

$$f : [a, b] \times (-\varepsilon_1, \varepsilon_1) \times \cdots \times (-\varepsilon_m, \varepsilon_m) \to M$$

is said to be C^∞ if all partial derivatives of all orders exist (for any derivative wrt the *first* variable, we consider the appropriate one-sided derivative when at an endpoint) and are continuous throughout $[a, b] \times (-\varepsilon_1, \varepsilon_1) \times \cdots \times (-\varepsilon_m, \varepsilon_m)$. Again by using the lemma of Borel mentioned above, one can prove that this is equivalent to requiring that f can be extended to a C^∞ function from the *open* set $(a - \varepsilon, b + \varepsilon) \times (-\varepsilon_1, \varepsilon_1) \times \cdots \times (-\varepsilon_m, \varepsilon_m)$ to M, for some $\varepsilon > 0$.

p. 4 (bottom): Here I plan to mention the concept of *isometry* in passing, without writing out the definition. See [5, Def. 1.4.5]; of course you should learn this definition!

p. 8, as stated here, we make (just as Jost, is seems) the simplifying assumption that all of γ is contained in a single coordinate chart. However this is not necessary for the derivation of the Euler-Lagrange equations, and the key fact to see this is the following: By linearity, for any given covering

[4]it is not even a topological manifold; however it is a C^∞ *manifold with boundary* (cf., e.g., [2, Ch. VI.4]), but we won't introduce this concept in this course.

[5]Of course we mean: "With respect to any C^∞ chart on M containing the point under consideration". — Our presentation here is somewhat sloppy, since anyway the main point we wish to make is that: "There is no serious complication involved and we will generally not worry about this technical issue".

of $[a, b]$ by open intervals $I_1, \ldots, I_n$, we have $\frac{d}{ds}E(\gamma(\cdot, s))_{s=0}$ for *all* proper variations of γ iff for each j, $\frac{d}{ds}E(\gamma(\cdot, s))_{s=0}$ holds for all proper variations of γ trivial outside I_j. We do not discuss this in further details, since we will anyway rederive the Euler-Lagrange equations again later in the course, working in a more intrinsic (coordinate independent) language.

4. Geodesics

#4. Geodesics

Let M be a Riemannian manifold.

<u>Recall</u>: <u>$d(p,q)$</u> $= \inf \{ L(\gamma) : \gamma : [a,b] \to M$ pw C^∞ curve with $\gamma(a) = p,\ \gamma(b) = q \}$

$$L(\gamma) = \int_a^b \|\dot\gamma(t)\|\, dt$$

$$E(\gamma) = \frac{1}{2} \int_a^b \|\dot\gamma(t)\|\, dt$$

{ d is a metric on M, and gives the manifold topology.

A <u>geodesic</u> is a C^∞ curve $\gamma : [a,b] \to M$ which satisfies

$$\underline{\ddot x^j(t) + \Gamma^j_{ik}\, \dot x^i(t)\, \dot x^k(t) = 0} \qquad \left(\text{with } x(t) := x(\gamma(t))\right)$$

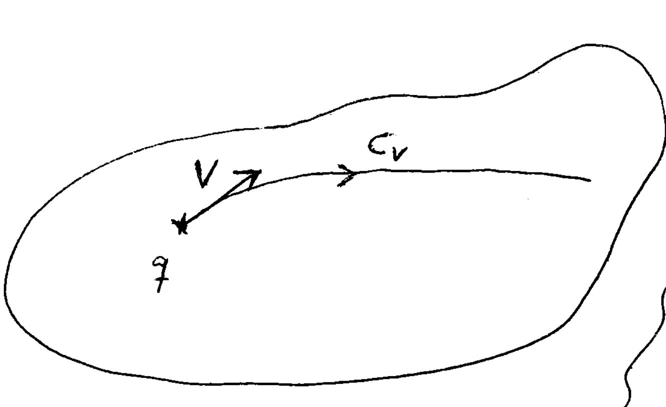
{ Euler-Lagrange equation for $E(\gamma)$, proper variations of γ }

<u>Theorem 1</u>: { $\approx$ Jost Theorem 1.4.2 }

Given $p \in M$ there exist an open neighborhood V of p and $\varepsilon > 0$, $r > 0$ such that for all $q \in V$ and all $v \in T_q M$ with $\|v\| < r$,

$\underline{\exists!\ \text{geodesic } c_v : (-\varepsilon, \varepsilon) \to M}$ with $\underline{c_v(0) = q}$, $\underline{\dot c_v(0) = v}$.

Also $c_v(t)$ depends smoothly on q, v, t.



<u>Proof</u>: Real analysis; existence theorem for ODEs. Extend to R^{2d} $\longrightarrow$ <u>1st order</u> ODE.

Theorem 1′ (more precise uniqueness):

If $I, J \subset \mathbb{R}$ are open intervals containing 0 and $c: I \to M$, $\tilde{c}: J \to M$ are geodesics with $c(0) = \tilde{c}(0)$ and $\dot{c}(0) = \dot{\tilde{c}}(0)$, then

$\exists\, \delta > 0$ s.t. $\underline{I_\delta := (-\delta, \delta) \subset I \cap J}$

and $\underline{c(t) = \tilde{c}(t),\ \forall t \in I_\delta}$.

Using Thm 1′ "iteratively" $\rightsquigarrow$ $\underline{c(t) = \tilde{c}(t),\ \forall t \in I \cap J}$,

Problem 2[redacted](a)

and from this one gets:

$\forall p \in M,\ v \in T_pM$: $\exists!$ $\underline{\underline{\text{maximal geodesic}}}$ starting at v, i.e. $\underline{\underline{c_v : I_v \to M}}$ with $\underline{c_v(0) = p,\ \dot{c}_v(0) = v}$ (I_v open interval, $0 \in I_v$)

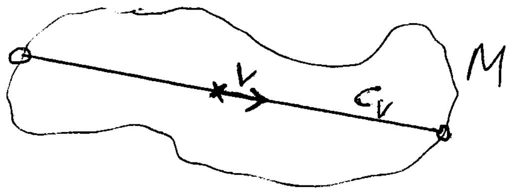
such that: For any open interval $J \subset \mathbb{R}$ with $0 \in J$ and any geodesic $\gamma: J \to M$ with $\gamma(0) = p$, $\dot{\gamma}(0) = v$, one has: $\underline{J \subset I_v \text{ and } \gamma = c_v|_J}$.

Proof: Problem 2[redacted](a)

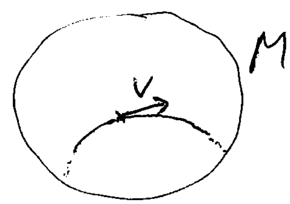
We fix this notation I_v, c_v from now on!

~~Ex: Often $I_v = \mathbb{R}$. But for e.g. M = bounded open subset of $\mathbb{R}^d$, I_v is bounded, $\forall v \in TM$.~~

Ex: Often $I_v = \mathbb{R}$. But if, e.g., $M =$ bounded open subset of $\mathbb{R}^d$ (with standard Riemannian metric), then I_v is bounded, $\forall v \in TM$; $(v \neq 0)$.



Of course this can change if we put other metric on $M \subset \mathbb{R}^d$; cf. e.g. the Poincaré disk model of hyperbolic plane;



Scaling: $\forall \lambda \in \mathbb{R}$: $c_{\lambda v}(t) = c_v(\lambda t)$, $I_{\lambda v} = \lambda^{-1} I_v$

$(I_0 = \mathbb{R})$

Natural notation which removes the "scaling redundancy"

Def 1: For any $v \in TM$ with $1 \in I_v$:

$\exp v := c_v(1)$.

Set also $\mathcal{D} := \{v \in TM : 1 \in I_v\}$; then $\exp : \mathcal{D} \to M$.

Thus: $\boxed{c_v(t) = \exp(tv)}$ for all $t \in \mathbb{R}$, $v \in TM$ where either is defined, i.e. $t \in I_v$

Theorem 2: $\mathcal{D}$ is open and $\exp : \mathcal{D} \to M$ is C^∞.

proof: See Problem 2 – basically one just "runs the machine" (Thm 1; local existence & uniqueness) to its limit!

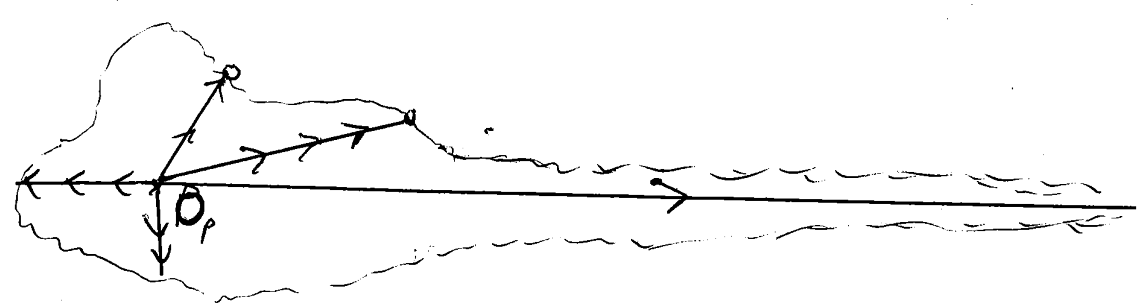
write $\mathcal{D}_p := \mathcal{D} \cap T_pM$

Def 1′: For $p \in M$, $\exp_p := \exp_{|\mathcal{D} \cap T_pM}$

Note that $\mathcal{D}_p$ is a star shaped open set in T_pM, containing 0_p. From now on, $0_p :=$ the 0 vector in T_pM!

Indeed, $\forall v \in T_pM : \{t \in \mathbb{R} : tv = \mathcal{D}_p\} = I_v$!

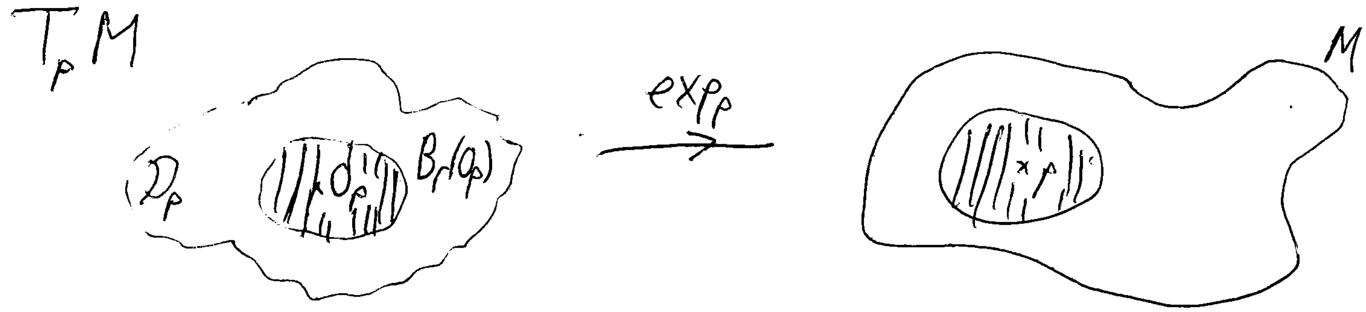
$\mathcal{D}_p$ in T_pM:



Theorem 3 {Jost Thm 1.4.3}

$\forall p \in M : \exists r > 0 : \quad B_r(0_p) \subset \mathcal{D}_p$ and $\underline{\exp_{p|B_r(0_p)}}$ is a <u>diffeomorphism</u> onto an open set in M.

{containing p!}



proof: Compute $\underbrace{(d\exp_p)_{0_p}} = 1_{T_pM}$

maps $\underbrace{T_{0_p}T_pM}_{= T_pM} \to T_pM$

See Jost for the proof; it is a direct consequence of $\frac{d}{dt}\exp(tv)_{|t=0} = \dot{c}_v(0) = v.$

This is a nonsingular linear map; hence Thm 3 follows by the <u>Inverse Function Theorem</u>.

$\square$

Theorem 3' (stronger version): $\forall p \in M : \exists r > 0$ and an open neighborhood U of p s.t. $\forall q \in U : \quad B_r(0_q) \subset \mathcal{D}_q$ and $\exp_{q|B_r(0_q)}$ is a diffeomorphism onto an open set in M.

proof ("standard") see Problem 21; one approach is to consider an appropriate "higher dimensional" function and apply the Inverse Function Theorem to it!

Corollary 1: Jost Cor 1.4.3

If M is compact then $\exists r > 0$ which "works for all M"!

Given p, r as in Thm 3, set $U = \exp_p(B_r(0_p))$.

Identify $T_pM \cong \mathbb{R}^d$ in some way "respecting $\langle , \rangle$" this is done by choosing an ON basis $e_1, \dots, e_d$ in T_pM and identify $e_j \leftrightarrow (0, \dots, 0, 1, 0, \dots, 0)$ pos j.

$$\begin{array}{ccc} B_r(0_p) & \underset{\exp_p^{-1}}{\overset{\exp_p}{\rightleftarrows}} & U \\ \cap & & \cap \\ T_pM = \mathbb{R}^d & & M \end{array}$$

Here $\exp_p^{-1}$ is sloppy notation for $\left(\exp_p|_{B_r(0_p)}\right)^{-1}$

Def. 2: Such a chart $(U, \exp_p^{-1})$ is called (Riemannian) normal coordinates with center p.

Lemma 1: In normal coordinates, $g_{ij}(0) = \delta_{ij}$, $\Gamma^i_{jk}(0) = 0$, $g_{ij,k}(0) = 0$ $(\forall\, i,j,k)$.

Jost Thm 1.4.4

proof: $g_{ij}(0) = \delta_{ij}$ is clear. Namely since $T_pM = \mathbb{R}^d$ respecting $\langle\,,\,\rangle$ and $(d\exp_p^{-1})_{0_p} = 1$.

For any $v \in \mathbb{R}^d$, $x(t) = tv$ $(|t| < \varepsilon)$ is a geodesic in our coordinates; hence

$$\underbrace{\ddot{x}^j(t)}_{0} + \Gamma^j_{ik}(x(t))\, \underbrace{\dot{x}^i(t)}_{v^i}\, \underbrace{\dot{x}^k(t)}_{v^k} \equiv 0$$

For $t=0$, get $\Gamma^j_{ik}(0)\, v^i v^k = 0$, $\forall v \in \mathbb{R}^d$; hence (using also $\Gamma^j_{ik} = \Gamma^j_{ki}$) $\Gamma^j_{ik}(0) = 0$.

The last claim, $g_{ij,k}(0) = 0$, now follows from the def. of Γ^j_{ik} and some manipulations.

□

≈ Jost Cor 1.4.2

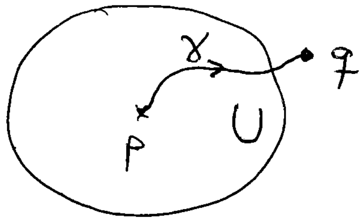
Theorem 4: For any p, r as in Thm. 3 (so that $(U, \exp_p^{-1})$ are normal coordinates, $U := \exp_p(B_r(0_p))$),

$\boxed{U = B_r(p)}$, and for every $v \in B_r(0_p)$, $q = \exp_p(v)$, the geodesic $\gamma(t) = \exp(tv)$, $t \in [0,1]$, is the unique (up to reparametrization) shortest curve in M from p to q.

proof (outline):

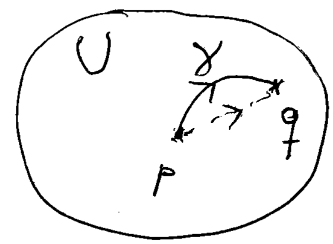
Must show:

①



$L(\gamma) \geq r$

②



$L(\gamma) \geq L(t \mapsto \exp(tv))$

Our task is to prove that ① any curve starting at p which goes outside U has length $\geq r$ and ② any curve from p to q which stays inside U has length $\geq$ this curve → $\boxed{t \to \exp(tv)}$ (with equality iff γ is a reparametrization of that curve)

Let $x: U \to \mathbb{R}^d$ be normal coordinates and let $(r, \varphi^1, \dots, \varphi^{d-1})$ be polar coordinates i.e. $r = \|x\| > 0$ and $(\varphi^1, \dots, \varphi^{d-1}) = \varphi(\|x\|^{-1} x)$.

This is for (V, φ) a chart on S^{d-1}; note that with a fixed "polar coordinate chart" we can only cover an open cone $\subset U$

Riemannian metric wrt $(r, \varphi^1, \dots, \varphi^{d-1})$:

$$(h_{ij}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{\text{pos. def.}} & \\ 0 & & & \end{pmatrix}$$

proof: Problem 23

= Jost Thm 1.4.5

Hence for any $\gamma : [a,b] \to M$ (pw C^∞) with $\gamma(t) \in U \setminus \{p\}$, $\forall t \in (a,b)$:

$$\boxed{\begin{array}{l} \|\dot\gamma(t)\| \geq |\dot r(t)| \quad \forall t \in (a,b) \\ \text{with equality iff } \dot\varphi(t) = 0. \end{array}}$$

Thus:

$$L(\gamma) = \int_a^b \|\dot\gamma(t)\|\, dt \geq \int_a^b |\dot r(t)|\, dt \geq |r(b) - r(a)|$$

limits if $\gamma(a), \gamma(b) = p$ or $\notin U$

Equality iff $\dot\varphi(t) = 0$ $\forall t \in (a,b)$, i.e. φ constant

Equality iff $\dot r(t) \geq 0$ $\forall t \in (a,b)$ or $\dot r(t) \leq 0$ $\forall t \in (a,b)$

This gives (easily): 1 and 2 proved!

Technicality: If γ does not stay in our open cone inside U: Split $\int_a^b$ appropriately, use several charts (V, φ) on S_1^{d-1}.

$\square$

4.1. **Notes.** .

In this lecture we continue to follow (in some sense) Jost [5, Sec. 1.4].

p. 1: For the proof of Theorem 1 we refer to the following basic theorem of analysis; this theorem plays an important role at several junctures in the development of the foundations of C^∞ manifolds.

Theorem (Existence theorem for ODEs; cf., e.g., [7, Ch. IV] or [2, Sec. IV.4].) *Consider the equations*

$$\frac{dx^i}{dt} = f^i(t,x), \qquad \textit{for } i = 1, \ldots, n, \tag{1}$$

where $f^1, \ldots, f^n$ are given real-valued C^r functions ($r \geq 1$) on $I_\varepsilon \times U$, with $U \subset \mathbb{R}^n$ being an open set and $\varepsilon > 0$, $I_\varepsilon := (-\varepsilon, \varepsilon)$. Then for each $x \in U$ there exist $\delta > 0$ and an open neighborhood V of x, $V \subset U$, such that there exists a C^r function $x : I_\delta \times V \to U$ such that for each $a \in V$ we have $x(0,a) = a$ and, writing $x(t,a) = (x^1(t,a), \ldots, x^n(t,a))$, the function $t \mapsto x(t,a)$ satisfies (1) *for all $t \in I_\delta$, $a \in V$.*

Uniqueness: *Any solution to* (1) *is unique in the following strong sense: If $I, J \subset \mathbb{R}$ are two open intervals both containing* 0*, and if*

- *(i)* $x : I \to U$ *is C^1 and satisfies* (1) *for all $t \in I$,*
- *(ii)* $\overline{x} : J \to U$ *is C^1 and satisfies* (1) *for all $t \in J$, and*
- *(iii)* $x(0) = \overline{x}(0)$;

then $\overline{x}(t) = x(t)$ for all $t \in I \cap J$.

Remark: In this course, we will only apply the theorem with $r = \infty$, i.e. dealing only with C^∞ functions.

In order to prove Theorem 1 (on p. 1 in the lecture), after passing to local coordinates the task is to prove existence of solutions to the ODE

$$\ddot{x}^j(t) + \Gamma^j_{ik}(x(t)) \cdot \dot{x}^i(t)\dot{x}^k(t) = 0.$$

In order to be able to apply the above existence theorem, one first applies the standard trick of vieweing also $\dot{x}^1, \ldots, \dot{x}^d$ as unknowns to be solved for (we call these unknowns $y^1, \ldots, y^d$). Thus one studies instead the system of $2d$ equations

$$\dot{x}^j(t) = y^j(t) \qquad (j = 1, \ldots, d);$$

$$\dot{y}^j(t) = -\Gamma^j_{ik}(x(t)) \cdot y^i(t)y^k(t) \qquad (j = 1, \ldots, d).$$

This system is of the form (1) above, with [new x] $= (x^1, \ldots, x^d, y^1, \ldots, y^d)$, thus $n = 2d$. For further details, cf., e.g., Boothby [2, Lemma 5.4].

p. 5, Theorem 3: We remark that the largest possible $r > 0$ which works in this theorem is called the *injectivity radius* of p, $i(p)$. (It is not immediately clear that this definition of injectivity radius agrees with the one in Jost, [5, Def. 1.4.6]; however we will prove later that the two definitions agree, as an application of the theory of Jacobi fields.)

p. 5: In the proof of Theorem 3 we refer to:

The Inverse Function Theorem: *Let M, N be C^∞ manifolds of the same dimension, let $f : M \to N$ be a C^∞ function, and let $p \in M$. Assume that the linear map df_p is nonsingular. Then there exists an open neighborhood V of p in M such that $f(V)$ is open in N and $f_{|V}$ is a C^∞ diffeomorphism of V onto $f(V)$.*

5. Geodesics: Hopf–Rinow etc.

#5. Geodesics; Hopf–Rinow Theorem

Def 1: Two curves $\gamma_0, \gamma_1 : I = [0,1] \to M$ with

$\gamma_0(0) = \gamma_1(0) = p$ and $\gamma_0(1) = \gamma_1(1) = q$

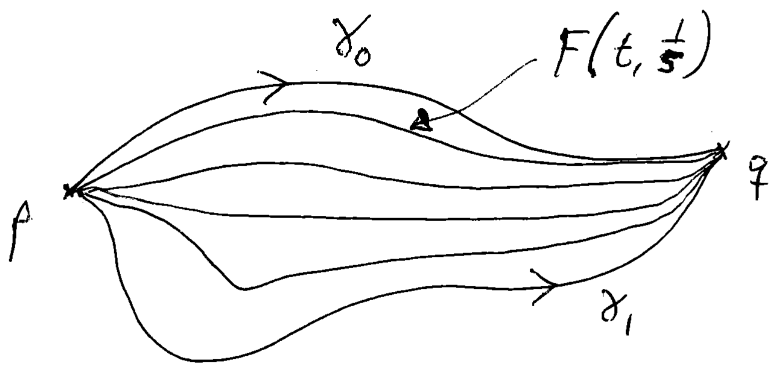
are called homotopic if $\exists$ continuous map

with fixed endpoints

$F : I \times I \to M$

with $F(t,0) = \gamma_0(t)$, $F(t,1) = \gamma_1(t)$, $\forall t \in I$,

$F(0,s) = p$, $F(1,s) = q$, $\forall s \in I$



Two closed curves $c_0, c_1 : S^1 \to M$ are called homotopic if $\exists$ continuous map

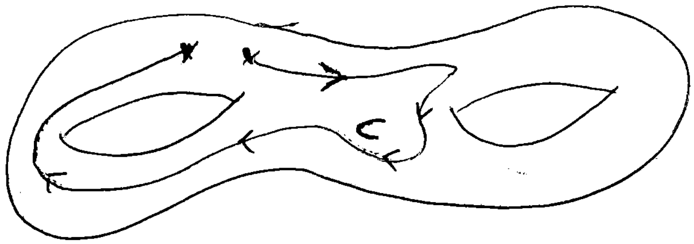
$F : S^1 \times I \to M$ with $F(t,0) = c_0(t)$
$F(t,1) = c_1(t)$, $\forall t \in S^1$

Theorem 1: {Jost Thm 1.5.1}

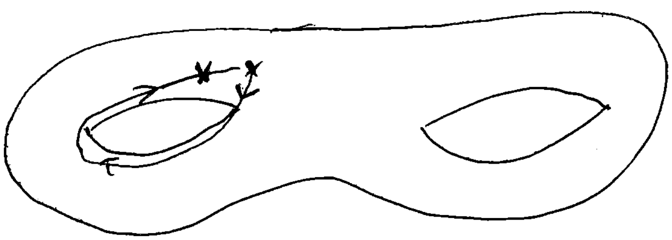
Let M be a compact Riemannian manifold. Let $c: I \to M$ be a curve. Then there is a <u>geodesic</u> homotopic to c, {thus: with same start- & end-points as c} and this geodesic can be chosen as a shortest curve in its homotopy class.

Similarly, any homotopy class of <u>closed</u> curves in M contains a curve which is <u>shortest</u> & <u>geodesic</u>.

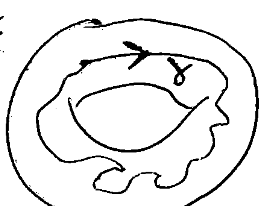
<u>Ex:</u>



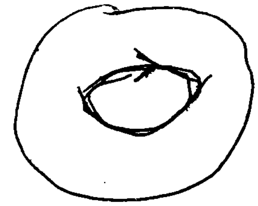
- geodesic homotopic to c would look something like:



<u>Ex:</u>



- a closed geodesic homotopic to γ would look something like:



Addendum {Jost Cor. 1.4.5} (Still: M a compact R. mfld.)

Any two points p, q can be connected by a curve of shortest length, and this curve is a geodesic.

<u>proof outline</u>: By Cor. 4.1 ← i.e., Cor 1 in Lecture #4

$\exists\, r_0 > 0$ such that normal coordinates with "radius r_0" work at every $p \in M$. Hence by Thm. 4.4 ← i.e., Thm 4 in Lecture #4,

for any $p, q \in M$ with $d(p,q) < r_0$ there is a unique shortest curve between p, q.

up to reparametrization

This is Jost Cor 1.4.4

namely the geodesic ~~$\exp$~~ ~~...~~

$\gamma_{p,q}(t) := \exp_p(t \cdot \exp_p^{-1}(q)) \quad (t \in I)$.

This $\gamma_{p,q}(t)$ depends smoothly on p, q, t.

by Problem 22(b)

We now outline a proof of <u>first part</u> of Thm 1; thus let $c : I \to M$ be a curve. Let $\mathcal{F}$ be the family of all <u>pw C^∞ curves</u> <u>homotopic</u> to c.

$\mathcal{F} \neq \emptyset$; cf. Problem 18, easy elaboration.

Consider $\inf_{c_0 \in \mathcal{F}} L(c_0)$. By definition of infimum, there is a sequence $\gamma_1, \gamma_2, \ldots \in \mathcal{F}$ with

$$L(\gamma_n) \longrightarrow L_0 := \inf_{c_0 \in \mathcal{F}} L(c_0) \quad \text{as} \quad n \to \infty.$$

{ This is what Jost calls a minimizing sequence }

We may assume $L(\gamma_n) < L_0 + 1$, $\forall n$.

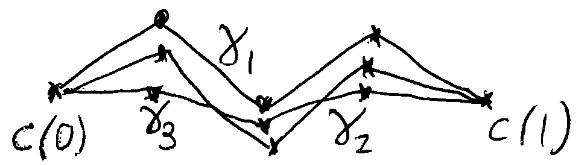
Take $m \in \mathbb{Z}^+$ so large that $\frac{L_0+1}{m} < r_0$.

Now split each γ_n into m parts of equal length, and replace each part by the unique shortest geodesic between the same endpoints!

Then $L(\text{new } \gamma_n) \leq L(\gamma_n)$ so we still have

$$L(\gamma_n) \xrightarrow{n \to \infty} L_0.$$

Since M is compact, after passing to a subsequence we may assume that the "j:th breakpoint" of γ_n converges to a limit as $n \to \infty$, for each $j \in \{1, 2, \ldots, m-1\}$.



Let γ be the curve formed by geodesic segments between the limit points.

One easily sees $\boxed{L(\gamma) = \lim_{n\to\infty} L(\gamma_n) = L_0}$ and γ is homotopic with all γ_n (for large n, hence for all n) see Jost Lemma 1.4.7

i.e. $\gamma \in \mathcal{F}$. Finally, the fact that γ minimizes $L(\gamma)$ within the homotopy class $\mathcal{F}$ implies that γ is a geodesic (and not just "piecewise geodesic") — indeed otherwise we could find p, q on γ with $d(p,q) < r_0$ such that γ between p, q is not geodesic and then we can make γ shorter — contradiction!

$\square$ $\square$

The argument at the end gives also: Details: Problem 24.

Theorem 2: If $\gamma : [a,b] \to M$ is a pw C^∞ curve and $L(\gamma) = d(\gamma(a), \gamma(b))$, then γ is a geodesic. (and γ is parametrized by arc-length)

When can every (or some) geodesic be extended indefinitely?

Theorem 3: <u>Theorem of Hopf–Rinow</u>

Jost Thm 1.7.1

Let M be a Riemannian manifold.
Then $\boxed{(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5)}$ with

(1) (M, d) is <u>complete</u>. ← viz., every Cauchy sequence converges.

The metric coming from the Riemannian structure.

(2) Any closed bounded subset of M is compact.

i.e., contained in a ball $B_r(p)$. $(p \in M, r > 0)$

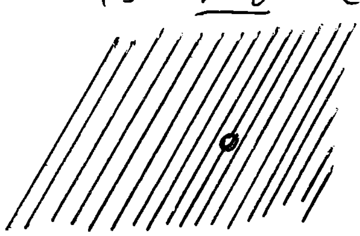
(3) $\exists\, p \in M$ s.t. $\mathcal{D}_p = T_pM$.

$\Leftrightarrow$ $\exp_p$ is defined on all T_pM
$\Leftrightarrow$ every geodesic starting at p can be indefinitely extended.

(4) $\forall\, p \in M : \quad \mathcal{D}_p = T_pM$.

(5) $\forall\, p, q \in M : \exists$ geodesic γ from p to q with $L(\gamma) = d(p,q)$.

<u>Ex</u>: $M = \mathbb{R}^d \setminus \{0\}$ (with standard Riemannian metric) is <u>not</u> complete. Also (5) fails!



<u>Ex</u>: More generally, any connected open subset $U \subset \mathbb{R}^d$ is a Riemannian manifold

{with Riemannian metric induced by the standard Riemannian metric on $\mathbb{R}^d$}

and is <u>not</u> complete unless $U = \mathbb{R}^d$.

However U satisfies (5) if U is ______.

<u>Ex</u>: M compact $\Longrightarrow$ M complete!

However, for a general C^∞ manifold M there can exist both complete and non-complete Riemannian metrics!

{See Problem 25!} home assignment!

<u>Ex</u>: If M is a <u>closed</u> <u>embedded</u> <u>submanifold</u> of a Riemannian manifold N, (thus M inherits a Riemannian structure from N; see Problem 17), then

<u>N complete</u> $\Longrightarrow$ <u>M complete</u>!

{See <u>Problem 26</u>; this is <u>not</u> immediate from the similar basic fact about metric spaces!}

Outline of proof of Theorem 3 (Hopf–Rinow):

We will focus on proving the following

KEY FACT:

$$\boxed{\begin{array}{l}\text{Given } p\in M, \text{ if } \mathcal{D}_p = T_pM \\ (\text{i.e. } \exp_p \text{ is defined on all } T_pM) \\ \text{then } \forall q\in M:\ \exists \text{ geodesic} \\ \gamma \text{ from } p \text{ to } q \text{ with } L(\gamma)=d(p,q)\end{array}}$$

Once the KEY FACT is proved, the proof of all Thm 3 is quite easy – see Jost's book!

One detail about this: I think Jost's proof of (1) $\Rightarrow$ (4) is unnecessarily complicated. Following his argument, once we have identified the limit point he calls "p", we may simply apply Theorem 4:3′ (= Thm 3′ in Lecture # 4); using this one easily shows that the geodesic may be continued beyond p.

Proof of KEY FACT: Assume $p, q \in M$, $\mathcal{D}_p = T_pM$.

Set $r = d(p,q)$.

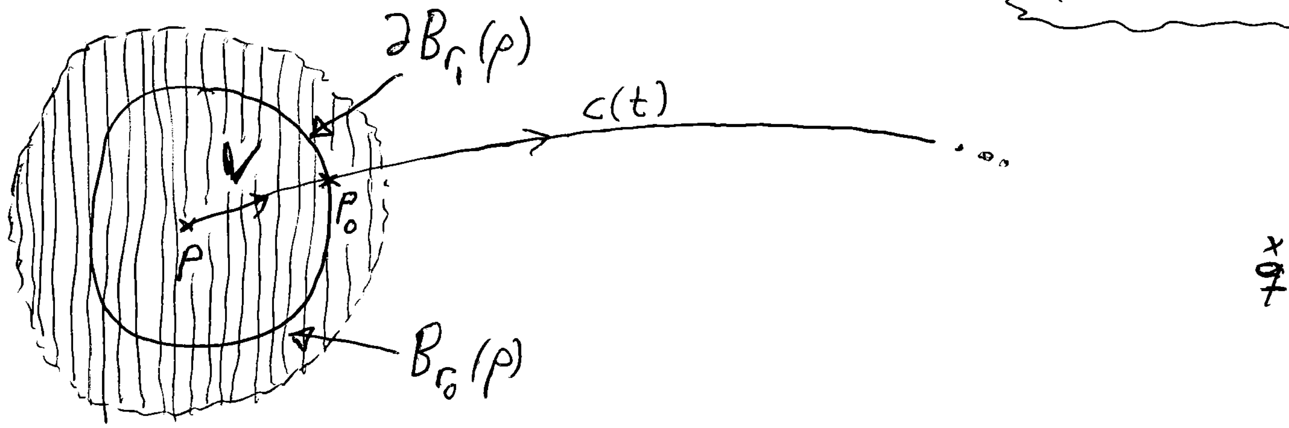
Take $r_0 > 0$ such that $\exp_{|B_{r_0}(0_p)}$ is a diffeomorphism onto an open set (Thm 4:3).

If $r < r_0$ then Thm 4:4 $\Rightarrow$ done!

So assume $r \geq r_0$.

Take $r_1 \in (0, r_0)$; thus $0 < r_1 < r_0 \leq r$.

$r_1 \leftrightarrow$ Jost's "ρ"



Take $p_0 :=$ a point where $d(\cdot, q)_{|\partial B_{r_1}(p)}$ is minimal.

Note: By Thm 4.4, $B_{r_0}(p) = \exp_p(B_{r_0}(0_p))$ and $\partial B_{r_1}(p) = \exp_p(\partial B_{r_1}(0_p))$ which is compact; and $d(\cdot, q)$ is a continuous function on this set; hence the minimum is attained, i.e. p_0 exists.

Set $V := \frac{1}{r_1} \exp_p^{-1}(p_0) \in T_p(M)$

Thus $\|V\| = 1$ and $p_0 = \exp(r_1 V)$.

Consider the geodesic $c(t) := \exp_p(tV)$.

Want to prove: $c(r) = q$ — then done!

Set $I = \{t \in [0,r] : d(c(t), q) = r - t\}$.

the set of ~~points~~ time points where our geodesic is "still on course".

Take $t_0 := \sup I$.

Note $r_1 \in I$, ← (Problem 26(c))
thus $t_0 \geq r_1$.

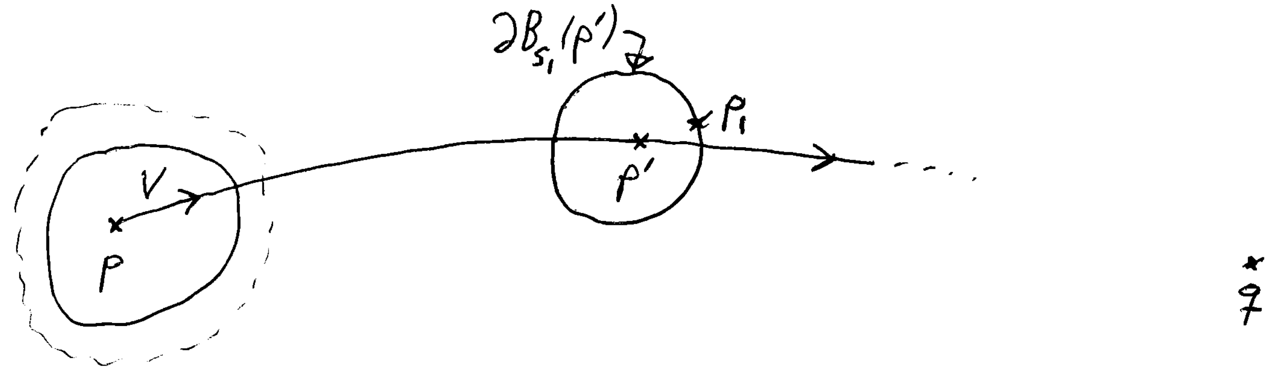
Also $d(c(t_0), q) = r - t_0$

By def. of I and t_0, and since $t \mapsto d(c(t), q)$ is continuous — cf. Probl. 27(a)

If $t_0 = r$ then done,
so $\boxed{\text{assume } r_1 \leq t_0 < r}$ ←

This will lead to a contradiction; namely we will prove that I must contain some $t > t_0$!

Set $p' = c(t_0)$.



Take $s_0 \in (0, r - t_0)$ so that $\exp_{|B_{s_0}(p')}$ is a diffeomorphism and take $s_1 \in (0, s_0)$.

Take $p_1 :=$ a point where $d(\cdot, q)_{|\partial B_{s_1}(p')}$ is minimal.

Then $r - t_0 = d(p', q) = s_1 + d(p_1, q)$.

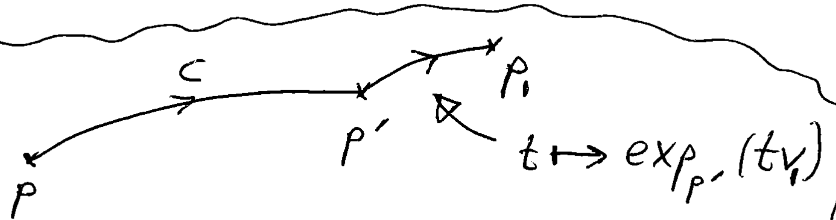
noted above

see Problem 27(c)

Hence $\underline{\underline{d(p, p_1)}} \geq \underbrace{d(p,q)}_{r} - \underbrace{d(p_1, q)}_{r - t_0 - s_1} = \underline{\underline{t_0 + s_1}}$

But we have an obvious pw C^∞ curve γ from p to p_1 with $L(\gamma) = t_0 + s_1$

— namely $\gamma =$



where $v_1 := \frac{1}{s_1} \exp_{p'}^{-1}(p_1)$

hence $\underline{\underline{d(p, p_1) = t_0 + s_1}}$ and, by Theorem 2 above (if we assume γ parametrized by arc length)

$\underline{\gamma \text{ is a geodesic}}$!

Hence by uniqueness of geodesic with given initial data $(V \in T_p M)$, Problem 21(a)

$\underline{\underline{\gamma \equiv c_{[0, t_0 + s_1]}}}$. Hence $\underline{c(t_0 + s_1) = p_1}$ and $\underline{\underline{t_0 + s_1 \in I}}$;

contradiction!

$\square$ KEY FACT proved

6. The fundamental group. The theorem of Seifert-van Kampen

#6. The Fundamental Group

Let X be an arbitrary topological space.
{actually more general now, since X is}

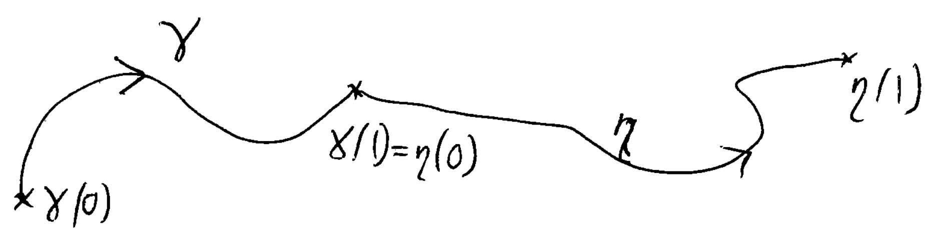
Recall from #5 that two curves $\gamma_0, \gamma_1 : I = [0,1] \to X$ with $\gamma_0(0) = \gamma_1(0) = p$ and $\gamma_0(1) = \gamma_1(1) = q$ are called <u>homotopic</u> if there exists a continuous map $F : I \times I \to X$ with $F(\cdot, 0) = \gamma_0$, $F(\cdot, 1) = \gamma_1$, $F(0,\cdot) \equiv p$, $F(1, \cdot) \equiv q$.

(Hatcher, Prop. 1.2)

We then write $\gamma_0 \simeq \gamma_1$. This is easily seen to be an <u>equivalence relation</u> (on the set of curves from p to q – for <u>given</u> $p, q \in X$).

For any curves $\gamma, \eta : I \to X$ with $\gamma(1) = \eta(0)$ we define the <u>product path</u> of γ and η, $\gamma \cdot \eta : I \to X$, by

$$\gamma\cdot\eta(t) = \begin{cases} \gamma(2t) & \text{for } t \in [0, \tfrac{1}{2}] \\ \eta(2t-1) & \text{for } t \in [\tfrac{1}{2}, 1]. \end{cases}$$



This operation <u>respects homotopy</u>!

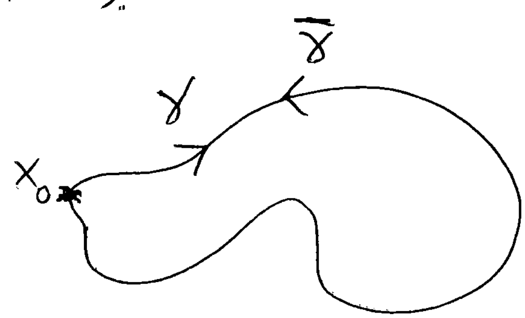
That is, <u>if</u> $\gamma_0 \simeq \gamma_1$ and $\eta_0 \simeq \eta_1$ <u>then</u>
$\gamma_0 \cdot \cancel{\gamma_1} \simeq \eta_0 \cdot \eta_1$.

Def 1: For $x_0 \in X$, the <u>fundamental group</u> of X at the basepoint x_0, $\underline{\underline{\pi_1(X, x_0)}}$, is the set of homotopy classes $[\gamma]$ of curves $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = x_0$, and product operation $\underline{[\gamma]\cdot[\eta] := [\gamma\cdot\eta]}$.

such a curve is called a <u>loop</u>.

well-defined, since the product path operation respects homotopy.

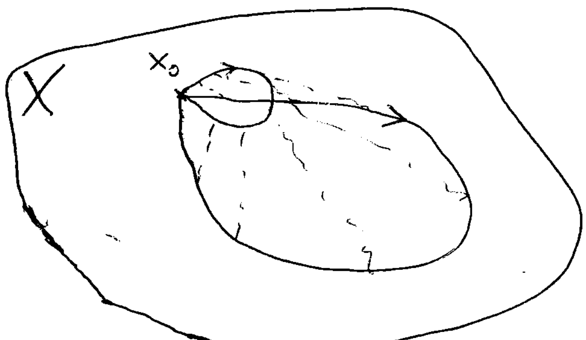
Of course one has to <u>prove</u> that $\pi_1(X, x_0)$ is a group; this is a basic exercise in constructing (simple) homotopies; see Hatcher p. 27. In particular the <u>identity element</u> is $\underline{e = [c]}$ where $\underline{c(t) = x_0, \forall t \in I}$; and the <u>inverse</u> of $[\gamma]$ is $\underline{[\gamma]^{-1} = [\bar{\gamma}]}$ where $\bar{\gamma}(t) = \gamma(1-t)$.



Think through how to see $[\gamma]\cdot[\bar{\gamma}] = e$!

Ex: For X any <u>convex set</u> $\subset \mathbb{R}^d$, $\underline{\pi_1(X, x_0) = \{e\}}$.

i.e., X is <u>simply</u> <u>connected</u> (DEF!)

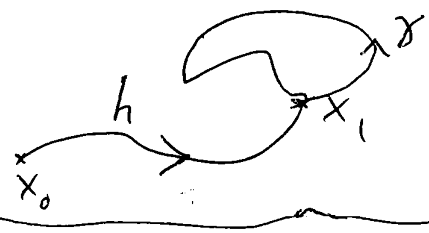


proof: use "linear homotopy" (e.g.)

Fact: If $h: I \to X$ is a curve with $h(0) = x_0$, $h(1) = x_1$, then $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$

change-of-basepoint map $\qquad \beta_h([\gamma]) := [h \cdot \gamma \cdot \bar{h}]$

is an isomorphism. $\qquad$ (Hatcher, Prop 1.5)



Hence if X is path-connected, the isomorphism class of $\pi_1(X, x_0)$ is independent of x_0, and we can write simply "$\pi_1(X)$".

FUNDAMENTAL PROPERTY: If X, Y are homotopy equivalent then $\pi_1(X) \cong \pi_1(Y)$ $\qquad$ (Hatcher Prop 1.18)

Ex: $\pi_1(S^1)$ = infinite cyclic group $= \mathbb{Z}$

Generator:  $\qquad$ Try to prove the fact directly! (See Hatcher p. 29-31 for a proof.)

On the other hand, $\pi_1(S^d) = \{e\}$ for $d \geq 2$. (see also p. 7) $\qquad$ (Hatcher Prop 1.14)

Fact: $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ if X, Y are path-connected.

(Hatcher, Prop 1.12)

Hence $\pi_1(\underbrace{T^d}_{d\text{-dim torus}}) \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{d \text{ copies}}$

Fact: Any continuous map $f: X \to Y$ with $f(x_0) = y_0$ $(x_0 \in X, y_0 \in Y)$ induces a homomorphism

$$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0); \qquad f_*([\gamma]) := [f \circ \gamma]$$

Hatcher p. 34

In fact π_1 (with "$f \mapsto f_*$") is a functor from the category of topological spaces with base point to the category of groups.

We will next state the (Seifert-) van Kampen Theorem, which gives a method for computing π_1 of many spaces. We first need some preparation.

view as pairwise disjoint!

Def 2: Let $\{G_\alpha\}_{\alpha \in A}$ be a family of groups. Then the free product $*_\alpha G_\alpha$ is the set

$$\{\underbrace{g_1 g_2 \cdots g_m}_{} : m \geq 0,\ g_i \in G_{\alpha_i} \setminus \{e\} \text{ for } i = 1, \dots, m \text{ where } \alpha_i \neq \alpha_{i+1} \text{ for } i = 1, \dots, m-1\}$$

A word, ie a sequence.

$m = 0$ means the empty word, "ε"

With group operation

$$(g_1 \cdots g_m)(h_1 \cdots h_n) := \text{reduce}(g_1 \cdots g_m h_1 \cdots h_n)$$

if $g_m, h_1 \in$ Same G_α then replace "g_m, h_1" by one entry "$g_m h_1$". Remove any e and repeat!

Fact(s): Each G_α can naturally be viewed as a subgroup of $*_\alpha G_\alpha$.

Let H be any group and assume given a homomorphism $\varphi_\alpha : G_\alpha \to H$ for each $\alpha \in A$.

Then these have a unique extension $\varphi : *_\alpha G_\alpha \to H$

(namely $\varphi(g_1 \cdots g_m) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_m}(g_m)$, if $g_i \in G_{\alpha_i}, i = 1,\dots,m$)

Now: Back to $\pi_1(X, x_0)$!

Let $\{A_\alpha\}$ be a family of subsets of X with $x_0 \in A_\alpha$, and let $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ be the induced homomorphisms.

Shorthand! $\pi_1(A_\alpha) := \pi_1(A_\alpha, x_0)$
$\pi_1(X) = \pi_1(X, x_0)$

These extend to a unique homomorphism

$$\boxed{\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X).}$$

For any indices α, β, let $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ be the homomorphism induced by $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$.

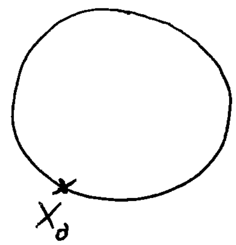
Note that $j_\alpha \circ i_{\alpha\beta} = j_\beta \circ j_{\beta\alpha}$.

Theorem 1 [the van Kampen Theorem]: Assume all A_α are open & path-connected, $X = \bigcup_\alpha A_\alpha$, and $\forall \alpha, \beta$: $A_\alpha \cap A_\beta$ is path-connected.

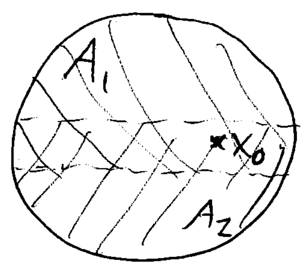
Then Φ is <u>surjective</u>.

If also $\forall \alpha, \beta, \gamma$: $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then $\ker(\Phi)$ equals the normal subgroup $N < *_\alpha \pi_1(A_\alpha)$ generated by all elements of the form $i_{\alpha\beta}(w)\, i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_\alpha \cap A_\beta)$, and so $\underline{\underline{\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N}}$.

Ex: Apply to S^1, with each A_α being an interval? — Not possible! (recall $x_0 \in A_\alpha, \forall \alpha$)



Ex: For $X = S^d$ $(d \geq 2)$, can write $X = A_1 \cup A_2$ with A_1, A_2 = enlargened hemispheres.



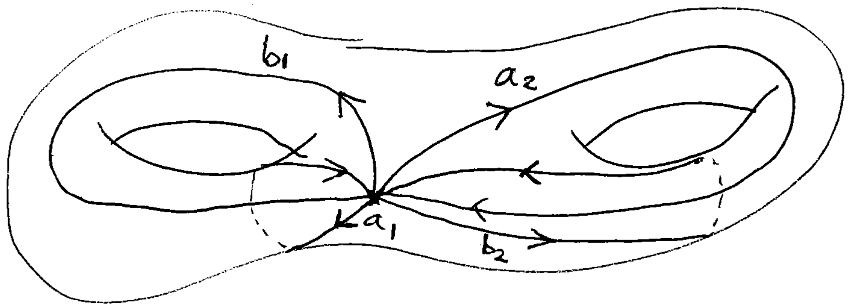
$\pi_1(A_1) = \{e\}$

$\pi_1(A_2) = \{e\}$

$\therefore$ $\underline{\underline{\pi_1(S^d) = \{e\}}}$

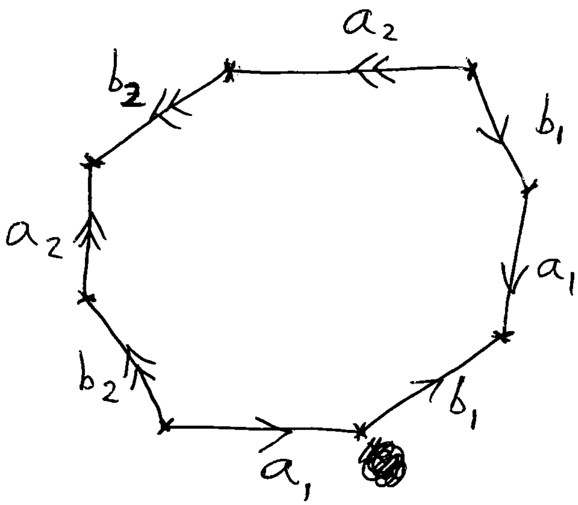
($\pi_1(A_1 \cap A_2) = \mathbb{Z}$ when $d = 2$ but this doesn't matter.)

Ex: For X a genus 2 surface, let a_1, b_1, a_2, b_2 be curves as here:



see Hatcher p.5 !

Then X is homeomorphic to:

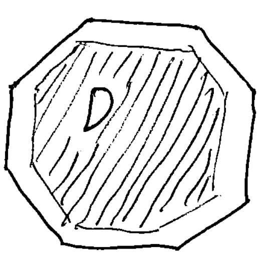


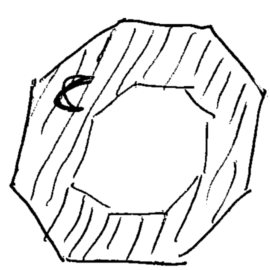
Sides identified in pairs as indicated; note that all vertices are equal!

Fix $\varepsilon > 0$ small, and let

$C =$ "[a $(10\,\varepsilon)$-neighborhood of $a_1 \cup a_2 \cup b_1 \cup b_2$]"

$D = X \setminus$ "[an ε-neighborhood of $a_1 \cup a_2 \cup b_1 \cup b_2$]"





$\pi_1(D) = \{e\}$

$\pi_1(C) = ??$

To determine $\pi_1(C)$, we split it into <u>4 parts</u>:

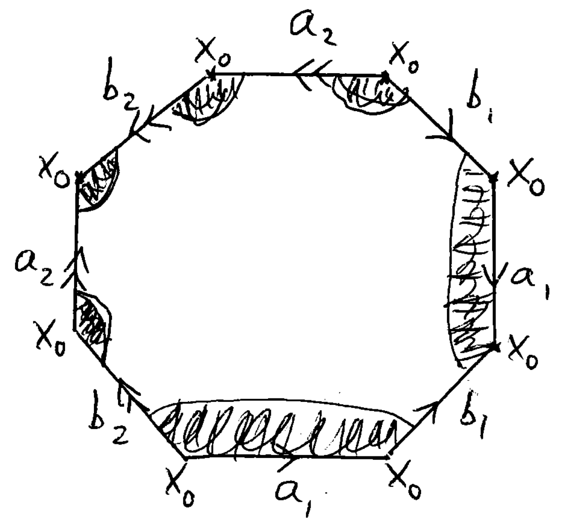
$A_1 = [\text{a } (10\varepsilon)\text{-neighborhood of } a_1]$

$B_1 = [\text{a } (10\varepsilon)\text{-neighborhood of } b_1]$

$A_2 = [\text{a } (10\varepsilon)\text{-neighborhood of } a_2]$

$B_2 = [\text{a } (10\varepsilon)\text{-neighborhood of } b_2]$

A_1

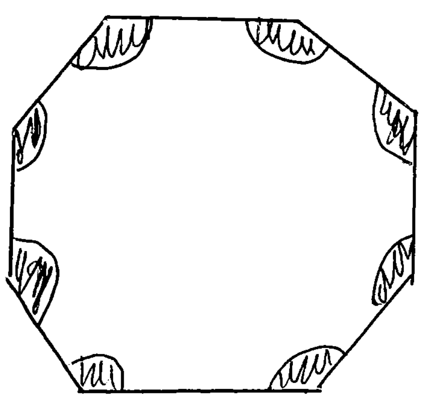


B_1, A_2, B_2 analogous

$\pi_1(A_1) \cong \mathbb{Z}$

generator: $[a_1]$

$A_1 \cap B_1 = A_1 \cap B_2 = A_1 \cap A_2 = \cdots =$ <u>a disc</u>



$\therefore \pi_1(A_1 \cap B_1) = \{e\}$

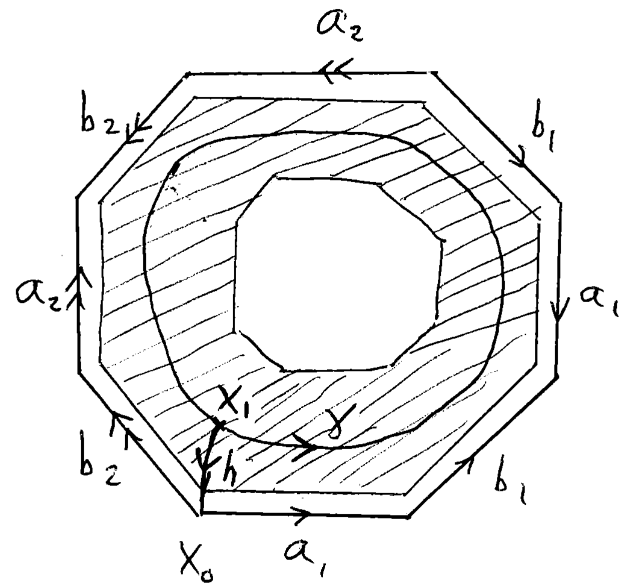
$\pi_1(A_1 \cap B_2) = \{e\}$

...

$\therefore \pi_1(C, x_0) = \langle a_1, b_1, a_2, b_2 \rangle =$ The free group generated by a_1, b_1, a_2, b_2.

$\simeq$ a genus 2 surface with 1 puncture!

Finally $\pi_1(C \cap D,\ x_1) \cong \mathbb{Z}$, generator $[\gamma]$:



Let $\beta_h : \pi_1(C, x_0) \to \pi_1(C, x_1)$ be the change-of-basepoint map.

$[\text{Image of } [\gamma] \text{ in } \pi_1(C, x_1)] = \beta_h([a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}])$.

Hence, using Theorem 1 to compute $\pi_1(X, x_1)$ via $X = C \cup D$, and then applying β_h^{-1} gives

$$\boxed{\pi_1(X, x_0) = \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = e \rangle}$$

That is, the group with generators a_1, b_1, a_2, b_2 ~~and~~ subject only to the relation $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = e$.

The above computation generalizes directly to a genus g surface X $(g \geqslant 1)$:

$$\underline{\pi_1(X, x_0) = \langle a_1, b_1, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = e \rangle}$$

Along the way one also finds that for $\underline{X}$ with 1 puncture:

$$\pi_1(X \setminus \{p\}, x_0) = \langle a_1, b_1, \dots, a_g, b_g \rangle$$

{free group with 2g generators}

In particular for $X = T^2$:

$$\underline{\underline{\pi_1(T^2, x_0)}} = \langle a_1, b_1 \mid a_1 b_1 a_1^{-1} b_1^{-1} = e \rangle$$

$$\underline{= \langle a_1, b_1 \mid a_1 b_1 = b_1 a_1 \rangle}$$

$$\underline{\cong \mathbb{Z} \times \mathbb{Z}},$$ {as noted before; see p. 3}

but $\underline{\underline{\pi_1(T^2 \setminus \{p\}, x_0) = \langle a_1, b_1 \rangle}}$!

6.1. **Notes.** .

In this lecture we follow Hatcher, [3, Ch. 1.1-2]. Note that this book is freely available from Allen Hatcher's web page. (In our lecture, "t" and "s" have switched roles versus Hatcher's presentation, since we follow Jost's usage.)

p. 2: The fact that $\pi_1(X, x_0)$ is indeed a group is [3, Prop. 1.3]; the proof occupies most of [3, p. 27]. (Regarding the proof of $[\gamma] \cdot [\overline{\gamma}] = e$; see also http://mathworld.wolfram.com for an animation illustrating this fact.)

p. 3, here we mention the fact that π_1 is an invariant that only depends on homotopy type, i.e. if X and Y are *homotopy equivalent* then $\pi_1(X) \cong \pi_1(Y)$ [3, Prop. 1.18]. Unfortunately we won't have time to introduce and discuss the notion of "homotopy equivalence" [3, p. 3] in the course, and I will simply say that intuitively speaking, two spaces are homotopy equivalent if they can be deformed continuously into one another. This means that some of the material in this lecture stands on a less firm ground than most of the other (non-expository) material in the course. For example, on p. 9 in the lecture, the reason for $\pi_1(A_1) \cong \mathbb{Z}$ is that A_1 is homotopy equivalent with S^1; similarly on p. 10 we have $\pi_1(C \cap D) \cong \mathbb{Z}$ for the same reason.

p. 3, bottom: Regarding the fact that $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$, again see http://mathworld.wolfram.com for an animation illustrating the fact that $[a] \cdot [b] = [b] \cdot [a]$, where $[a]$ and $[b]$ are the two standard generators of $\pi_1(T^2)$.

p. 8: Note that we never give a careful definition of the "genus" of a surface in this course; we'll simply say (e.g.) "a compact surface of genus g is any surface that can be obtained as the connected sum of g tori. (Cf. Wikipedia: Connected sum and here.)

pp. 8–10: Let us note that the computation of $\pi_1(X)$ which we give here is basically the same as in [3, p. 51 (above Cor. 1.27)], although we do not make use of notions such as cell complexes, wedge sums and homotopy equivalence. (As extracurricular reading we recommend learning about these concepts from Hatcher's book!) Namely: Our open subset $C \subset X$ is homotopy equivalent to the CW complex consisting of a point (viz., a 0-cell) with $2g$ 1-cells attached to it; this is equivalent to a wedge sum of $2g$ circles, and as discussed in [3, Ex. 1.21] van Kampen's Theorem easily implies that its fundamental group is a free group on $2g$ generators; this application of van Kamplen's Theorem is completely analogous to what we do on p. 9. Next our discussion on p. 10 corresponds exactly to the proof of [3, Prop. 1.26(a)], in the special case of attaching a single 2-cell to the 1-skeleton just discussed.

Finally, as extracurricular material, we recommend reading Hatcher's [3, Ch. 1.3] about *covering spaces.* Covering spaces are very closely related to fundamental groups, and they will appear later in the course when we discuss classification of Riemannian manifolds of constant curvature. We give below a brief summary of some of the most pertinent facts from [3, Ch. 1.3], and indicate how they apply to (C^∞ or topological) manifolds, as opposed to general topological spaces.

A *covering space* of a topological space X is a topological space $\widetilde{X}$ together with a continuous map $\pi : \widetilde{X} \to X$ satisfying the following condition: Each point $x \in X$ has an open neighborhood U in X such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by π. It turns out that if M is a topological manifold, then also any connected covering space $\widetilde{M}$ of M is a topological manifold, of the same dimension (Problem 32(a)). Any additional structure carried by M is often inherited by any covering space; for example if M is a C^∞ (or Riemannian) manifold then also $\widetilde{M}$ gets equipped with a natural structure of a C^∞ (resp., Riemannian) manifold, such that π is a local diffeomorphism (resp., local isometry); cf. Problem 32(b),(c).

Two covering spaces $\pi_1 : \widetilde{M_1} \to M$ and $\pi_2 : \widetilde{M_2} \to M$ are said to be *isomorphic* if there is a homeomorphism $h : \widetilde{M_1} \to \widetilde{M_2}$ satisfying $\pi_1 = \pi_2 \circ h$. An isomorphism of a covering $\pi : \widetilde{M} \to M$ with itself is called a *deck transformation.* If M is a C^∞ (or Riemannian) manifold then each deck transformation of $\pi : \widetilde{M} \to M$ is a diffeomorphism (resp., an isometry) of $\widetilde{M}$ onto itself.[6]

Any topological manifold M has a *universal cover,* i.e. a covering space $\pi : \widetilde{M} \to M$ with $\widetilde{M}$ *simply connected.* The universal cover is unique up to isomorphism [3, Prop. 1.37]. If $\pi : \widetilde{M} \to M$ is a universal cover then given any two points $\widetilde{p}_1, \widetilde{p}_2 \in \widetilde{M}$ with $\pi(\widetilde{p}_1) = \pi(\widetilde{p}_2)$, there exists a unique deck transformation of $\pi : \widetilde{M} \to M$ which maps $\widetilde{p}_1$ to $\widetilde{p}_2$. The set of deck transformations of $\pi : \widetilde{M} \to M$ clearly forms a group under composition, and after

[6]Proof: Suppose that M is a C^∞ manifold; then also $\widetilde{M}$ has a natural C^∞ manifold structure, as mentioned above. Suppose that $h : \widetilde{M} \to \widetilde{M}$ is a deck transformation, and let $p \in \widetilde{M}$. Then since $\pi(p) = \pi(h(p))$, there is an open neighborhood U of $\pi(p)$ in M, and two open sets $\widetilde{U}_1, \widetilde{U}_2$ in $\widetilde{M}$, either disjoint or equal, such that $p \in \widetilde{U}_1$, $h(p) \in \widetilde{U}_2$, and π maps each of $\widetilde{U}_1, \widetilde{U}_2$ diffeomorphically onto U (cf. Problem 32(b)). We can assume that U is path-connected. Then by unique lifting property [3, Prop. 1.34], $h_{|\widetilde{U}_1} = (\pi_{|\widetilde{U}_2})^{-1} \circ \pi_{|\widetilde{U}_1}$ (since the two maps agree at the point p, and they are both lifts of the map $\pi_{|\widetilde{U}_1}$). Hence $h_{|\widetilde{U}_1}$ is a diffeomorphism, being a composition of two diffeomorphisms. Since each point $p \in \widetilde{M}$ has such a neighborhood $\widetilde{U}_1$ (and we know from start that h is a homeomorphism of $\widetilde{M}$ onto itself), it follows that h is indeed a diffeomorphism of $\widetilde{M}$ onto itself. The proof in the Riemannian case is completely similar.

fixing a point $\widetilde{p}_0 \in \widetilde{M}$ and setting $p_0 := \pi(\widetilde{p}_0)$, one obtains an *identification* between the group of deck transformations and the fundamental group $\pi_1(M, p_0)$ [3, Prop. 1.39]. In particular, with this identification, $\pi_1(M, p_0)$ is a subgroup of Homeo(M), the group of homeomorphisms of M. Now for any subgroup Γ of $\pi_1(M, p_0)$, the *quotient manifold* $\Gamma\backslash\widetilde{M}$ (cf. Problem 9 [7]) is a covering space of M, and this gives a bijective correspondence between the family of all isomorphism classes of connected covering spaces of M, and the family of conjugacy classes of subgroups of $\pi_1(M, p_0)$ [3, Thm. 1.38]. In particular we have an identification

$$M = \pi_1(M, p_0)\backslash\widetilde{M}.$$

[7]Problem 9 applies, since $\pi_1(M, p_0)$ can be verified to act freely and properly discontinuously on $\widetilde{M}$. Indeed the fact that the action is free is an immediate consequence of the unique lifting property, [3, Prop. 1.34]. In order to prove the proper discontinuity, let $K \subset \widetilde{M}$ be a compact set. Then also $\pi(K)$ is compact, and so $\pi(K)$ can be covered by a finite family of connected open sets $U_1, \ldots, U_n$ such that each U_j has the property that $\pi^{-1}(U_j)$ is a union of disjoint open sets in $\widetilde{M}$ each of which is mapped homeomorphically onto U_j by π. Since K is compact, it follows that K can be covered by a family of open sets $\widetilde{U}_1, \ldots, \widetilde{U}_m$ such that for each $j \in \{1, \ldots, m\}$, $\pi_{|\widetilde{U}_j}$ is a homeomorphism of $\widetilde{U}_j$ onto U_k for some $k = k(j) \in \{1, \ldots, n\}$. Note that for any two $j, j' \in \{1, \ldots, m\}$ with $k(j) = k(j')$, there is exactly one deck transformation $\gamma \in \pi_1(M, p_0)$ satisfying $\gamma(U_j) \cap U_{j'} \neq \emptyset$. (Indeed, take $p \in U_j$ with $\gamma(p) \in U_{j'}$; then $\pi(p) = \pi(\gamma(p))$ since γ is a deck transformation, and thus $\gamma(p) = (\pi_{|U'_j})^{-1} \circ \pi_{|U_j}(p)$. Hence $\gamma_{|U_j} = (\pi_{|U'_j})^{-1} \circ \pi_{|U_j}$ (cf. footnote 6). Therefore if $\gamma, \gamma' \in \pi_1(M, p_0)$ both satisfy $\gamma(U_j) \cap U_{j'} \neq \emptyset$ and $\gamma'(U_j) \cap U_{j'} \neq \emptyset$, then $\gamma'_{|U_j} = \gamma_{|U_j}$, and so by the unique lifting property $\gamma' \equiv \gamma$.) Let us call the deck transformation whose uniqueness we have just proved $\gamma[j, j']$. Then $\{\gamma \in \pi_1(M, p_0) : \gamma(K) \cap K \neq \emptyset\} \subset \{\gamma[j, j'] : j, j' \in \{1, \ldots, m\},\, k(j) = k(j')\}$, which is a finite set. Done!

7. Vector bundles

#7 Vector bundles

Def 1: A (C^∞) <u>vector bundle</u> of <u>rank n</u> is a triple (E, π, M) {often write just "E"} where E and M are C^∞ manifolds and $\pi : E \to M$ is a C^∞ map, and

(1) $\forall x \in M$: $E_x := \pi^{-1}(x)$ {the <u>fiber</u> over x} is equipped with a structure of an n-dim vector space over $\mathbb{R}$;

(2) $\forall x \in M$: There exist an open set $U \subset M$ with $x \in U$ and a diffeomorphism $\varphi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$ such that $\forall y \in U$: $\varphi_y := \varphi_{|E_y}$ is a <u>bijection $E_y \xrightarrow{\sim} \{y\} \times \mathbb{R}^n$</u> with <u>$pr_2 \circ \varphi_y$ linear</u>

Any such pair (U, φ) is called a <u>bundle chart</u>.

$$
\begin{tikzcd}
\pi^{-1}(U) = \coprod_{x \in U} E_x \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}
$$

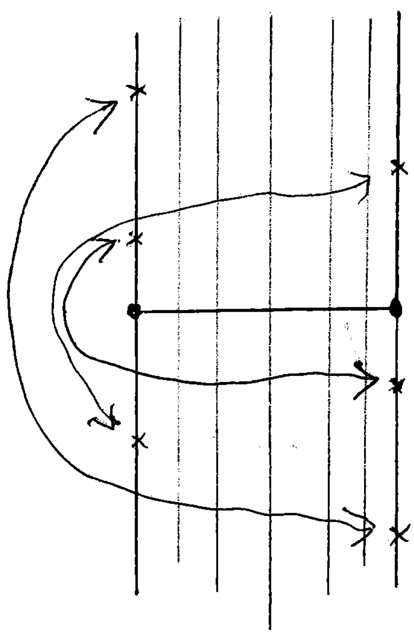
{Thus: <u>Locally</u>, <u>$\pi : E \to M$</u> can be identified with <u>$M \times \mathbb{R}^n \xrightarrow{pr_1} M$</u>

Ex
- $\underline{\underline{TM}}$ is a vector bundle over M of rank $= \dim M$.

- $\underline{E = M \times \mathbb{R}^n}$ is a vector bundle over M of rank n
  — <u>trivial</u> vector bundle.

- <u>Möbius bundle over S^1</u>: $E = [0,1] \times \mathbb{R} / \sim$

where $(a,b) \sim (c,d) \overset{\text{def}}{\iff} \left[\{a,c\} = \{0,1\} \text{ and } b = -d \right]$



$\pi : E \to S^1$

$\pi(a,b) = \left(\cos(2\pi a), \sin(2\pi a) \right)$

This vector bundle is <u>not</u> trivial.

— see Problem 37.

More generally, a (C^∞) <u>fiber bundle</u> consists of a C^∞ map $\pi : E \to M$ (where E, M are C^∞ manifolds) and a C^∞ manifold F ← the <u>standard fiber</u>

such that for every $x \in M$ there exist an open set $U \subset M$ with $x \in U$ and a diffeomorphism $\varphi : \pi^{-1}(U) \xrightarrow{\sim} U \times F$ such that $pr_1 \circ \varphi = \pi$.

Thus <u>locally</u> $\pi : E \to M$ looks like $M \times F \xrightarrow{pr_1} M$

In this language, a <u>vector bundle of rank n</u> is the same as a <u>fiber bundle with standard fiber $\mathbb{R}^n$</u> and <u>structure group $GL_n(\mathbb{R})$</u> ← we don't explain this now - may return to it later.

Def 2: Let (E, π, M) be a vector bundle. A **section** of E is a C^∞ map $s: M \to E$ with $\pi \circ s = 1_M$. The set of all sections of E is called ΓE.

$$\begin{tikzcd} E \arrow[d, "\pi"', shift right] \\ M \arrow[u, "s"', shift right, bend right] \end{tikzcd}$$

Thus: To give a section $s \in \Gamma E$ means choosing one vector $((s(p) \in E_p))$ in the fiber over p, for each $p \in M$.

Note: ΓE is a $C^\infty M$-module!

write $\mathcal{F} := C^\infty M$

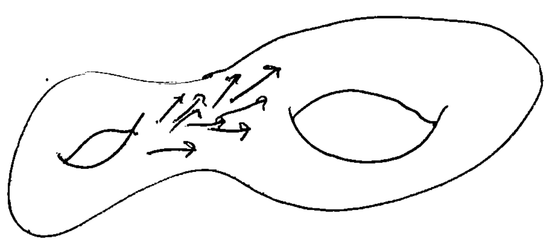
Namely, for any $s_1, s_2 \in \Gamma E$, also $s_1 + s_2 \in \Gamma E$

pointwise addition

and for any $f \in \mathcal{F}$ and $s \in \Gamma E$, also $fs \in \Gamma E$.

pointwise multiplication

Ex: A section $s \in \Gamma(TM)$ is called a vector field.



Ex: $\Gamma(M \times \mathbb{R}) = C^\infty M$.

Def 3: A <u>bundle homomorphism</u> between two vector bundles over M, (E_1, π_1, M) and (E_2, π_2, M), is a C^∞ map $h : E_1 \to E_2$ such that $\pi_2 \circ h = \pi_1$ {i.e. h is <u>fiber preserving</u>} and $h_p := h|_{E_{1,p}} : E_{1,p} \to E_{2,p}$ is <u>linear</u>, $\forall p \in M$.

$$
\begin{tikzcd}
E_1 \arrow[rr, "h"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\
& M &
\end{tikzcd}
$$

– Now it is also clear what a <u>bundle isomorphism</u> is!

(<u>Ex</u>: On p.1, $\varphi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$ is a bundle isomorphism)

Def 4: A <u>subbundle</u> of a vector bundle (E, π, M) of rank n is a subset $E' \subset E$ such that for every $p \in M$ there is a bundle chart (U, φ) for E such that $p \in U$ and $\underline{\varphi(E' \cap \pi^{-1}(U)) = U \times \mathbb{R}^m}$ for some $m \leq n$.

{view $\mathbb{R}^m = \{(*, \dots, *, 0, \dots, 0)\} \subset \mathbb{R}^n$}

– see <u>Problem 41</u>.

Next we will define pull-back of a vector bundle.

This we will use a lot later, e.g. pullback of TM to a geodesic!

Let $f: M \to N$ be a C^∞ map (with M, N C^∞ manifolds) and let (E, π, N) be a vector bundle.
We will define a vector bundle $(f^*E, \tilde{\pi}, M)$.

$$\begin{array}{ccc} f^*E & & E \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ M & \xrightarrow{f} & N \end{array}$$

Intuitively:
"$f^*E := \bigsqcup_{p \in M} E_{f(p)}$"

– however this is ok only if f is injective.

Def 5: As a set, $f^*E := \{(p,v) : p \in M, v \in E_{f(p)}\} \subset M \times E$, and $\tilde{\pi} = pr_1 : f^*E \to M$.
(Thus $(f^*E)_p = \cancel{\{p\}} \times E_{f(p)}$ $(\forall p \in M)$.)

Bundle charts: For any bundle chart (U, φ) for (E, π, N), $(f^{-1}(U), \tilde{\varphi})$ is a bundle chart for $(f^*E, \tilde{\pi}, M)$,
where $\tilde{\varphi} : \tilde{\pi}^{-1}(f^{-1}(U)) \xrightarrow{\sim} f^{-1}(U) \times \mathbb{R}^n$

$$\tilde{\varphi}(p,v) := (p, pr_2(\varphi(v)))$$

well-defined: see Problem 42

Special case: If M is an immersed submanifold of N, i.e. $f: M \hookrightarrow N$ is an injective immersion, then f^*E can be defined as $\coprod_{p \in M} E_{f(p)}$. We then also write $E_{|M}$ for f^*E (at least if $M \subset N$ and f = inclusion map.)

Very common: $E_{|U}$ for U open $\subset N$.

Ex: More general notion of "bundle homomorphism".

For $f: M \to N$ C^∞ as above, and given vector bundles (E_1, π_1, M) and (E_2, π_2, N), a bundle homomorphism along f is a C^∞ map $h: E_1 \to E_2$ such that $\pi_2 \circ h = f \circ \pi_1$ and $h_p := h_{|E_{1,p}} : E_{1,p} \to E_{2,f(p)}$ is linear, $\forall p \in M$.

$$\begin{tikzcd} E_1 \arrow[r, "h"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Example: $df: TM \to TN$ is a bundle homomorphism along f!

Now, there is a canonical bijection between bundle homomorphisms $E_1 \to E_2$ along f, and bundle homomorphisms $E_1 \to f^*E_2$

See problems 44 & 45

(Namely $h \mapsto \tilde{h}$, $\tilde{h}(v) = (\pi_1(v), h(v)) \in f^*E_2$, $\forall v \in E_1$)

~~Special case: If M is an immersed subbundle of N, i.e. $f: M \hookrightarrow N$ is an injective immersion, then f^*E can be defined as $\bigsqcup_{p\in M} E_{f(p)}$. We also write $E_{|M}$ for f^*E (at least if $M \subset N$ and f = inclusion map)~~

~~Very common: U open $\subset N$; then $E_{|U}$.~~

Def. 6: If (E_1, π_1, M) and (E_2, π_2, M) are vector bundles over M, then $E_1 \oplus E_2$, $E_1 \otimes E_2$, E_1^*, $\mathrm{Hom}(E_1, E_2)$ are also vector bundles over M.

Precise definition in the case of $E_1 \otimes E_2$:

As a set, $E_1 \otimes E_2 := \bigsqcup_{p\in M} (E_{1,p} \otimes E_{2,p})$,

with $\pi: E_1 \otimes E_2 \to M$, $\pi(v) = p$, $\forall v \in E_{1,p} \otimes E_{2,p}$.

If (U, φ_j), $j = 1, 2$ are bundle charts for E_1, E_2, then set $\tau: \pi^{-1}(U) \to U \times \underbrace{(\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})}_{\text{"}= \mathbb{R}^{n_1 n_2}\text{"}}$ (n_j = rank E_j)

$$\tau(v) := (p, (\varphi_{1,p} \otimes \varphi_{2,p})(v)), \quad \forall p \in U, \ v \in E_{1,p} \otimes E_{2,p}$$

Then (U, τ) is a bundle chart for $E_1 \otimes E_2$.

(thus $\tau_p = \varphi_{1,p} \otimes \varphi_{2,p}$)

Details: Problem 39

Similarly also: $T^k(E)$, $\Lambda^k(E)$.

Note: Canonical identifications which hold for vector spaces (finite dimensional, over $\mathbb{R}$), immediately carry over to vector bundles (since they apply fiber by fiber):

$$(E^*)^* = E,$$

$$\mathrm{Hom}(E_1, E_2) = E_1^* \otimes E_2$$

$$(E_1 \otimes E_2)^* = E_1^* \otimes E_2^*$$

$$E_1 \otimes (E_2 \oplus E_3) = (E_1 \otimes E_2) \oplus (E_1 \otimes E_3)$$

Note: $s \in \Gamma(\mathrm{Hom}(E_1, E_2)) \iff \begin{bmatrix} s \text{ is a bundle} \\ \text{homomorphism } E_1 \to E_2 \end{bmatrix}$

– see Problem 40

Theorem 1: "$E \mapsto \Gamma E$" is a functor from the category of vector bundles over M to the category of $C^\infty M$-modules. (This functor takes any bundle homomorphism $h: E_1 \to E_2$ to the $C^\infty M$-linear map $h_*: \Gamma E_1 \to \Gamma E_2$; $h_*(s) := h \circ s$.) It satisfies

$\Gamma(E_1 \oplus E_2) = \Gamma E_1 \oplus \Gamma E_2$, $\quad \Gamma(E^*) = (\Gamma E)^*$,

$\Gamma(E_1 \otimes E_2) = \Gamma E_1 \otimes \Gamma E_2$, $\quad \Gamma(\mathrm{Hom}(E_1, E_2)) = \mathrm{Hom}(\Gamma E_1, \Gamma E_2)$.

Above, all "$=$" really stand for "canonical isomorphism of $C^\infty M$-modules."

<u>IMPORTANT</u>: Hom, $\otimes$, etc. on spaces of sections are <u>operations</u> on $C^\infty M$-modules! In particular:

- $\Gamma E_1 \otimes \Gamma E_2$ means $\Gamma E_1 \otimes_{C^\infty M} \Gamma E_2$
- $\mathrm{Hom}(\Gamma E_1, \Gamma E_2)$ is the $C^\infty M$-module of all <u>$C^\infty M$-linear</u> maps $\Gamma E_1 \to \Gamma E_2$.

See Problem 43.

<u>Ex. of the above formalism</u>:

To give a section $\alpha \in \Gamma(E^*)$ is the same thing as giving $\alpha \in \Gamma(E)^*$, i.e. <u>a $C^\infty M$-linear form $\Gamma(E) \to C^\infty M$.</u>

$$\forall s \in \Gamma(E): \quad \underline{(\alpha, s)(p) = (\alpha(p), s(p))} \in \mathbb{R} \quad (\forall p \in M).$$

Also if E, E_1, E_2 are vector bundles over M, there is a ($C^\infty M$-linear) <u>contraction</u> map

$$\underline{\Gamma(E \otimes E_1 \otimes E^* \otimes E_2) \longrightarrow \Gamma(E_1 \otimes E_2)}$$

This comes from the $C^\infty M$-multilinear map

$$\Gamma(E)^* \times \Gamma(E) \times \Gamma(E_1 \otimes E_2) \longrightarrow \Gamma(E_1 \otimes E_2)$$

$$\langle \alpha, s, \sigma \rangle \longmapsto \underbrace{(\alpha, s)}_{\in C^\infty M} \cdot \sigma$$

7.1. **Notes.** .

In Lectures #7 and #8 we wish to cover the material in [5, Sec. 2.1], up to and including [5, Thm. 2.1.5].

p. 7, Def. 6: Cf. [5, Def. 2.1.8] and below. Note that we write "$E_1 \oplus E_2$" (which is also standard notation) for the vector bundle which Jost calls "$E_1 \times E_2$" ("product bundle"). This is also called the "direct sum bundle" or "Whitney sum" of E_1 and E_2.

7.2. **Review of tensor products and exterior algebra.** .

In the following we work in the setting of *R-modules,* where R is an arbitrary (fixed) commutative ring[8]. Recall that if R is a field then any R-module is a vector space over R. In the course, we will need the theory developed below in for two choices of R, namely $R = \mathbb{R}$ (the field of real numbers), and $R = C^\infty(M)$ (the ring of C^∞ functions $M \to \mathbb{R}$).

Tensor product. (Cf. Lang [8, Ch. XVI.1-2].)

Prop 1. *Given any two R-modules V, W there exists an R-module*

$$\text{“}V \otimes W\text{”}$$

and an R-bilinear map

$$\varphi : V \times W \to V \otimes W$$

such that for any R-module Z and any R-bilinear map $h : V \times W \to Z$, there exists a unique R-linear map $g : V \otimes W \to Z$ such that $h = g \circ \varphi$ [9]. The pair $\langle V \otimes W, \varphi \rangle$ is unique in the following sense: If $\langle \widetilde{V \otimes W}, \widetilde{\varphi} \rangle$ is another pair satisfying the same conditions, then there exists a unique isomorphism of R-modules, $J : \widetilde{V \otimes W} \xrightarrow{\sim} V \otimes W$, such that $\varphi = J \circ \widetilde{\varphi}$.

More generally, given n R-modules $V_1, \ldots, V_n$, there exists an R-module $V_1 \otimes \cdots \otimes V_n$ and an R-bilinear map $\varphi : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$ with the completely analogous properties as in the case $n = 2$ described above.

For a proof of Prop. A see e.g. [8, Ch. XVI.1]. The standard construction of a tensor product $V \otimes W$ is to define it to be the quotient M/N, where M is the free R-module generated by the set $V \times W$, and N is the R-submodule of M generated by all the elements

$$\begin{aligned}
&(v + v', w) - (v, w) - (v', w) \\
&(v, w + w') - (v, w) - (v, w') \\
&(av, w) - a(v, w) \\
&(v, aw) - a(v, w)
\end{aligned}$$

for all $v, v' \in V$, $w, w' \in W$, $a \in R$. We also remark that the uniqueness statement in Proposition A is proved by standard “abstract nonesense”.

In the above situation, we write “$v \otimes w$” for $\varphi(v, w)$ (for any $v \in V$, $w \in W$). An element of $V \otimes W$ that can be written in the form $v \otimes w$ is called a *pure tensor.* A general element in $V \otimes W$ can always be expressed (in a non-unique way) as a finite sum of pure tensors.

---

[8]We always assume that R has a multiplicative identity, 1. Of course R also has an identity element for addition, 0. (And $0 \neq 1$, for non-triviality.)

[9]This property is called the universal property of the tensor product.

Using the universal property of the tensor product one easily proves that there exists a unique isomorphism of R-modules

$$V \otimes W \xrightarrow{\sim} W \otimes V$$

mapping $v \otimes w \mapsto w \otimes v$ for all $v \in V$, $w \in W$ [8, Prop. 1.2]. Similarly, if also U is an R-module, then there exist unique isomorphism

$$U \otimes (V \otimes W) \xrightarrow{\sim} (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes V \otimes W$$

mapping $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$ and $(u \otimes v) \otimes w \mapsto u \otimes v \otimes w$, respectively ($\forall u \in U, v \in V, w \in W$) [8, Prop. 1.1]. In view of the last fact, we will often identify the three R-modules $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$ and $U \otimes V \otimes W$.

Next, if V, W, X, Y are R-modules, and $f : V \to X$ and $g : W \to Y$ are R-linear maps, then there exists a unique R-linear map

$$f \otimes g : V \otimes W \to X \otimes Y$$

satisfying

$$(f \otimes g)(v \otimes w) = f(v) \otimes g(w) \qquad (\forall v \in V,\ w \in W).$$

(Cf. [8, pp. 605–606], where this map is first denoted "$T(f, g)$".) This construction satisfies the following "functoriality property": If also $h : X \to Z$ and $i : Y \to U$ are R-linear maps then

$$(h \otimes i) \circ (f \otimes g) = (h \circ f) \otimes (i \circ g).$$

(Obviously we also have $1_V \otimes 1_W = 1_{V \otimes W}$, where 1_U denotes the identity map on the R-module U. In this way, the tensor product becomes a *bifunctor* from the category of R-modules to itself, covariant in both arguments.)

Let us recall some facts which hold when V and W are *free and finite dimensional over* R (in particular these facts hold for finite dimensional vector spaces over $\mathbb{R}$, or over any other field):

Prop 2. *Let V and W be free and finite dimensional modules over R. Then:*
(a) $V \otimes W$ is also free and finite dimensional over R, and if V and W have bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$, respectively, then $V \otimes W$ has a basis

$$\{v_i \otimes w_j \ : \ i \in \{1, \ldots, n\},\ j \in \{1, \ldots, m\}\}.$$

(Thus $\dim(V \otimes W) = (\dim V)(\dim W)$*.)*
(b) There is a natural isomorphism of R-modules $V^* \otimes W \cong Hom(V, W)$, [10] *under which $\alpha \otimes w \in V^* \otimes W$ corresponds to $v \mapsto \alpha(v)w$ in $Hom(V, W)$.*
(c) There is also a natural isomorphism of R-modules $V^* \otimes W^* \cong (V \otimes W)^*$, under which $\alpha \otimes \beta \in V^* \otimes W^*$ corresponds to the element in $(V \otimes W)^*$ which maps $v \otimes w$ to $\alpha(v)\beta(w)$ for all $v \in V$, $w \in W$.

(Cf. [8, Cor. 2.4 and Cor. 5.5, Cor. 5.6].)

[10] Here $\mathrm{Hom}(V, W)$ is the R-module of R-linear maps $V \to W$.

The tensor algebra. (Cf. [8, Ch. XVI.7].) Let V be an R-module as before. Then for each integer $r \geq 0$, we let

$$T^r(V) := \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}} \text{ (if } r \geq 1), \qquad \text{and } T^0(V) = R.$$

The *tensor algebra* of V is defined to be the direct sum

$$T(V) := \bigoplus_{r=0}^{\infty} T^r(V).$$

Thus, $T(V)$ as a set consists of all infinite sequences $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ such that $\alpha_r \in T^r(V)$ for each r and $\alpha_r = 0$ for all except finitely many r's. $T(V)$ is an R-module, where addition and R-multiplication is defined "entry by entry". We often write "$\alpha_0 + \alpha_1 + \alpha_2 + \cdots$" in place of $(\alpha_0, \alpha_1, \alpha_2, \ldots)$, and in that sum we can leave out any term α_r which is 0. Note that there is a natural R-bilinear map

$$T^r(V) \times T^s(V) \to T^{r+s}(V), \qquad (\alpha, \beta) \mapsto \alpha \otimes \beta. \tag{2}$$

This map extends by R-linearity to endow $T(V)$ with the structure of a ring (where the multiplication operation is denoted "$\otimes$"); thus $T(V)$ is an *R-algebra.* (In fact $T(V)$ is a graded R-algebra, exactly since it can be written as a direct sum of R-submodules $T^0(V), T^1(V), T^2(V), \ldots$ satisfying $T^r(V) \otimes T^s(V) \subset T^{r+s}(V)$ for all $r, s \geq 0$.)

The exterior algebra. We here only discuss the exterior algebra of V^* for V a finite dimensional vector space over $\mathbb{R}$, since this is the case that is most relevant for differential forms – and it seems to be the only case which we will be concerned with in this course. Details can be found in, e.g., Boothby [2, Ch. V.5–6] and Lee, [10, Ch. 14].

Note that outside of differential geometry – and always when dealing with general modules – one most often defines $\bigwedge^r(V^*)$ differently, namely as a certain *quotient* (as opposed to a subspace) of $T^r(V^*)$. Also it is then more natural to speak directly about $\bigwedge^r(V)$ rather than $\bigwedge^r(V^*)$. Anyway the definitions can be shown to be equivalent in the case of free modules of finite dimension (thus in particular for finite dimensional vector spaces over $\mathbb{R}$), *except* that there exist different conventions for the normalizing factor in (3). This is carefully explained in [10, Ch. 14]. Cf. also Lang [8, Ch. XIX.1 (esp. Exercise 3)].

Let V be a finite dimensional vector space over $\mathbb{R}$. By the definition of tensor product together with Prop. 2(c) (extended to r-fold tensor products), the space $T^r(V^*)$ can be identified with the space of multilinear forms

$$F : V^{(r)} := \underbrace{V \times \cdots \times V}_{r \text{ times}} \longrightarrow \mathbb{R}.$$

Under this identification, $\alpha_1 \otimes \cdots \otimes \alpha_r \in T^r(V^*)$ corresponds to the multilinear form

$$F(v_1, \ldots, v_r) = \prod_{j=1}^{r} \alpha_j(v_j), \qquad \forall \langle v_1, \ldots, v_r \rangle \in V^{(r)}.$$

Note also that under our identification, the product operation $T^r(V^*) \times T^s(V^*) \to T^{r+s}(V^*)$ (cf. (2)) is given by

$$(F_1 \otimes F_2)(v_1, \ldots, v_{r+s}) = F_1(v_1, \ldots, v_r) F_2(v_{r+1}, \ldots, v_{r+s})$$

(also when F_1, F_2 do not correspond to pure tensors).

Now we define $\bigwedge^r(V^*)$ to be the subspace of *alternating* forms in $T^r(V^*)$, i.e. forms $F \in T^r(V^*)$ such that $F(v_1, \ldots, v_r) = 0$ whenever $v_i = v_j$ for some $i \neq j$. In particular $\bigwedge^0(V^*) = \mathbb{R}$ and $\bigwedge^1(V^*) = V^*$. We also define the *exterior algebra of* V^*, to be the direct sum

$$\bigwedge(V^*) := \bigoplus_{r=0}^{\infty} \bigwedge^r(V^*).$$

(In fact this sum turns out to be finite, since $\Lambda^r(V^*) = \{0\}$ whenever $r > \dim V$; it's a nice exercise to prove this fact already here; cf. also Prop. 3 below.) Thus $\bigwedge(V^*)$ is a linear subspace of the tensor algebra $T(V^*)$; however it is certainly *not* a subalgebra of $T(V^*)$, since typically $F \otimes G \notin \bigwedge(V^*)$ even if $F, G \in \bigwedge(V^*)$. Instead we will introduce a different product operation, "$\wedge$" ("wedge product"), on $\bigwedge(V^*)$.

Let $\mathfrak{S}_r$ be the group of permutations of $\{1, \ldots, r\}$. For $\sigma \in \mathfrak{S}_r$ and $F \in T^r(V^*)$ we define the form $\sigma \cdot F \in T^r(V^*)$ by

$$(\sigma \cdot F)(v_1, \ldots, v_r) := F(v_{\sigma(1)}, \ldots, v_{\sigma(r)}), \qquad \forall \langle v_1, \ldots, v_r \rangle \in V^{(r)}.$$

Then we have that $F \in T^r(V^*)$ is alternating if and only if $\sigma \cdot F = (\operatorname{sgn} \sigma) F$ for all $\sigma \in \mathfrak{S}_r$. We define the following linear map:

$$\mathcal{A} : T^r(V^*) \to \textstyle\bigwedge^r(V^*); \qquad \mathcal{A}(F) = \frac{1}{r!} \displaystyle\sum_{\sigma \in \mathfrak{S}_r} (\operatorname{sgn} \sigma)\, (\sigma \cdot F).$$

One verifies that $\mathcal{A}$ is indeed a linear map from $T^r(V^*)$ into $\bigwedge^r(V^*)$, and that $\bigwedge^r(V^*)$ is exactly the set of those $F \in T^r(V^*)$ satisfying $\mathcal{A}(F) = F$. Using the map $\mathcal{A}$, we now define the following product operation "$\wedge$" ("wedge product"), for any $r, s \geq 0$:

$$\begin{aligned} &\textstyle\bigwedge^r(V^*) \times \bigwedge^s(V^*) \to \bigwedge^{r+s}(V^*), \\ &\langle F_1, F_2 \rangle \mapsto F_1 \wedge F_2 := \frac{(r+s)!}{r!s!} \mathcal{A}(F_1 \otimes F_2). \end{aligned} \tag{3}$$

This map extends by $\mathbb{R}$-linearity It is more or less immediate that this product operation is $\mathbb{R}$-bilinear, hence it has a unique extension to an $\mathbb{R}$-bilinear map

$$\textstyle\bigwedge(V^*) \times \bigwedge(V^*) \to \bigwedge(V^*).$$

By a somewhat longer computation one also verifies that $\wedge$ is *associative.* (In fact one finds that

$$(F_1 \wedge F_2) \wedge F_3 = \frac{(r+s+t)!}{r!s!t!} \mathcal{A}(F_1 \otimes F_2 \otimes F_3) = F_1 \wedge (F_2 \wedge F_3)$$

for all $F_1 \in \bigwedge^r(V^*)$, $F_2 \in \bigwedge^s(V^*)$, $F_3 \in \bigwedge^t(V^*)$.) Hence $\bigwedge(V^*)$ with the multiplication operation "$\wedge$" is an associative graded $\mathbb{R}$-algebra.

Prop 3. *If $n = \dim V$ and $\beta_1, \ldots, \beta_n$ is any basis for V^* then $\bigwedge^r(V^*) = \{0\}$ for all $r > n$, while for $0 \leq r \leq n$ one has $\dim \bigwedge^r(V^*) = \binom{n}{r}$ and a basis for $\bigwedge^r(V^*)$ is given by*

$$\{\beta_{i_1} \wedge \cdots \wedge \beta_{i_r} \;:\; 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}.$$

In particular $\bigwedge^n(V^*)$ is 1-dimensional and spanned by

$$\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n.$$

Explicitly this form is given by:

$$\big[\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n\big](v_1, \ldots, v_n) = \det(\beta_i(v_j))_{i,j} = \begin{vmatrix} \beta_1(v_1) & \cdots & \beta_1(v_n) \\ \vdots & & \vdots \\ \beta_n(v_1) & \cdots & \beta_n(v_n) \end{vmatrix}.$$

8. Vector bundles; exterior calculus

#8. Vector bundles; exterior calculus

Def 1: Let M be a C^∞ manifold. Then
$\underline{\underline{T^*M}} := \underline{(TM)^*}$ $\left(\text{thus } T_p^*M = (TM)_p^* = (T_pM)^*,\ \forall p \in M\right)$.

Elements of T_p^*M are called <u>cotangent vectors</u>.
Sections of T^*M are called <u>1-forms</u> (= "cotangent vector fields").

For $r \geq 0$, $s \geq 0$: $\underline{\underline{T_s^r(M)}} := \underbrace{TM \otimes \cdots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s \text{ times}}$.

Sections of $T_s^r(M)$ are called <u>"r times contravariant, s times covariant tensor (fields) on M"</u>

<u>Ex</u>: For any C^∞ map $f: M \to \mathbb{R}$, $\underline{df \in \Gamma(T^*M)}$

namely, $\forall p \in M$: $df(p) = df_p$ is a linear map $T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$, hence $df(p) \in T_p^*M$.

<u>Ex</u>: A Riemannian metric is a tensor field $\underline{\underline{m \in \Gamma T_2^0(M)}}$ of a special kind (namely m has to be symmetric and positive definite – meaning that m_p is symmetric and positive definite, $\forall p \in M$).

Indeed, view $m \in \Gamma((TM \otimes TM)^*)$, meaning $\forall p \in M$: m_p "is" a bilinear map $T_pM \times T_pM \to \mathbb{R}$; and we write $\underline{\underline{\langle v, w \rangle := m_p(v \otimes w)}}$, $\forall v, w \in T_pM$.

Related fact: A Riemannian metric equips us with a singled out bundle isomorphism

$$TM \xrightarrow{\sim} T^*M, \quad v \mapsto v^b$$

defined by

$$v^b(w) := \langle v, w \rangle, \quad \forall w \in T_pM \quad (\text{for } v \in T_pM)$$

Inverse map:

$$T^*M \xrightarrow{\sim} TM, \quad \omega \mapsto \omega^\#;$$

thus $\langle \omega^\#, u \rangle = \omega(u)$, $\forall u \in T_pM$ (for $\omega \in T_p^*M$).

In local coordinates, if $v = v^j \frac{\partial}{\partial x^j}$ then $v^b = v_j \, dx^j$ with $v_j = g_{kj} v^k$. One says v^b is obtained from v by lowering an index. Similarly $\omega \mapsto \omega^\#$ raising an index.

Ex: For $f \in C^\infty(M)$, $\mathrm{grad}(f) =: \nabla f := (df)^\# \in \Gamma(TM)$

(see Jost, p. 89-90)

Aside now, important later:

Def 2: A bundle metric on a vector bundle (E, π, M) is a section $m \in \Gamma((E \otimes E)^*)$ which is symmetric and positive definite (we again write $\langle v, w \rangle$ for $m_p(v \otimes w)$, $\forall v, w \in E_p$, $p \in M$)

In local coordinates

Given a chart (U, x) for M, recall that at each $p \in U$, $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}$ is a basis for T_pM.

Hence $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}$ is a basis of sections in $\Gamma(TU) = \Gamma(TM_{|U})$, and for every vector field $X \in \Gamma(TM)$, $\exists!\ f^1, \dots, f^d \in C^\infty(U)$ such that $X_{|U} = f^j \frac{\partial}{\partial x^j} \in \Gamma(TU)$

see Problems 33, 34

Also $dx^1, \dots, dx^d$ is a basis of sections in $\Gamma(T^*U)$, indeed $dx^j\left(\frac{\partial}{\partial x^k}\right) = \delta_{j,k}$, i.e. $dx^1, \dots, dx^d$ is the dual basis of $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}$ at each $p \in U$.

Transformation formulas

If (V, y) is another chart for M, then

$$\underbrace{\frac{\partial}{\partial x^j}}_{\in \Gamma(TU)} = \underbrace{\frac{\partial y^k}{\partial x^j}}_{\in C^\infty(U \cap V)} \cdot \underbrace{\frac{\partial}{\partial y^k}}_{\in \Gamma(TV)} \quad \text{in } U \cap V.$$

we noted this in Lecture #2, "at each $p \in U \cap V$"

and

$$\underbrace{dx^j}_{\in \Gamma(T^*U)} = \underbrace{\frac{\partial x^j}{\partial y^k}}_{\in C^\infty(U \cap V)} \underbrace{dy^k}_{\in \Gamma(T^*V)} \quad \text{in } U \cap V.$$

(Check: This gives $dx^j\left(\frac{\partial}{\partial x^l}\right) = \left(\frac{\partial x^j}{\partial y^k}\, dy^k\right)\left(\frac{\partial y^i}{\partial x^l}\cdot\frac{\partial}{\partial y^i}\right)$

$$= \frac{\partial x^j}{\partial y^k}\frac{\partial y^i}{\partial x^l}\cdot\delta^k_i = \frac{\partial x^j}{\partial y^k}\cdot\frac{\partial y^k}{\partial x^l} = \delta^j_l,$$ as it should!)

~~Note~~ It also follows that, e.g.,

$dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$ $\quad (i,j,k,l \in \{1,\dots,d\})$

is a basis of sections in $\Gamma(T^1_3 U)$.

Hence for any ~~vector~~ ~~field~~ tensor (field) $A \in \Gamma(T^1_3 M)$ there exist functions $A^l_{ijk} \in C^\infty(U)$ such that

$$A_{|U} = A^l_{ijk}\cdot dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$$

later example: the curvature tensor!

And then in $U \cap V$ we get

$$A_{|U\cap V} = \underbrace{A^l_{ijk}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\frac{\partial x^k}{\partial y^c}\frac{\partial y^d}{\partial x^l}}_{=:\tilde{A}^d_{abc}}\, dy^a \otimes dy^b \otimes dy^c \otimes \frac{\partial}{\partial y^d}$$

$=: \tilde{A}^d_{abc}$, the coefficients of A w.r.t. (V,y)

Def 3: A "weak algebra bundle" is a vector bundle (E, π, M) together with a "multiplication rule" i.e. a vector bundle homomorphism $m: E \otimes E \to E$.

Thus for any $p \in M$ we have an $\mathbb{R}$-bilinear map

$$E_p \times E_p \to E_p$$
$$\langle v, w \rangle \mapsto v \cdot w := m(v \otimes w)$$

one uses different notation for this product depending on context, e.g. $[v, w]$, $v \otimes w$, $v \wedge w$, ...

Note: Also for any $s_1, s_2 \in \Gamma E$, get $s_1 \cdot s_2 \in \Gamma E$, defined by $(s_1 \cdot s_2)(p) := s_1(p) \cdot s_2(p)$.

Examples: Let (E, π, M) be an arbitrary vector bundle.

- $End(E) := Hom(E, E)$ is a ((weak)) algebra bundle with product $\langle S, T \rangle \mapsto S \circ T \in End\, E_p$ (for $S, T \in End\, E_p$)

  we often write simply "ST" for $S \circ T$!

- The vector bundle $End(E)$ also has another natural ((weak)) algebra bundle structure, namely

  $$[S, T] := ST - TS \quad \text{for } S, T \in End\, E_p$$

  This algebra bundle is called $gl(E)$, it is a Lie algebra bundle.

- The <u>tensor algebra bundle</u>

$$T(E) := \underbrace{T^0(E)}_{C^\infty(M)} \oplus \underbrace{T^1(E)}_{E} \oplus \underbrace{T^2(E)}_{E \otimes E} \oplus \cdots$$

is an ∞-dim algebra bundle, product rule $\otimes$.

- The <u>exterior algebra bundle</u>

$$\Lambda(E) = \underbrace{\Lambda^0(E)}_{C^\infty(M)} \oplus \underbrace{\Lambda^1(E)}_{E} \oplus \Lambda^2(E) \oplus \cdots$$

To follow the notes in Sec. 7.2, view $\Lambda(E) := \Lambda((E^*)^*)$; thus each $\alpha \in \Lambda^r(E)_p$ $(p \in M)$ is an alternating multilinear form $\underbrace{E_p^* \times \cdots \times E_p^*}_{r \text{ copies}} \to \mathbb{R}$

$\underline{\text{rank } \Lambda(E)} = 1 + n + \binom{n}{2} + \cdots + \binom{n}{n} + 0 + 0 + \cdots = 2^n$,

where $n = \text{rank } E$.

Product rule $\wedge$. <u>exterior product</u> or <u>wedge product</u>.

<u>Def 4:</u> $\Lambda^r(M) := \Lambda^r(T^*M)$

$\Lambda(M) := \Lambda(T^*M) = \bigoplus_{r=0}^{\infty} \Lambda^r(T^*M)$ for $d = \dim M$

$\Omega^r(M) := \Gamma \Lambda^r(T^*M)$ such sections are called <u>r-forms</u> on M

$\Omega(M) := \Gamma \Lambda(M) = \bigoplus_{r=0}^{\infty} \Omega^r(M)$.

<u>In local coordinates:</u> For (U, x) a chart on M, a basis of sections in $\Gamma(\Lambda^r(M)_{|U}) = \Omega^r(U)$ is given by

$$dx^I := dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

where I runs through all r-tuples in $\{1, \dots, d\}^r$ with $i_1 < i_2 < \dots < i_r$.

Thus: Any $\omega \in \Omega^r(U)$ can be uniquely expressed as $\omega = \sum_I \omega_I \, dx^I$ with $\omega_I \in C^\infty(U)$.

Generalization of $\Omega^r(M)$:

$\Omega^r(E) := \Gamma(E \otimes \Lambda^r M)$ for any vector bundle (E, π, M)

– see Problem 49

Pullback of covariant tensors

"intro": Given a C^∞-map $f: M \to N$ we have

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ M & \xrightarrow{f} & N \end{array}$$

but there is in general no "pushforward" by f of a vector field $X \in \Gamma(TM)$ to $\Gamma(TN)$.

(Instead we get $df \circ X \in \Gamma_f(TN) = \Gamma(f^*TN)$.)

However, we can use the dual of df to pullback any cotangent vector field on N to M.

$$\begin{array}{ccc} T^*M & \xleftarrow{(df)^*} & T^*N \\ \downarrow \pi_1 & & \uparrow\downarrow \pi_2 \\ M & \longrightarrow & N \end{array}$$

Def 5: Let $f: M \to N$ be a C^∞ map.

For $w \in \Gamma(T^*N)$, define $f^*(w) \in \Gamma(T^*M)$

by $f^*(w)_p(v) := w_{f(p)}(df_p(v))$ $\quad \forall p \in M, \ v \in T_pM$.

Here I write "$w_{f(p)}$" for $w(f(p)) \in T^*_{f(p)}N$, and "$f^*(w)_p$" for $f^*(w)(p) \in T_p^*M$.

More generally for any $w \in \Gamma(T_s^0(N))$, define $f^*(w) \in \Gamma(T_s^0(M))$ by

$$f^*(w)_p(v_1, \dots, v_s) := w_{f(p)}(df_p(v_1), \dots, df_p(v_s)) \qquad \forall p \in M, \ v_1, \dots, v_s \in T_pM.$$

Also for $w \in C^\infty(N) = $ "$\Gamma(T_0^0(N))$", define

$f^*(w) := w \circ f \in C^\infty(M)$.

Extending by linearity, get a map on sections of the full tensor algebra of covariant tensors:

$$\underline{f^*: \Gamma\left(\bigoplus_{s=0}^{\infty} T_s^0(N)\right) \longrightarrow \Gamma\left(\bigoplus_{s=0}^{\infty} T_s^0(M)\right)}$$

Clearly f^* "respects $\otimes$", i.e.

$$\underline{f^*(\omega_1 \otimes \omega_2) = f^*(\omega_1) \otimes f^*(\omega_2)}, \quad \forall \begin{array}{l} \omega_1 \in \Gamma(T_s^0(N)) \\ \omega_2 \in \Gamma(T_r^0(N)) \end{array}$$

Also, clearly: $\underline{\underline{f^*(\Omega^s(N)) \subset \Omega^s(M)}}$ $\quad (\forall s \geq 0)$,

and $\underline{\underline{f^*(\omega_1 \wedge \omega_2) = f^*(\omega_1) \wedge f^*(\omega_2)}}$ $\quad \forall \begin{array}{l} \omega_1 \in \Omega^s(N), \\ \omega_2 \in \Omega^r(N). \end{array}$

(E.g. for $\omega_1, \omega_2 \in \Omega^1(N) = \Gamma(T^*N)$:

$$\omega_1 \wedge \omega_2 = \frac{(1+1)!}{1!1!} \mathcal{A}(\omega_1 \otimes \omega_2) = \omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1$$

thus $\underline{f^*(\omega_1 \wedge \omega_2)} = f^*(\omega_1) \otimes f^*(\omega_2) - f^*(\omega_2) \otimes f^*(\omega_1)$

$$= \underline{f^*(\omega_1) \wedge f^*(\omega_2)}\ )$$

<u>Note also</u>: $\forall g \in C^\infty(N)$: $\underline{f^*(dg) = d(g \circ f)} \in \Omega^1(M)$

(Indeed, for any $p \in M$, $v \in T_p M$:

$$(f^*(dg))_p(v) = dg_{f(p)}(df_p(v)) \underset{\text{chain rule!}}{=} d(g \circ f)_p(v)\ )$$

Def 6: The <u>exterior derivative</u> is the ($\mathbb{R}$-linear) map $\underline{d : \Omega^r(M) \to \Omega^{r+1}(M)}$ (any $r \geq 0$) given by $\underline{\underline{d\left(\sum_I \omega_I dx^I\right) = \sum_I d\omega_I \wedge dx^I}}$.

In more precis terms: If $\omega \in \Omega^r(M)$ and if (U, x) is any C^∞ chart on M, take $\omega_I \in C^\infty(U)$ so that $\omega_{|U} = \sum_I \omega_I dx^I$; then $\underline{(d\omega)_{|U} = \sum_I d\omega_I \wedge dx^I}$.

Well-defined: See <u>Problem 48</u>

<u>Properties</u>:

- Note: $df \in \Omega^1(M)$ for $f \in \Omega^0(M) = C^\infty(M)$; this is the same object as before! (("dx^I with $I = \emptyset$"))
- $\underline{d(\omega \wedge \vartheta) = d\omega \wedge \vartheta + (-1)^r \omega \wedge d\vartheta}$, $\forall \omega \in \Omega^r(M)$, $\vartheta \in \Omega(M)$.
- $d(f^*(\omega)) = f^*(d\omega)$ for $f : M \to N$ (C^∞), $\omega \in \Omega(N)$

  – <u>Problem 48</u>
- $\underline{\underline{d \circ d = 0}}$ Jost Thm 2.1.5

~~Wedge~~ **Brief** survey: For d-dim, ~~to give~~ a ~~non-vanishing~~ d-form $\omega \in \Omega^d(M)$ is called a volume form on M "top dimensional"

and ~~$\int_M$~~ $\int_K \omega \in \mathbb{R}$ is well-def $\forall K$ compact $\subset M$

Stoke's Theorem:
For M an oriented compact mfld ~~with boundary~~

$\omega \in \Omega^{d-1}(M)$, $\boxed{\int_M d\omega = \int_{\partial M} i^* \omega}$

$i : \partial M \to M$
& appropriate orientation

de Rham Cohomology groups

$$H^k(M) = Z^k(M) / B^k(M)$$

the de Rham cohomology gp of dimension k.

$$Z^k(M) = \{ \omega \in \Omega^k(M) : d\omega = 0 \}$$

closed k-forms

$$B^k(M) = \{ d\omega : \omega \in \Omega^{k-1}(M) \}$$

~~From alg top:~~
Cohomology groups of M - def over $\mathbb{Z}$,
dual to homology groups M ~~$\forall$~~

and $H^k(M) = H^k_{\mathbb{Z}}(M) \otimes \mathbb{R}$!

Duality easy to see using Stokes

wedge product $\leftrightarrow$ cup prod

8.1. **Notes.** .

p. 5, Def. 3: This definition of "weak algebra bundle" is not standard as far as I know[11], I introduce it only to give, with minimal effort, a conceptual framework for the material that comes later. A much more common concept is that of an *algebra bundle.* The definition is as follows: Let A be a fixed (finite dimensional) algebra over $\mathbb{R}$ (that is, a vector space over $\mathbb{R}$ provided with a "product rule", i.e. an $\mathbb{R}$-bilinear map $A \times A \to A$). Then an "*algebra bundle with standard fiber A*" is a weak algebra bundle (E, π, M) with the property that for each point $p \in M$ there exists a bundle chart (U, φ) with $p \in U$ and an $\mathbb{R}$-linear bijection $j : \mathbb{R}^n \to A$ ($n = \operatorname{rank} E$) such that $j \circ \varphi_p$ is an *algebra isomorphism* $E_p \xrightarrow{\sim} A$ for each $p \in U$. (Cf., e.g., [11, Def. 1.40].)

We remark that all the "weak algebra bundles" which we give as examples later on p. 5, are in fact ("genuine") algebra bundles!

Note that Jost in his book does not introduce the notion of an algebra bundle explicitly; however he often works with objects which are in fact algebra bundles.

[11]However the notion of a "weak Lie algebra bundle" appears to be standard, and our definition corresponds naturally with it. Namely: In our notation, a weak Lie algebra bundle is a weak algebra bundle such that E_p (with the given product operation) is a Lie algebra for each $p \in M$.

#9: Connections

(E, π, M) – a vector bundle

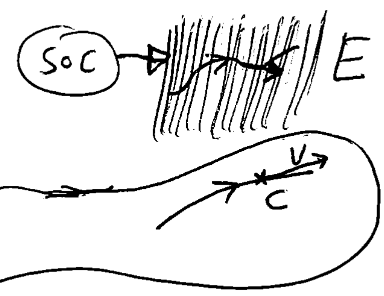
Given $s \in \Gamma E$, $p \in M$, $v \in T_pM$, <u>want</u> to define

$D_v s$ = "rate of change of $s(p)$ in direction v"

$$= \overset{?}{\lim_{h\to 0}} \frac{s(c(h)) - s(c(0))}{h}$$

Nonsense! ⊛ for any curve $c : (-\varepsilon, \varepsilon) \to M$ with $c'(0) = v$.

⊛ Nonsense! Since $s(c(h)) \in E_{c(h)}$, $s(c(0)) \in E_{c(0)}$; cannot subtract!



We'll see that there are <u>many</u> ways to introduce a reasonable such "D" (a covariant derivative = a connection). One <u>naive</u> way is of course to <u>fix a bundle chart</u> containing p; then $s(c(h)) \in$ "$E_{c(h)} = \mathbb{R}^n$" and we can compute $s(c(h)) - s(c(0)) \in \mathbb{R}^n$; however <u>the result depends on the choice of coordinates</u>!

Def 1: A connection (or covariant derivative) on (E, π, M) is an $\mathbb{R}$-linear map

$$D : \Gamma E \to \Gamma(E \otimes T^*M) \qquad \{= \Omega^1(E)\}$$

such that

$$D(fs) = f \cdot Ds + s \otimes df, \qquad \forall f \in C^\infty(M),\ s \in \Gamma E.$$

Thus for $s \in \Gamma E$, $p \in M$:

$$(Ds)(p) \in (E \otimes T^*M)_p \overset{\Delta}{=} \mathrm{Hom}(T_pM, E_p)$$

{natural & standard identification (cf. Lecture #7)}

Notation: $\underline{D_v s} := (Ds)(v)$ for $v \in TM$.

{cf. p. 1, the thing we wanted to define!}

Also $D_X s \in \Gamma E$ for $X \in \Gamma(TM)$ {contraction!}

Lemma 1: Let D be a connection on (E, π, M) and let U be an open subset of M.

a) $\forall s_1, s_2 \in \Gamma E$: $\quad s_{1|U} = s_{2|U} \Rightarrow (Ds_1)_{|U} = (Ds_2)_{|U}$.

b) $\exists!$ connection "$D_{|U}$" on $E_{|U}$ such that
$(Ds)_{|U} = D_{|U}(s_{|U})$, $\forall s \in \Gamma E$.

c) Let $(U_\alpha)_{\alpha \in A}$ be an open covering of M and for each $\alpha \in A$ let D_α be a connection on $E_{|U_\alpha}$. Assume $\forall \alpha, \beta \in A$: $D_{\alpha|U_\alpha \cap U_\beta} = D_{\beta|U_\alpha \cap U_\beta}$. Then there exists a unique connection D on E such that $D_{|U_\alpha} = D_\alpha$, $\forall \alpha \in A$.

Summary of Lemma 1: A connection can be described locally in a natural way!

Note: Later we'll often write "D" for "$D_{|U}$".

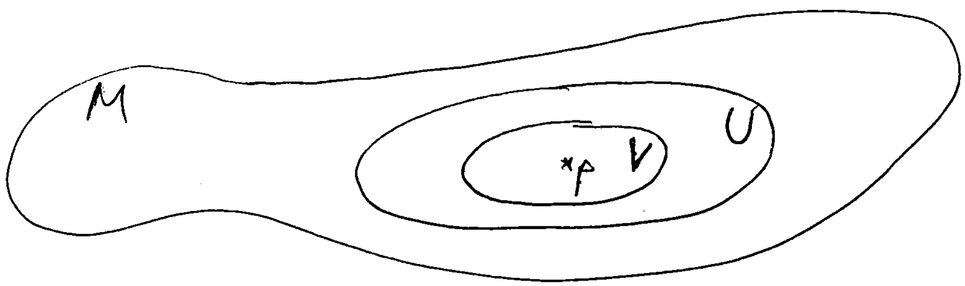
proof of Lemma 1: Parts b, c are discussed in Problem 52. Note that (b) is not immediate.

a) Assume $s_{1|U} = s_{2|U}$.

Take $p \in U$. It suffices to prove $(Ds_1)(p) = (Ds_2)(p)$.

Take $f \in C^\infty(M)$ with $\operatorname{supp} f \subset U$ (e.g. compact support)

and $f_{|V} \equiv 1$ for some open neighborhood $V \subset U$ of p.



Then $fs_1 = fs_2$ in ΓE (!)

$\therefore D(fs_1) = D(fs_2)$ in $\Gamma(E \otimes T^*M)$

$\therefore fDs_1 + s_1 \otimes df = fDs_2 + s_2 \otimes df$

Evaluate this at p, use $f(p) = 1$ and $df_p = 0$.

$\therefore (Ds_1)(p) = (Ds_2)(p)$ in $E_p \otimes T_p^*M$.

□

"Coefficients" of a connection

Assume U open $\subset M$ and bases of sections $X_1, \dots, X_d \in \Gamma(TU)$ and $s_1, \dots, s_n \in \Gamma(E_{|U})$.

Define $\Gamma_{ij}^k \in C^\infty(U)$ by $D_{X_i} s_j = \Gamma_{ij}^k s_k$, $\forall i, j$.

{the Christoffel symbols of D}

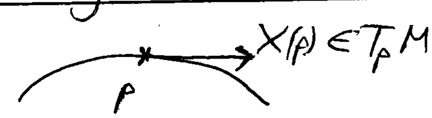
Details: "D_{X_i}" really means "$(D_{|U})_{X_i}$" (cf. Lemma 1(b)).
We have $D_{X_i} s_j \in \Gamma(E_{|U})$, hence $\exists !\ \Gamma_{ij}^k \in C^\infty(U)$ such that the above holds; cf. Problem 34.

Now for arbitrary $X \in \Gamma(TM)$ (or $X \in \Gamma(TU)$) and $s \in \Gamma E$, $\exists !\ c^1, \dots, c^d \in C^\infty(U)$, $a^1, \dots, a^n \in C^\infty(U)$ such that $X_{|U} = c^j X_j$ and $s_{|U} = a^j s_j$

and then

$$D_X(s)_{|U} = D_X(a^k s_k) = X(a^k) \cdot s_k + a^k \cdot D_X(s_k) =$$
$$= X(a^k) \cdot s_k + c^j a^k D_{X_j}(s_k)$$
$$= X(a^k) s_k + c^j a^k \Gamma_{jk}^\ell s_\ell .$$

Hence $\forall p \in U$: $D_X(s)(p)$ depends only on the values of s along any "ε-curve"



Indeed, given $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\dot\gamma(0) = X(p)$, we have

$(X a^k)(p) = (a^k \circ \gamma)'(0)$ {Depends only on "s along γ"!} (see Problem 53)

Note also: Any choice of $\Gamma_{ij}^k \in C^\infty(U)$ defines a connection on $E_{|U}$!

More abstract "coefficients" of D

If D and $\bar{D}$ are connections on E, then $D-\bar{D}$ is a $C^\infty(M)$-linear map $\Gamma E \to \Gamma(E \otimes T^*M)$; hence

$$\begin{aligned} D-\bar{D} &\in \Gamma\, Hom(E, E\otimes T^*M) \\ &= \Gamma(E^* \otimes E \otimes T^*M) \\ &= \Gamma(End(E) \otimes T^*M) \\ &= \Omega^1(End\, E) \end{aligned}$$

We are using "standard identifications", see Problem 42(c) and Lecture #7, p.9.

Conversely, given any fixed connection D on E then for every $A \in \Omega^1(End\ E)$, $D+A$ is also a connection on E. Hence the space of connections on E is an "affine space modeled on $\Omega^1(End\ E)$"!

Given a basis of sections $s_1, \dots, s_n \in \Gamma(E_{|U})$ (thus "$E_{|U} = U \times \mathbb{R}^n$") we get a "naive" connection on $E_{|U}$ by just "differentiating coordinate by coordinate" as on p.1; Jost calls this connection "d"; thus $d(a^k s_k) := s_k \otimes da^k$ for any $a^1, \dots, a^n \in C^\infty(U)$.

— see Problem 54

Given also a fixed connection D on E, we set $A := D - d \in \Omega^1(End\ E_{|U})$.

Thus $\boxed{D = d + A}$ in U.

Of course, A depends not only on D but also on the trivialization $s_1, \dots, s_n$; thus we certainly do not get an intrinsic section of $\mathrm{End}\, E \otimes T^*M$!

More explicit coefficients of A (and thus of D)

① Write $A = \begin{pmatrix} A^1_1 & \cdots & A^1_n \\ \vdots & & \vdots \\ A^n_1 & \cdots & A^n_n \end{pmatrix} = (A^k_j)$ where each $A^k_j \in \Omega^1(U)$.

matrix for A wrt the basis $s_1, \dots, s_n$

Defining relation: $A(a^k s_k) = a^j s_k \otimes A^k_j$, $\forall a^1, \dots, a^n \in C^\infty U$.

Thus $D(a^k s_k) = s_k \otimes da^k + a^j s_k \otimes A^k_j$,

and in particular $D(s_j) = s_k \otimes A^k_j$

This can alternatively be taken as the defining relation for A^k_j.

Note: If also $X_1, \dots, X_d \in \Gamma(TU)$ is a fixed basis of sections, then $\Gamma^k_{ij} = A^k_j(X_i)$.

② Given a basis $\gamma^1, \dots, \gamma^d \in \Gamma(T^*U)$

e.g. $\gamma^j = dx^j$ if (U, x) is a chart for M.

then $\exists!\ A_1, \dots, A_d \in \Gamma(\mathrm{End}\, E_{|U})$ such that $A = A_j \otimes \gamma^j$.

Note: If we also have a fixed basis $s_1, \dots, s_n \in \Gamma E_{|U}$, then $A_i = \begin{pmatrix} \Gamma^1_{i1} & \cdots & \Gamma^1_{in} \\ \vdots & & \\ \Gamma^n_{i1} & \cdots & \Gamma^n_{in} \end{pmatrix} = (\Gamma^k_{ij})$

Other ways to think about a connection

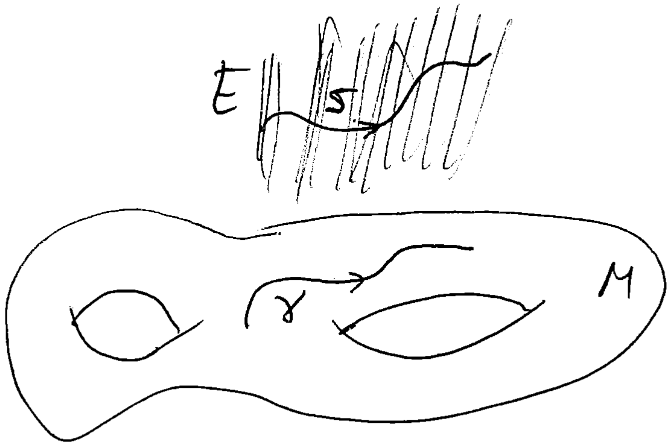
(1) Parallel transport

Let $\gamma:[a,b]\to M$ be a C^∞ curve and let $\underline{\underline{s\in\Gamma_\gamma E}}$

(viz., $s:[a,b]\to E$, a C^∞ map with $\pi\circ s=\gamma$

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ {[a,b]} \arrow[ur, "s"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

"s is a lift of γ to E")



Set $\boxed{\begin{array}{l} \dot s(t) := D_{\dot\gamma(t)}(\tilde s) \in E_{c(t)} \\ \text{for any } \tilde s\in\Gamma E \text{ with } \tilde s(\gamma(t_1)) = s(t_1) \\ \text{for all } t_1 \text{ near } t. \end{array}}$

This makes $\dot s(t)$ well-defined at any t where $\dot\gamma(t)\neq 0$, and there is a natural extension of the definition also to the case $\dot\gamma(t)=0$ – see Problem 45 and formula below!

Let (U,x) be a chart on M; then $\underline{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^d}}$ is a basis of sections in $\Gamma(TU)$; also let $\underline{s_1,\dots,s_n}$ be a basis of sections in $\Gamma(E_{|U})$. $\Rightarrow$ Get Christoffel symbols $\underline{\Gamma_{jk}^{\ell} \in C^\infty(U)}$.

Take $b_k \in C^\infty(U)$ so that $\underline{\tilde{s}_{|U} = b^k s_k}$. Write $x(\gamma(t)) = (\gamma^1(t),\dots,\gamma^d(t))$; then $\underline{\dot{\gamma}(t) = \dot{\gamma}^j(t)\frac{\partial}{\partial x^j}}$ Problem 14

By our formula on p. 4

$$\therefore \underline{D_{\dot{\gamma}(t)}(\tilde{s}) = \underbrace{(\dot{\gamma}(t))(b^k)}_{=(b^k\circ\gamma)'(t)} \cdot \underbrace{s_k(\gamma(t))}_{=:S_k(t)} + \dot{\gamma}^j(t)\cdot b^k(\gamma(t))\cdot \underbrace{\Gamma_{jk}^{\ell}(\gamma(t))}_{=:\Gamma_{jk}^{\ell}(t)}\cdot s_\ell(\gamma(t))}$$

Write $\underline{s(t) = a^k(t)s_k(t)}$ $\quad (a^k \in C^\infty([a,b]))$

Hence if $\tilde{s}$ exists at all:

$$\boxed{\dot{s}(t) = \dot{a}^k(t)\,s_k(t) + \dot{\gamma}^j(t)a^k(t)\,\Gamma_{jk}^{\ell}(t)\,s_\ell(t)}$$

We take this as the <u>def</u> of $\dot{s}(t)$, even if $\tilde{s}$ does not exist.

<u>Lemma 2</u>: $\forall v \in E_{\gamma(a)}$: $\exists!\ s \in \Gamma_\gamma E$ such that $s(a)=v$ and $\dot{s}(t)=0$, $\forall t \in [a,b]$.

<u>proof</u>: Cover γ with charts as above. In each chart, we seek C^∞-functions $a^1(t),\dots,a^d(t)$ satisfying

$$\underline{\dot{a}^{\ell}(t) + \dot{\gamma}^j(t)\,\Gamma^{\ell}_{jk}(t)\,a^k(t) \equiv 0} \qquad (\forall \ell)$$

This is a <u>linear</u> system of ODE's of order 1. Hence given any initial values $a^1(t_0), \dots, a^d(t_0)$ there exist unique C^∞ solutions $a^1, \dots, a^d$ in <u>any</u> interval around t_0 where $\dot{\gamma}^j(t)$ and $\Gamma^{\ell}_{jk}(t)$ are defined and C^∞.

$\square$

The <u>parallel transport</u> of v along the curve γ.

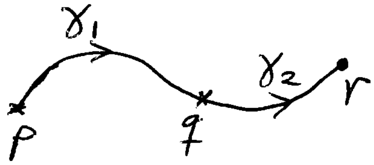
<u>Def 2</u>: We write $\underline{\mathbb{P}_\gamma v \in \Gamma_\gamma E}$ for the "s" in the above lemma. Also, if $p = \gamma(a)$, $q = \gamma(b)$, we let $\underline{\mathbb{P}_{p \xrightarrow{\gamma} q} : E_p \to E_q}$ be the map given by

$$\underline{\mathbb{P}_{p \xrightarrow{\gamma} q}(v) := (\mathbb{P}_\gamma v)(b)}$$

<u>Lemma 3</u>: $\mathbb{P}_{p \xrightarrow{\gamma} q}$ is <u>independent</u> of the <u>parametrization</u> of γ (but it certainly depends on the curve γ itself!).

$\mathbb{P}_{p \xrightarrow{\gamma} q}$ is a <u>linear isomorphism</u> with $\left(\mathbb{P}_{p \xrightarrow{\gamma} q}\right)^{-1} = \mathbb{P}_{q \xrightarrow{\bar{\gamma}} p}$

see #6, p. 2 for "$\bar{\gamma}$"!

Also if 

$\gamma = \gamma_1 \cdot \gamma_2$ a C^∞ curve,

then $$\underline{\mathbb{P}_{p \xrightarrow{\gamma} r} = \mathbb{P}_{q \xrightarrow{\gamma_2} r} \circ \mathbb{P}_{p \xrightarrow{\gamma_1} q}}$$

The last property leads to a natural definition of $\mathbb{P}_{p \xrightarrow{\gamma} q}$ for any <u>pw</u> C^∞ curve γ!

Now we can express D in terms of parallel transport, and make sense of the formula on p. 1 (modified):

Proposition 1: For any $\mu \in \Gamma E$ and any C^∞ curve $\gamma : (-\varepsilon, \varepsilon) \to M$,
$$D_{\dot\gamma(0)}(\mu) = \lim_{h \to 0} \frac{P_{\gamma,h}(\mu(\gamma(h))) - \mu(\gamma(0))}{h} \qquad \circledast$$
where $P_{\gamma,h} : E_{\gamma(h)} \to E_{\gamma(0)}$ is parallel transport along γ.

proof: Fix a basis $v_1, \dots, v_n$ of $E_{\gamma(0)}$; then set
$$\mu_j := P_\gamma v_j \in \Gamma_\gamma E.$$
Then $\mu_1, \dots, \mu_n$ is a basis of sections in $\Gamma_\gamma E = \Gamma\gamma^*(E)$.
Now $\mu \circ \gamma \in \Gamma_\gamma E$; hence $\exists!\ a^1, \dots, a^n \in C^\infty(-\varepsilon, \varepsilon)$ such that
$\mu \circ \gamma = a^j \mu_j$ in $\Gamma_\gamma E$.

Then $P_{\gamma,h}(\mu_j(h)) = \mu_j(0)$ and thus
$$P_{\gamma,h}(\mu(\gamma(h))) = P_{\gamma,h}(a^j(h)\mu_j(h)) = a^j(h)\mu_j(0).$$
Hence
$$[\text{Right hand side of } \circledast] = \dot a^j(0)\mu_j(0).$$
Now let's use the pulled back connection $\gamma^* D$ on $\gamma^* E$. (Cf. Problem ___.) We have
$$D_{\dot\gamma(0)}(\mu) = (\gamma^* D)_{1_0}(\mu \circ \gamma) = (\gamma^* D)_{1_0}(a^j \mu_j)$$
$$= \dot a^j(0)\mu_j(0) + a^j(0) \cdot \underbrace{(\gamma^* D)_{1_0}(\mu_j)}_{= \dot\mu_j(0) = 0}$$
$$= \dot a^j(0)\mu_j(0). \quad \text{DONE!} \ \square\square$$

The tangent vector "1" in $T_0(-\varepsilon, \varepsilon) = \mathbb{R}$

② "Horizontal space"

Given a vector bundle (E, π, M) $(\dim M = d,\ \mathrm{rank}\, E = n)$, and $\Psi \in E$, we have $\underline{\dim T_\Psi E = d+n}$ and $T_\Psi E$ contains a distinguished "vertical subspace" $\underline{\underline{V_\Psi}} \subset T_\Psi E$ with $\underline{\dim V_\Psi = n}$.

{ V_Ψ = "all directions which don't point out of the fiber $E_{\pi(\Psi)}$".

If we are given a connection D on E, then there's also a distinguished "horizontal space" $\underline{\underline{H_\Psi}} \subset T_\Psi E$ $(\dim H_\Psi = d)$ such that $\underline{T_\Psi E = V_\Psi \oplus H_\Psi}$,

namely

$$\underline{\underline{H_\Psi := \left\{ \frac{d}{dt} (\underbrace{\mathbb{P}_\gamma \Psi)(t)}\Big|_{t=0} \ :\ \gamma : [0,b] \to M \text{ any } C^\infty \text{ curve with } \gamma(0) = \pi(\Psi) \right\}}}$$

{ one can prove that this only depends on $\dot\gamma(0)$, and one gets a linear isomorphism $T_{\pi(\Psi)} M \xrightarrow{\sim} H_\Psi$.

Both $\underline{\underline{V := \bigcup_\Psi V_\Psi}}$ and $\underline{\underline{H := \bigcup_\Psi H_\Psi}}$ are vector $\underline{\text{subbundles}}$ of $TE \to E$; $\underline{TE = V \oplus H}$, and H is "homogeneous". Such an H is called a "$\underline{\text{connection}}$". One proves: It is $\underline{\text{equivalent}}$ to give D and to give H!

9.1. **Notes.** .

p. 7: Note that $\dot{s}(t)$ is well-defined at any t where $\dot{\gamma}(t) \neq 0$, by Problem 46 together with Problem 53. Also at t where $\dot{\gamma}(t) = 0$, the formula on p. 8 makes sense – indeed, in this case the formula says simply that

$$\dot{s}(t) = \dot{a}^k(t) s_k(\gamma(t)) \tag{4}$$

– and we take this as the definition of $\dot{s}(t)$. Here one must verify that this definition does not depend on the choice of the basis of sections, $s_1, \ldots, s_n$ (the formula is obviously independent of the choice of the chart (U, x)). This is done as follows: Assume that $\sigma_1, \ldots, \sigma_n$ is another basis of sections in some open neighborhood V of the point $\gamma(t)$; then there exist $\tau_k^\ell \in C^\infty(U \cap V)$ $(k, \ell \in \{1, \ldots, n\})$ such that $s_k = \tau_k^\ell \sigma_\ell$ in $U \cap V$. Hence $s(t_1) = a^k(t_1) s_k(\gamma(t_1)) = a^k(t_1) \tau_k^\ell(\gamma(t_1)) \sigma_\ell(\gamma(t_1))$ for all t_1 near t, and so the above formula, applied with respect to the basis of sections $\sigma_1, \ldots, \sigma_n$, says that

$$\begin{aligned}
\dot{s}(t) &= \left(\frac{d}{dt} \big(a^k(t) \tau_k^\ell(\gamma(t)) \big) \right) \cdot \sigma_\ell(\gamma(t)) \\
&= \left(\dot{a}^k(t) \cdot \tau_k^\ell(\gamma(t)) + a^k(t) \cdot (\tau_k^\ell \circ \gamma)'(t) \right) \cdot \sigma_\ell(\gamma(t)) \\
&= \dot{a}^k(t) \cdot \tau_k^\ell(\gamma(t)) \cdot \sigma_\ell(\gamma(t)) \\
&= \dot{a}^k(t) \cdot s_k(\gamma(t)),
\end{aligned}$$

where the third equality holds since $\dot{\gamma}(t) = 0$, thus $(\tau_k^\ell \circ \gamma)'(t) = 0$. This proves that $\dot{s}(t)$ is well-defined also when $\dot{\gamma}(t) = 0$.

[Let us also note that using the pullback connection $\gamma^* D$ which we will introduce later in Problem 57, $\dot{s}(t)$ can be defined by the simple and intrinsic formula

$$\dot{s}(t) := (\gamma^* D)_{1_t}(s), \tag{5}$$

where 1_t is the tangent vector "1" in $T_t([a, b]) = \mathbb{R}$. Indeed, for any t with $\dot{\gamma}(t) \neq 0$, the fact that the above formula gives the same answer as the definition on p. 7 in the lecture is clear from the defining relation for $\gamma^* D$ (cf. Problem 57(a)), applied in an appropriate small neighborhood of t in $[a, b]$. On the other hand for t with $\dot{\gamma}(t) = 0$, one verifies the claim by comparing the explicit formula (4) above with the explicit formula for $\gamma^* D$ in terms of such a local basis of sections $s_1, \ldots, s_n$ for E (cf. equation (150) in the solution to Problem 57(a)).]

p. 9(top): The statement about unique existence of a C^∞ solution to a first order linear system of ODEs; see [4, p. 399 (Corollary)] or [1, Sec. 1.2].

p. 9, Def 2: It is possible to build up the theory of connections by *starting* from the notion of parallel transport. One then defines a *system of parallel transport* on a vector bundle E to be a system of lifts "$\mathbb{P}_\gamma v$" of the C^∞

curves on M – such a "system of parallel transport" is assumed to satisfy certain conditions (some of which appear in Lemma 3). One can then get back "our" D by Prop. 1 on p. 10. Cf., e.g., Poor [11, Ch. 2].

p. 10, Prop. 1: This is the formula from Jost [5, p. 135 (just above (4.1.8))]. Our proof is in principle the same as Jost's; however by using the pulled back connection $\gamma^* D$ (cf. Problem 57) we avoid the following slight issue: Jost refers to (4.1.6) for the deduction of (4.1.9), but (4.1.6) concerns the case when $\mu_1, \ldots, \mu_n$ is a basis of sections in an *open subset* of M; and this assumption is used in the deduction of (4.1.6); if we wish to make sense of (4.1.6) when $\mu_1, \ldots, \mu_n$ is merely a basis of sections *along* γ [12] then certain questions on interpretation arise, and these are exactly taken care of by introducing the pulled back bundle $\gamma^* E$ and connection $\gamma^* D$.

Following the computation at the bottom of our p. 10: The equality $D_{\dot\gamma(0)}(\mu) = (\gamma^* D)_{1_0}(\mu \circ \gamma)$ holds by the defining relation for $\gamma^* D$, cf. Problem 57(a). The next two equalities are clear. Finally we use the fact that $(\gamma^* D)_{1_0}(\mu_j) = \dot\mu_j(0) = 0$; the first of these equalities holds by (5) above, and the second by our choice of μ_j.

[12] or along c, in Jost's notation.

#10. Connections (II)

Let (E, π, M) be a vector bundle.

Proposition 1: Given a connection D on E, there is a unique connection D^* on E^* such that

$$\circledast \quad d(\mu, \nu) = (D\mu, \nu) + (\mu, D^*\nu), \qquad \forall \mu \in \Gamma E,\ \nu \in \Gamma E^*.$$

Here $(\cdot,\cdot)$ denotes contraction, $E_p \times E_p^* \to \mathbb{R}$; e.g. in "$(D\mu, \nu)$": $\Gamma(E \otimes T^*M) \otimes \Gamma(E^*) \longrightarrow \Gamma(T^*M)$.

Remark: Note that we get $(D^*)^* = D$ on $E^{**} = E$.

Motivation for the formula: ① Generalized Leibniz' rule! ② Via parallel transport; see Problem 56.

proof: Not completely trivial! Just ignores any difficulty regarding "well-definedness"!

Given $\nu \in \Gamma(E^*)$, consider the map

$$H : \Gamma E \longrightarrow \Gamma(T^*M),$$
$$H(\mu) := d(\mu, \nu) - (D\mu, \nu)$$

This H is $C^\infty M$-linear; indeed additivity is obvious, and for any $f \in C^\infty M$ we have:

$$\begin{aligned} H(f\mu) &= d(f\mu, \nu) - (D(f\mu), \nu) \\ &= (\mu, \nu) \otimes df + f \cdot d(\mu, \nu) - (\mu \otimes df + f \cdot D\mu, \nu) \\ &= f \cdot H(\mu). \end{aligned}$$

Hence by Problem 43(c), H corresponds to a unique element in $\Gamma(\mathrm{Hom}(E, T^*M)) = \Gamma(E^* \otimes T^*M)$ which we call $D^*(\nu)$. This means:

$$\underbrace{(\mu, D^*(\nu))}_{\text{contraction } E_p \times E_p^* \to \mathbb{R}, \text{ as above!}} = H(\mu) = d(\mu,\nu) - (D\mu, \nu), \qquad \forall \mu \in \Gamma E$$

We have thus defined a map

$$D^* : \Gamma(E^*) \to \Gamma(E^* \otimes T^*M)$$

This map satisfies ⊛ by construction, and conversely ⊛ forces the above definition of D^*!

Hence it only remains to prove that D^* is a connection. D^* is clearly additive. Next, for any $f \in C^\infty M$ and $\nu \in \Gamma(E^*)$:

$$D^*(f\nu) \overset{?}{=} \nu \otimes df + f \cdot D^*(\nu) \qquad ?$$

$$\Leftrightarrow \forall \mu \in \Gamma E: \quad (\mu, D^*(f\nu)) = (\mu, \nu \otimes df + f \cdot D^*(\nu))$$

$$\Leftrightarrow \forall \mu \in \Gamma E: \quad d\underbrace{(\mu, f\nu)}_{\mathbb{R}} - \underbrace{(D\mu, f\nu)}_{\mathbb{R}} = (\mu,\nu)\cdot df + f \cdot \big(d(\mu,\nu) - (D\mu,\nu)\big)$$

$\boxed{\text{YES!}}$ since $d(f \cdot (\mu,\nu)) = (\mu,\nu) \cdot df + f \cdot d(\mu,\nu)$.

Hence D^* is a connection.

□

Proposition 2: Let E_1, E_2 be vector bundles over M with connections D_1, D_2, respectively. Then there is a unique connection D ($=:$ "$D_1 \otimes D_2$") on $E_1 \otimes E_2$ such that

$$D(\mu_1 \otimes \mu_2) = (D_1\mu_1)\otimes\mu_2 + \mu_1 \otimes (D_2\mu_2), \quad \forall \mu_1 \in \Gamma E_1,\ \mu_2 \in \Gamma E_2.$$

Proof: See Problem 58. It is easy to see that ~~these~~ D is uniquely defined if it exists, namely since every $s \in \Gamma(E_1 \otimes E_2)$ can be written as a finite sum of "pure tensors" $\mu_1 \otimes \mu_2$. (This follows from $\Gamma(E_1 \otimes E_2) = \Gamma E_1 \otimes \Gamma E_2$.) Thus what remains to prove is the existence of D...

Note: Now also get a connection on $\mathrm{Hom}(E_1, E_2) = E_1^* \otimes E_2$!

By similar methods one also proves:

Proposition 3: Let M, N be C^∞ manifolds and let $f: M \to N$ be a C^∞ map. Also let E be a vector bundle over N equipped with a connection D. Then there exists a unique connection f^*D on f^*E such that for all $s \in \Gamma E$,

$$(f^*D)(s \circ f) = D_{df(\cdot)}(s) \in \Gamma(\mathrm{Hom}(TM, f^*E)) = \Gamma(f^*E \otimes T^*M)$$

— see Problem 57!

Extension of D to E-valued forms (i.e. $\Omega^r(E)$)

Recall $\Omega^r(E) := \Gamma(E \otimes \Lambda^r M)$; thus D is a map $\Omega^0(E) \to \Omega^1(E)$. We now extend to $\Omega^r(E) \to \Omega^{r+1}(E)$.

Proposition 4: Let D be a connection on a vector bundle (E, π, M). Then (for any $r \geq 0$) there exists a unique $\mathbb{R}$-linear map $D : \Omega^r(E) \longrightarrow \Omega^{r+1}(E)$ satisfying

$$\underline{\underline{D(\mu \otimes \omega) = (D\mu) \wedge \omega + \mu \otimes d\omega}}, \quad \underline{\forall \mu \in \Gamma E,\ \omega \in \Omega^r(M)}.$$

Here $D\mu \in \Omega^1(E) = \Gamma(E) \otimes \Omega^1(M)$, and "$(D\mu) \wedge \omega$" stands for the image of $D\mu$ under the map

$$\text{"} 1_{\Gamma E} \otimes (\cdot \wedge \omega) \text{"} : \Gamma(E) \otimes \Omega^1(M) \longrightarrow \Gamma(E) \otimes \Omega^{r+1}(M).$$

proof: By same basic strategy as for Propositions 2 and 3. – See Problem 60.

Note: We use (as Jost) the same symbol "D"! Another (more!?) common notation is d^D (thus for ∇ : d^∇)

This $D : \Omega^r(E) \to \Omega^{r+1}(E)$ is often called an exterior covariant derivative.

Remark: (Generalized Leibniz' rule):

$$\underline{\underline{D(f\mu) = df \wedge \mu + f \cdot D\mu}}, \quad \underline{\forall f \in C^\infty(M),\ \mu \in \Omega^r(E)}.$$

("easy exercise").

Def 1: The <u>curvature</u> of a connection D is the operator $\underline{\underline{F = F_D := D \circ D : \Omega^0(E) \to \Omega^2(E)}}$.

Very elegant & <u>abstract</u> def! We'll soon see more concrete reformulations!

Also: D is called <u>flat</u> if $F = 0$.

Lemma 1: $\underline{\underline{F \in \Omega^2(\mathrm{End}\, E)}}$

i.e. "F transforms like a tensor".

Proof: $F : \Omega^0(E) \to \Omega^2(E)$ is clearly <u>$\mathbb{R}$-linear</u>.

In fact F is even <u>$C^\infty(M)$-linear</u>, since

$\forall f \in C^\infty(M),\ \mu \in \Omega^0(E)$:

$$\underline{\underline{F(f \cdot s)}} = D(D(fs)) = D(s \otimes df + f \cdot Ds) =$$

$$= \underbrace{Ds \wedge df + s \otimes d(df)}_{\text{By def of } D : \Omega^1(E) \to \Omega^2(E) \text{ (Prop 4)}} + \underbrace{df \wedge Ds + f \cdot D(Ds)}_{\text{By the remark p. 4 (bottom).}}$$

Now use $d \circ d = 0$ and $Ds \wedge df = -df \wedge Ds$.

$$= f \cdot D(Ds) = \underline{\underline{f \cdot F(s)}}.$$

Hence

$$F \in \mathrm{Hom}(\Omega^0(E), \Omega^2(E)) \overset{\text{Problem 42(c)}}{=} \Gamma\, \mathrm{Hom}(E, E \otimes \Lambda^2 M)$$

$$= \Gamma(E^* \otimes E \otimes \Lambda^2 M) = \Gamma(\mathrm{End}(E) \otimes \Lambda^2 M)$$

$$= \underline{\underline{\Omega^2(\mathrm{End}(E))}}. \qquad \square$$

F in local coordinates?

Take $U \subset M$ open with both a bundle chart (U, φ) and a chart (U, x). Using these we write $\underline{D = d + A}$ on U and $\underline{A = A_j dx^j}$ with $\underline{A_1, \ldots, A_n \in \Gamma(\mathrm{End}\, E_{|U})}$ In fact $E_{|U} \cong U \times \mathbb{R}^n$ via φ; thus we can view each A_j as $\underline{A_j : U \to M_n(\mathbb{R})}$.

Now for any $s \in \Gamma E$, we have on U:

$$\underline{Ds = ds + A_k s \otimes dx^k} \quad \in \Omega^1(E_{|U})$$

and so

$$\underline{\underline{D(Ds)}} = \underline{d(Ds) + (A_j \otimes dx^j) \wedge (Ds)}$$

See Problem 60(b); here $(A_j dx^j) \wedge (Ds)$ is vector-wedge-prod $\Omega^1(\mathrm{End}\, E_{|U}) \times \Omega^1(E_{|U}) \to \Omega^2(E_{|U})$ coming from the standard "evaluation" map $\Gamma(\mathrm{End}\, E_{|U}) \times \Gamma(E_{|U}) \to \Gamma(E_{|U})$.

$$= \underline{d(ds) + d(A_k s \otimes dx^k) + A_j dx^j \wedge ds + (A_j dx^j) \wedge (A_k s \otimes dx^k)}$$

Here d is the "naive" exterior covariant derivative $\Omega^1(E_{|U}) \to \Omega^2(E_{|U})$; via φ this is the same as $\Omega^1(U \times \mathbb{R}^n) \longrightarrow \Omega^2(U \times \mathbb{R}^n)$, standard exterior derivative on each coordinate! Hence $\underline{d(ds) = 0}$.

Also $d(A_k s \otimes dx^k) = d(A_k s) \wedge dx^k =$

$= \underline{(dA_k \wedge s) \wedge dx^k + (A_k \wedge ds) \wedge dx^k}$

By Problem 60(c); here $dA_k \wedge s$ is vector-wedge-prod $\Omega^1(\mathrm{End}\, E_{|U}) \times \Omega^0(E_{|U}) \to \Omega^1(E_{|U})$, and $A^k \wedge ds$ is $\Omega^0(\mathrm{End}\, E_{|U}) \times \Omega^1(E_{|U}) \to \Omega^1(E_{|U})$.

Here we have associativity, and $s \wedge dx^k = dx^k \wedge s$ and $ds \wedge dx^k = -dx^k \wedge ds$; see Problem 49(c),(d).

Hence get:

$$D(Ds) = dA_k \wedge dx^k \wedge s \underbrace{- A_k \wedge dx^k \wedge ds + A_j\, dx^j \wedge ds}_{} + A_j A_k s \otimes (dx^j \wedge dx^k)$$

These cancel! (The $\wedge$ between A^k and dx^k is redundant since A^k is a 0-form.)

$$= \left(dA_k \wedge dx^k + A_j A_k \otimes (dx^j \wedge dx^k)\right) s$$

$$= \left(\left(\frac{\partial A_k}{\partial x^j} + A_j A_k\right) \otimes (dx^j \wedge dx^k)\right) s$$

Note: No "ds" left; hence F is "C^∞-linear on U"

and $$F_{|U} = \left(\frac{\partial A_k}{\partial x^j} + A_j A_k\right) \otimes (dx^j \wedge dx^k) \in \Omega^2(\mathrm{End}\, E_{|U})$$

Using the above for a covering family of U's we get back $C^\infty(M)$-linearity and so, again, $F \in \Omega^2(\mathrm{End}\, E)$ as in Lemma 5!

To get more familiar working with local coordinates, let us verify that our expression for $F_{|U}$ <u>transforms correctly</u> (as a section in $\Omega^2(\mathrm{End}\, E)$) <u>under coordinate changes</u>!

Consider any two bundle charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) for E. Then there is a unique C^∞-map $\underline{\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})}$ translating between the two!

transition map

$$\boxed{\underline{\varphi_{\beta,p} = \varphi_{\beta\alpha}(p) \circ \varphi_{\alpha,p}} : \underline{E_p \to \mathbb{R}^n} \qquad (\forall p \in U_\alpha \cap U_\beta)}$$

Recall
$$E_{|U_\alpha} \xrightarrow[\sim]{\varphi_\alpha} U_\alpha \times \mathbb{R}^n$$
$$E_{|U_\beta} \xrightarrow[\sim]{\varphi_\beta} U_\beta \times \mathbb{R}^n$$

Thus if any section $s \in \Gamma E$ is represented by $s_\alpha : U_\alpha \to \mathbb{R}^n$ wrt $(U_\alpha, \varphi_\alpha)$ and by $s_\beta : U_\beta \to \mathbb{R}^n$ wrt (U_β, φ_β)

precisely this means that $\underline{s_\alpha = pr_2 \circ \varphi_\alpha \circ s_{|U_\alpha}} : U_\alpha \to \mathbb{R}^n$
and $\underline{s_\beta = pr_2 \circ \varphi_\beta \circ s_{|U_\beta}} : U_\beta \to \mathbb{R}^n$.

then $\underline{s_\beta(p) = \varphi_{\beta\alpha}(p) \cdot s_\alpha(p)}$ $\forall p \in U_\alpha \cap U_\beta$.

Next, any section $\underline{\underline{s \in \Gamma(\text{End } E)}}$ is represented by some $\underline{s_\alpha : U_\alpha \to M_n(\mathbb{R})}$ wrt $(U_\alpha, \varphi_\alpha)$ and some $\underline{s_\beta : U_\beta \to M_n(\mathbb{R})}$ wrt (U_β, φ_β).

Translation between these two?

Take $p \in U_\alpha \cap U_\beta$, $v \in E_p$.

really: $\varphi_{\beta\alpha}(p)$

Write $v_\alpha = \varphi_{\alpha,p}(v) \in \mathbb{R}^n$ and $v_\beta = \varphi_{\beta,p}(v) = \varphi_{\beta\alpha}(v_\alpha) \in \mathbb{R}^n$; then both $(s_\alpha(p))(v_\alpha)$ and $(s_\beta(p))(v_\beta)$ represent the vector $(s(p))(v)$; thus

$$s_\beta(p) \cdot v_\beta = \varphi_{\beta\alpha}(p) \cdot s_\alpha(p) \cdot v_\alpha$$

$$\Rightarrow s_\beta(p) \cdot \varphi_{\beta\alpha}(p) \cdot v_\alpha = \varphi_{\beta\alpha}(p) \cdot s_\alpha(p) \cdot v_\alpha$$

True $\forall v_\alpha \in \mathbb{R}^n$! Hence get:

$$\underline{\underline{s_\beta(p) = \varphi_{\beta\alpha}(p) \cdot s_\alpha(p) \cdot \varphi_{\beta\alpha}(p)^{-1}}}$$

(Matrix multiplication!)

In short-hand:

$$\boxed{\underline{\underline{s_\beta = \varphi_{\beta\alpha} \cdot s_\alpha \cdot \varphi_{\beta\alpha}^{-1}}} \quad \underline{\text{in } U_\alpha \cap U_\beta}}$$

Similarly for $s \in \Omega^r(\text{End } E)$, <u>again</u> get

$$\underline{\underline{s_\beta = \varphi_{\beta\alpha} \cdot s_\alpha \cdot \varphi_{\beta\alpha}^{-1}}} \quad \underline{\text{in } U_\alpha \cap U_\beta}$$

acting on the <u>bundle</u> part only (not the form part)

Now back to our connection D and $F = D \circ D$.

Write $D = d + A_\alpha$ wrt $(U_\alpha, \varphi_\alpha)$ and $D = d + A_\beta$ wrt (U_β, φ_β).

Thus $A_\alpha \in \Omega^1(U_\alpha \times M_n(\mathbb{R}))$ and $A_\beta \in \Omega^1(U_\beta \times M_n(\mathbb{R}))$.

For $s \in \Gamma E$ (repr. by $s_\alpha : U_\alpha \to \mathbb{R}^n$ and $s_\beta : U_\beta \to \mathbb{R}^n$):

$(Ds)_{|U_\alpha}$ is repr. by $ds_\alpha + A_\alpha \cdot s_\alpha \in \Omega^1(U_\alpha \times \mathbb{R}^n)$

and

$(Ds)_{|U_\beta}$ is repr. by $ds_\beta + A_\beta \cdot s_\beta \in \Omega^1(U_\beta \times \mathbb{R}^n)$.

Hence at any $p \in U_\alpha \cap U_\beta$:

$$\varphi_{\beta\alpha} \cdot (ds_\alpha + A_\alpha \cdot s_\alpha) = ds_\beta + A_\beta s_\beta =$$

$$= d(\varphi_{\beta\alpha} \cdot s_\alpha) + A_\beta \cdot \varphi_{\beta\alpha} \cdot s_\alpha$$

$$= (d\varphi_{\beta\alpha}) \cdot s_\alpha + \varphi_{\beta\alpha} \cdot ds_\alpha + A_\beta \varphi_{\beta\alpha} s_\alpha$$

Here $\varphi_{\beta\alpha} \cdot ds_\alpha$ cancels on both sides, and we get:

$$\varphi_{\beta\alpha} A_\alpha s_\alpha = (d\varphi_{\beta\alpha}) s_\alpha + A_\beta \varphi_{\beta\alpha} s_\alpha.$$

True $\forall s_\alpha(p) \in \mathbb{R}^n$! Hence

$$\boxed{A_\alpha = \varphi_{\beta\alpha}^{-1} d\varphi_{\beta\alpha} + \varphi_{\beta\alpha}^{-1} A_\beta \varphi_{\beta\alpha}} \text{ in } U_\alpha \cap U_\beta.$$

Now also assume that x is a local coordinate on $U := U_\alpha \cap U_\beta$ (viz., (U, x) is a chart on M), and write $\underline{A_\alpha = A_{\alpha,j}\, dx^j}$ and $\underline{A_\beta = A_{\beta,j}\, dx^j}$ in U

(thus $A_{\alpha,j}$ and $A_{\beta,j}$ are C^∞ maps $U \to M_n(\mathbb{R})$, for $j = 1, \dots, d$).

Then
$$\boxed{A_{\alpha,j} = \varphi_{\beta\alpha}^{-1} \frac{\partial \varphi_{\beta\alpha}}{\partial x^j} + \varphi_{\beta\alpha}^{-1} A_{\beta,j}\, \varphi_{\beta\alpha}} \quad \text{in } U.$$

Now $\underline{\text{in } U}$ and $\underline{\text{w.r.t. } \varphi_\alpha}$, $\underline{\underline{F}}$ is represented by:

write $\underline{\underline{\varphi := \varphi_{\beta\alpha}}}$

$$\underline{\underline{F_\alpha = \left(\frac{\partial A_{\alpha,k}}{\partial x^j} + A_{\alpha,j} A_{\alpha,k}\right) \otimes (dx^j \wedge dx^k)}} =$$

$$= \left[\frac{\partial}{\partial x^j}\left(\varphi^{-1} \frac{\partial \varphi}{\partial x^k} + \varphi^{-1} A_{\beta,k}\varphi\right) + \left(\varphi^{-1}\frac{\partial \varphi}{\partial x^j} + \varphi^{-1} A_{\beta,j}\varphi\right)\left(\varphi^{-1}\frac{\partial \varphi}{\partial x^k} + \varphi^{-1} A_{\beta,k}\varphi\right)\right] \otimes (dx^j \wedge dx^k)$$

$$= \Big[\frac{\partial(\varphi^{-1})}{\partial x^j}\frac{\partial \varphi}{\partial x^k} + \varphi^{-1}\frac{\partial^2 \varphi}{\partial x^j \partial x^k} + \frac{\partial(\varphi^{-1})}{\partial x^j} A_{\beta,k}\varphi + \varphi^{-1}\frac{\partial A_{\beta,k}}{\partial x^j}\varphi +$$

$$+ \varphi^{-1} A_{\beta,k}\frac{\partial \varphi}{\partial x^j} + \varphi^{-1}\frac{\partial \varphi}{\partial x^j}\varphi^{-1}\frac{\partial \varphi}{\partial x^k} + \varphi^{-1} A_{\beta,j}\frac{\partial \varphi}{\partial x^k} +$$

$$+ \varphi^{-1}\frac{\partial \varphi}{\partial x^j}\varphi^{-1} A_{\beta,k}\varphi + \varphi^{-1} A_{\beta,j} A_{\beta,k}\varphi\Big] \otimes (dx^j \wedge dx^k)$$

$\underline{\text{9 terms...}}$ Note $\varphi \cdot \varphi^{-1} \equiv I \Rightarrow \dfrac{\partial \varphi}{\partial x^j}\cdot\varphi^{-1} + \varphi \cdot \dfrac{\partial(\varphi^{-1})}{\partial x^j} \equiv 0$

$$\Rightarrow \boxed{\frac{\partial(\varphi^{-1})}{\partial x^j} = -\varphi^{-1}\frac{\partial \varphi}{\partial x^j}\varphi^{-1}}$$

Hence term 1 $= -\varphi^{-1}\frac{\partial\varphi}{\partial x^j}\varphi^{-1}\frac{\partial\varphi}{\partial x^k}$ which cancels against term 6.

Also term 2 $=\varphi^{-1}\frac{\partial^2\varphi}{\partial x^j\partial x^k} = \varphi^{-1}\frac{\partial^2\varphi}{\partial x^k\partial x^j}$, symmetric in $j\leftrightarrow k$; hence cancels when adding over $j,k\in\{1,\dots,d\}$.

(Use $dx^j\wedge dx^k = -dx^k\wedge dx^j$).

Term 3 $= -\varphi^{-1}\frac{\partial\varphi}{\partial x^j}\varphi^{-1}A_{\beta,k}\varphi$, cancels against term 8.

Also term 5 + term 7 symmetric in $j\leftrightarrow k$, hence cancel when adding over j,k.

We are thus left with (only!) terms 4 and 9, i.e.

$$\underline{F_\alpha} = \underline{\left[\varphi^{-1}\frac{\partial A_{\beta,k}}{\partial x^j}\varphi + \varphi^{-1}A_{\beta j}A_{\beta,k}\varphi\right]\otimes\left[dx^j\wedge dx^k\right]}$$

$$= \underline{\varphi_{\beta\alpha}^{-1}F_\beta\varphi_{\beta\alpha}}$$

Hence F indeed transforms like a section in $\Omega^2(\operatorname{End}E)$ (cf. p. 9) !!

□□

10.1. **Notes.** .

p. 5: Note that the symbol "R" is also used for the curvature "F", especially when viewing curvature as an element of $\Omega^2(\operatorname{End} E)$, and even more often in Riemannian geometry, when $E = TM$ and D is the Levi-Civita connection. It should also be noted right from the start that

$$F(X,Y) = -F(Y,X) \qquad \text{in } \Gamma(\operatorname{End} E), \qquad \forall X,Y \in \Gamma(TM). \tag{6}$$

(Or equivalently: $R(X,Y) = -R(Y,X)$.) Indeed this holds for *any* section in $\Omega^2(\operatorname{End} E)$, since any such section by definition is an *alternating* map from $\Gamma(TM) \otimes \Gamma(TM)$ to $\Gamma(\operatorname{End} E)$. Cf. Problem 51(a).

(Since the relation (6) is immediate from start, it seems somewhat strange that Jost refers to [5, Thm. 4.1.2] in his proof of that relation; cf. [5, Cor. 4.1.1].)

p. 7 (bottom): Here we arrive at the formula $F_{|U} = F_{jk} \otimes (dx^j \wedge dx^k)$ with

$$F_{jk} = \frac{\partial A_k}{\partial x^j} + A_j A_k \qquad (\text{in } \Gamma(\operatorname{End} E_{|U})).$$

Of course, since $dx^j \wedge dx^k = -dx^k \wedge dx^j$, we then also have $F_{|U} = \widetilde{F}_{jk} \otimes (dx^j \wedge dx^k)$ for any choice of $\widetilde{F}_{jk} \in \Gamma(\operatorname{End} E_{|U})$ $(j,k \in \{1,\ldots,d\})$ satisfying $\widetilde{F}_{jk} - \widetilde{F}_{kj} = F_{jk} - F_{kj}$ $(\forall j,k)$. One natural choice is to require $\widetilde{F}_{jk} = -\widetilde{F}_{kj}$ $(\forall j,k)$; this determines $\widetilde{F}_{jk}$ uniquely as:

$$\widetilde{F}_{jk} = \frac{1}{2}(F_{jk} - F_{kj}) = \frac{1}{2}\Big(\frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} + A_j A_k - A_k A_j\Big).$$

This choice of $\widetilde{F}_{jk}$ appears in [5, (4.1.27–28), and also (4.1.31–32)]. Note also that this $\widetilde{F}_{jk}$ can be defined by

$$\widetilde{F}_{jk} = \frac{1}{2} F\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big).$$

(In the right hand side we view F as a bilinear map $\Gamma(TM) \times \Gamma(TM) \to \Gamma(\operatorname{End} E)$; cf. Problem 51(a).)

p. 8, transition maps: We could ((should?)) have discussed these early on when we introduced vector bundles in #7! Indeed, Jost discusses transition maps already on the second page of [5, Ch. 2].

pp. 11–12: Here we carry out the calculation which Jost refers to (without showing it) just above [5, (4.1.29)].

11. More on curvature. Metric connections.

#11. More on curvature. Metric connections

Let D be a connection on a vector bundle (E, π, M). Recall $\underline{F = F_D := D \circ D \in \Omega^2(\mathrm{End}\, E)}$.

Let (U, φ) be a bundle chart for E and write w.r.t. (U,φ): $\underline{D = d + A}$, $\underline{A \in \Omega^1(\mathrm{End}\, E_{|U})}$.

Then for any $\mu \in \Gamma E$:

Now: Repeat the computation from #10, p. 6–7, but in more abstract & elegant and less explicit notation!

Acting on $\Omega^1(E)$; see Problem 60(b).

$$\underline{\underline{F(\mu)}} = (d + A) \circ (d+A)\mu$$

$$= (d+A)(d\mu + A\mu)$$

$$= \underbrace{d(d\mu}_{=0 \text{ cf. \#10. p. 6 (bottom)}} + A\mu) + A \wedge (d\mu + A\mu)$$

Associativity; see Problem 49(d).

$$= d(A\mu) + A\wedge d\mu + A \wedge A\mu$$

See Problem 60(c)

$$= dA \wedge \mu - A \wedge d\mu + A\wedge d\mu + A \wedge A\mu$$

$$= \underline{\underline{(dA + A\wedge A)\,\mu}}.$$

or "$\wedge \mu$"

Hence: ⊛ $\boxed{F = dA + A \wedge A \quad \text{in } \Omega^2(\mathrm{End}\, E_{|U})}$

If we name more explicit "coefficients" for A then ⊛ "immediately (via basic properties of vector-wedge-product) give correspondingly explicit formulas for F:

① If $A = A_j dx^j$ (with (U,x) some chart on M)

$$\Rightarrow \quad F = \left(\frac{\partial A_k}{\partial x^j} + A_j A_k\right) dx^j \wedge dx^k$$

as in #10, p. 7

② If $A = \mu^{j*} \otimes \mu_k \otimes A_j^k$ where $\mu_1, \dots, \mu_n \in \Gamma E_{|U}$ is the basis of sections coming from (U,φ), and $A_j^k \in \Omega^1(U)$, then

cf. Problem 50

$$F = \mu^{j*} \otimes \mu_k \otimes \left(dA_j^k + A_\ell^k \wedge A_j^\ell\right)$$

Equivalently: $F(a^j \mu_j) = a^j \mu_k \otimes \left(dA_j^k + A_\ell^k \wedge A_j^\ell\right)$ for any $a^1, \dots, a^n \in C^\infty(U)$.

③ If $\underline{A_j^k = \Gamma_{ij}^k dx^i}$ (with notation as in ①, ②;

thus $D_{\frac{\partial}{\partial x^i}} \mu_j = \Gamma_{ij}^k \mu_k$, as usual) then

$$\underline{\underline{dA_j^k + A_\ell^k \wedge A_j^\ell}} = d\Gamma_{mj}^k \wedge dx^m + (\Gamma_{i\ell}^k dx^i) \wedge (\Gamma_{mj}^\ell dx^m)$$

$$= \underline{\underline{\left(\frac{\partial \Gamma_{mj}^k}{\partial x^i} + \Gamma_{i\ell}^k \Gamma_{mj}^\ell \right) dx^i \wedge dx^m}}$$

and thus, using $dx^i \wedge dx^m = dx^i \otimes dx^m - dx^m \otimes dx^i$,

$$\underline{R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^m}\right)(\mu_j) = \left(\frac{\partial \Gamma_{mj}^k}{\partial x^i} - \frac{\partial \Gamma_{ij}^k}{\partial x^m} + \Gamma_{i\ell}^k \Gamma_{mj}^\ell - \Gamma_{m\ell}^k \Gamma_{ij}^\ell \right) \cdot \mu_k}$$

$\underline{\underline{\text{"}R\text{"} := F}}$ (especially when viewed as an element in $\Omega^2(\mathrm{End}\, E)$ and even more in Riemannian geometry, when $E = TM$ and D = the Levi-Civita connection)

One writes

$$\underline{\underline{R_{jim}^k}} := \underline{\frac{\partial \Gamma_{mj}^k}{\partial x^i} - \frac{\partial \Gamma_{ij}^k}{\partial x^m} + \Gamma_{i\ell}^k \Gamma_{mj}^\ell - \Gamma_{m\ell}^k \Gamma_{ij}^\ell \in C^\infty(U)}.$$

Thus

$$\underline{\underline{F = R = \tfrac{1}{2} R_{jim}^k \otimes \mu^{j*} \otimes \mu_k \otimes (dx^i \wedge dx^m)}}.$$

Theorem 1: For any $X, Y \in \Gamma(TM)$ and $\mu \in \Gamma E$,

$$F(X,Y)(\mu) = D_X D_Y \mu - D_Y D_X \mu - D_{[X,Y]} \mu \quad \in \Gamma E$$

{Lie Product; see Problem 47}

Proof: Note that both sides are $C^\infty(M)$-linear in X and Y! {Indeed for "$F(X,Y)(\mu)$" we know this, and for the right hand side $\mathbb{R}$-linearity is clear, and we note that for any $f \in C^\infty(M)$,

$$D_{fX} D_Y \mu - D_Y D_{fX} \mu - D_{[fX,Y]} \mu = \text{\{Use Problem 47(d)\}}$$

$$= f \cdot D_X D_Y \mu - D_Y(f \cdot D_X \mu) + (Yf) \cdot D_X \mu - f \cdot D_{[X,Y]} \mu$$

$$= f \cdot D_X D_Y \mu - (Yf) \cdot D_X \mu - f \cdot D_Y(D_X \mu) + (Yf) \cdot D_X \mu - f \cdot D_{[X,Y]} \mu$$

$$= f \cdot \left(D_X D_Y \mu - D_Y D_X \mu - D_{[X,Y]} \mu \right).$$

This proves $C^\infty(M)$-linearity w.r.t. X, and now $C^\infty(M)$-linearity w.r.t. Y follows by noticing that the ~~...~~ expression is antisymmetric in X, Y.}

Hence it suffices (via Problem 34) to prove the formula for X, Y = basis vectors in a basis of sections (for TM) (which we are free to choose) over an open set. We use the basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}$ (notation as in ① on p. 2 above); thus we assume

$\underline{\underline{X = \frac{\partial}{\partial x^i}}}$ and $\underline{\underline{Y = \frac{\partial}{\partial x^j}}}$ for some $i, j \in \{1, \dots, d\}$.

Then $\underline{[X, Y] = 0}$. Also (in U):

$$\underline{\underline{D_X D_Y \mu - D_Y D_X \mu}} = D_X\left(\frac{\partial \mu}{\partial x^j} + A_j \mu\right) - D_Y\left(\frac{\partial \mu}{\partial x^i} + A_i \mu\right)$$

Notation as p. 2; 1, and we are using $\underline{D = d + A_k dx^k}$.
Note that $\mu_{|U}$ "is" a C^∞-function $U \to \mathbb{R}^n$
(via (U, φ)), and A_i "is" a C^∞-function $U \to M_n(\mathbb{R})$.

$$= \underbrace{\frac{\partial^2 \mu}{\partial x^i \partial x^j}} + A_i \frac{\partial \mu}{\partial x^j} + \frac{\partial}{\partial x^i}(A_j \mu) + A_i A_j \mu - \boxed{\text{same with } j \leftrightarrow i}$$

cancel against $\frac{\partial^2 \mu}{\partial x^j \partial x^i}$

$$= \underbrace{A_i \frac{\partial \mu}{\partial x^j}} + \frac{\partial A_j}{\partial x^i} \cdot \mu + \underbrace{A_j \frac{\partial \mu}{\partial x^i}} + A_i A_j \mu - \boxed{\text{same with } j \leftrightarrow i}$$

cancel!

$$= \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + A_i A_j - A_j A_i\right) \mu$$

$$= \underline{\underline{F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)(\mu)}} = \underline{\underline{F(X, Y)(\mu)}}$$

By p. 2 1!

Done!

□

Theorem 2: the second Bianchi identity

For any connection D on any vector bundle (E, π, M)

$$D(F_D) = 0.$$

proof: We will work with the explicit representation $D = d + A$ in a given bundle chart (U, φ).

How does D act on $\text{End } E = E^* \otimes E$?

By Problem 59(c), for any $s \in \Gamma(\text{End } E)$ and $\alpha \in \Gamma E$:

$$D(s(\alpha)) = (Ds)(\alpha) + s(D\alpha).$$

and also $d(s(\alpha)) = (ds)(\alpha) + s(d\alpha)$ in U

Hence if we write $D = d + \tilde{A}$ on $\text{End } E_{|U}$ (thus $\tilde{A} \in \Omega^1(\text{End}(\text{End } E)_{|U})$), then:

$$(ds + \tilde{A}s)(\alpha) = (Ds)(\alpha) = D(s(\alpha)) - s(D\alpha) =$$

$$= d(s(\alpha)) + A \cdot s(\alpha) - s(d\alpha + A\alpha)$$

$$\Rightarrow (\tilde{A}s)(\alpha) = A(s(\alpha)) - s(A\alpha)$$

True $\forall \alpha \in \Gamma E$; hence ← using Problem 35(c)

$$\tilde{A}s = \underbrace{A \circ s - s \circ A}_{=[A,s]}, \quad \forall s \in \Gamma(\text{End } E)$$

$$\therefore \quad \tilde{A} = [A, \cdot\,] \quad \text{in} \quad \Omega^1(\text{End}(\text{End } E)_{|U}).$$

Hence: using $\widetilde{A} = [A, \cdot]$ and Problem 60(b)

$$\underline{\underline{DF = dF + [A, F]}}$$

Now we have $\underline{\text{two}}$ product operations $\Gamma(\mathrm{End}\, E) \times \Gamma(\mathrm{End}\, E) \to \Gamma(\mathrm{End}\, E)$, namely $\underline{\circ \text{ (composition)}}$ and $\underline{[\ ,\ ] \text{ (Lie product)}}$.

These are related by $\underline{\underline{[B_1, B_2] = B_1 \circ B_2 - B_2 \circ B_1}}$ ⊛

for any $B_1, B_2 \in \Gamma(\mathrm{End}\, E)$. We get two corresponding $\underline{\text{vector-wedge-products}}$ $\Omega^r(\mathrm{End}\, E) \times \Omega^s(\mathrm{End}\, E) \to \Omega^{r+s}(\mathrm{End}\, E)$,

which we $\underline{\text{also}}$ denote by $\circ$ and $[\ ,\ ]$. It follows from ⊛ above that we have

$$\underline{\underline{[B_1, B_2] = B_1 \circ B_2 - (-1)^{rs} B_2 \circ B_1}}, \quad \forall\, B_1 \in \Omega^r(\mathrm{End}\, E),\ B_2 \in \Omega^s(\mathrm{End}\, E).$$

— similar to Problem 49(b),(c).

Computation:

$$DF = d(dA + A\circ A) + [A, dA + A \circ A]$$

$$= 0 + \underbrace{(dA)\circ A - A \circ dA}_{\text{see Problem 60(c)}} + A \circ (dA) \overset{(-1)^{2\cdot 1} = 1}{-} (dA)\circ A + A \circ (A \circ A) \overset{(-1)^{2\cdot 1} = 1}{-} (A \circ A) \circ A$$

$$= A \circ (A \circ A) - (A \circ A) \circ A$$

$$= 0.$$

Associativity; cf. $\underline{\text{Problem 49(d)!}}$

$\square$

Applications of $D(F_D) = 0$

① In Riemannian geometry, other format; see #14.

② Definition of Chern classes:

Let M be a compact C^∞ manifold and let E be a complex vector bundle of rank m over M.

{"extracurricular"; however it should be clear what this means}

Fix an invariant homogeneous polynomial $P: M_n(\mathbb{C}) \to \mathbb{C}$ of degree k.

{Meaning: $P(\varphi^{-1}B\varphi) = P(B)$, $\forall \varphi \in GL_m(\mathbb{C})$, $B \in M_n(\mathbb{C})$}

Examples: $P(B) = \operatorname{tr} B$, $P(B) = \det B$,

$P(B) = (\operatorname{tr} B)^2 - \operatorname{tr}(B^2) = 2 \sum_{i<j} \lambda_i \lambda_j$ ← {$\lambda_1, \ldots, \lambda_m$ = the eigenvalues of B}

More generally: $P = p^k$ the k:th elementary symmetric polynomial in $\lambda_1, \ldots, \lambda_m$ {then $\det(B + t \cdot I) = \sum_{k=0}^{m} p^k(B) \cdot t^{m-k}$}

Then $P(B)$ is well-defined for any $B \in \operatorname{End} E$.

Hence since $F_D \in \Omega^2(\operatorname{End} E)$, $\boxed{P(F_D) \in \Omega^{2k}(M)}$!

To make sense of "$P(F_D)$", let $\tilde{P}$ be the symmetric linear form $\underbrace{M_m(\mathbb{C}) \otimes \cdots \otimes M_m(\mathbb{C})}_{k \text{ times}} \longrightarrow \mathbb{C}$ corresponding to

P and set

$$\underline{\underline{P(F_D)}} := \tilde{P}\underbrace{(F_D \otimes \cdots \otimes F_D)} \in \Omega^{2k}(M; \mathbb{C}).$$

$\in \Omega^{2k}((\mathrm{End}\, E) \otimes \cdots \otimes (\mathrm{End}\, E))$, a vector-wedge product.

Now, $\boxed{\text{using} \quad D(F_D) = 0}$ one proves that $\underline{\underline{d(P(F_D))=0}}$ (i.e. $P(F_D)$ is a <u>closed</u> differential form) <u>and</u> that the corresponding cohomology class $[P(F_D)] \in H^{2k}(M)$ is <u>independent of the choice of D</u>. ← Jost, Lemma 4.2.4

The <u>Chern classes</u> of E are defined as

$$\underline{\underline{c_j(E) := \left[P^j\left(\frac{i}{2\pi} F\right)\right] \in H^{2j}(M)}}, \qquad j = 1, 2, 3, \ldots$$

See p. 8 regarding "P^j"

In fact $c_j(E)$ is <u>real</u> and even <u>integral</u>;
$\underline{c_j(E) \in H^{2j}(M, \mathbb{Z})}$!

Metric connections

Now assume E is equipped with a bundle metric $\langle\, , \rangle$.

Recall from #8, p. 2: A bundle metric is a section $m \in (E \otimes E)^*$ which is symmetric and positive definite ($\forall p \in M$). We write $\langle v, w\rangle := m_p(v \otimes w)$, $\forall\, v, w \in E_p$.

Def 1: A connection D on E is called metric if $Dm = 0$, or equivalently (cf. Problem 60(d))

$$d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle, \quad \forall\, s_1, s_2 \in \Gamma E.$$

i.e. D respects the bundle metric; cf. def. in Problem 60(c).

Lemma 1: D is metric iff $\mathbb{P}_{p \xrightarrow{\gamma} q} : E_p \to E_q$ is an isometry for any C^∞-curve $\gamma : I \to M$ $(\gamma(0) = p,\ \gamma(1) = q)$.

Def 2: For E a vector bundle equipped with a bundle metric, a bundle chart (U,φ) is called **Metric** if $\varphi_p^{-1}(e_1),\dots,\varphi_p^{-1}(e_n)$ is an ON-basis of E_p, $\forall p\in U$.

Jost Thm 2.1.3: (E,π,M) can be covered by metric bundle charts!

Proof: Given any bundle chart, apply Gram-Schmidt orthogonalisation in each fiber!

Furthermore we set

$$\mathrm{Ad}\,E=\{B\in \mathrm{End}\,E\ :\ \langle Bv,w\rangle=-\langle v,Bw\rangle,\ \forall v,w\in E_{\pi(B)}\}$$

i.e., the matrix for B w.r.t. some ($\Leftrightarrow$ every) ON-basis for $E_{\pi(B)}$ is skew symmetric.

Facts: $\mathrm{Ad}\,E$ is a vector subbundle of $\mathrm{End}\,E$ of rank $\frac{n(n-1)}{2}$, and also a Lie algebra subbundle of ~~End E~~ $gl(E)$ (standard fiber: $o(n)\subseteq gl(n)$).

Furthermore, if D is a metric connection on E (and we write D also for the corresponding connection on End E) then D respects the subbundle $\mathrm{Ad}\,E$, and so D gives a well-defined connection also on $\mathrm{Ad}\,E$. See Problems 55, 63.

Lemma 2: Let D be a metric connection on E and let (U,φ) be a metric bundle chart on E. Write $D = d + A$ w.r.t. (U,φ). Then $A \in \Omega^1(\mathrm{Ad}\, E_{|U})$.

proof: Let $\mu_1, \dots, \mu_n$ be the basis of sections in $\Gamma E_{|U}$ corresponding to (U,φ). Thus $\mu_1(p), \dots, \mu_n(p)$ is an ON-basis of E_p $(\forall p \in U)$, and so:

$$\langle \mu_i, \mu_j \rangle = \delta_{ij} \quad \text{(constant function on } U\text{)}.$$

This implies:

$$0 = d\langle \mu_i, \mu_j \rangle = \langle \underbrace{D\mu_i}_{= d\mu_i + A\mu_i = A\mu_i}, \mu_j \rangle + \langle \mu_i, D\mu_j \rangle$$

$$= \langle A\mu_i, \mu_j \rangle + \langle \mu_i, A\mu_j \rangle \qquad (\text{in } \Omega^1(U))$$

Hence $A \in \Omega^1(\mathrm{Ad}\, E_{|U})$. $\square$

Indeed, for any $X \in T_pM$ $(p \in U)$, we get $\langle A(X)(\mu_i(p)), \mu_j(p) \rangle + \langle \mu_i(p), A(X)(\mu_j(p)) \rangle = 0$, $\forall i,j$.
Hence if $A(X) = \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_n^n \end{pmatrix}$ w.r.t. $\mu_1(p), \dots, \mu_n(p)$, then $a_i^j + a_j^i = 0$, ie. the matrix is skew-symmetric!

Cor. 1: For any $A \in \Omega^1(\mathrm{End}\, E)$, the connection $D + A$ on E is metric iff $A \in \Omega^1(\mathrm{Ad}\, E)$.

proof: Lemma 2 gives $[D+A \text{ metric}] \Rightarrow A \in \Omega^1(\mathrm{Ad}\, E)$. The reverse implication is proved by a completely similar computation.

Cor 2: $F_D \in \Omega^2(\mathrm{Ad}\, E)$.

See Jost, Cor. 4.2.1

11.1. **Notes.** .

p. 3: Note here that $R^k_{jim} = -R^k_{jmi}$; this is immediate from the definition of R^k_{jim} and the antisymmetry $R(X,Y) = -R(Y,X)$ (cf. (6) above; also [5, Cor. 4.1.2]).

p. 4, Theorem 1: This is Jost's [5, Thm. 4.1.2] and we follow Jost's proof (in principle), expanding on some details.

p. 6: Here we derive Jost's formula [5, (4.1.24)] without introducing explicit "coefficients" for A as Jost does.

p. 7: Note that our computation here is the same as in Jost, p. 139 (just above Theorem 4.1.1), [5], except that there is no need to introduce the explicit expansion "$A = A_i dx^i$", since we can refer to the associativity relation in Problem 49(d). Here are some more details from the end of Jost's computation, i.e. not using Problem 49(d) but instead working directly with "$[A_i dx^i, A_j dx^j \wedge A_k dx^k]$" and only using the definition of vector-wedge-product:

$$
\begin{aligned}
&[A_i dx^i, A_j dx^j \wedge A_k dx^k] = [A_i \otimes dx^i, (A_j \circ A_k) \otimes (dx^j \wedge dx^k)] \\
&= [A_i, A_j \circ A_k] \otimes (dx^i \wedge dx^j \wedge dx^k) \\
&= (A_i \circ A_j \circ A_k) \otimes (dx^i \wedge dx^j \wedge dx^k) - (A_j \circ A_k \circ A_i) \otimes (dx^i \wedge dx^j \wedge dx^k) \\
&= (A_i \circ A_j \circ A_k) \otimes (dx^i \wedge dx^j \wedge dx^k) - (A_i \circ A_j \circ A_k) \otimes (dx^k \wedge dx^i \wedge dx^j) \\
&= 0,
\end{aligned}
$$

since $dx^i \wedge dx^j \wedge dx^k = dx^k \wedge dx^i \wedge dx^j$.

In connection with the last computation, let us stress again that

$$
\begin{aligned}
&[B_1, B_2] = B_1 \circ B_2 - (-1)^{rs} B_2 \circ B_1, \\
&\qquad\qquad \forall B_1 \in \Omega^r(\operatorname{End} E),\ B_2 \in \Omega^s(\operatorname{End} E),
\end{aligned}
$$

as we pointed out on p. 7.

12. The Yang-Mills functional

#12. The Yang-Mills functional

Def 1: Let M be a compact Riemannian manifold and let E be a vector bundle over M equipped with a bundle metric. Then the **Yang-Mills functional** is the map

$$D \mapsto \underline{\underline{YM(D) := \int_M \|F_D\|^2 \, dvol}}$$

on the set of metric connections on E.

Needs explanation: $dvol$ & $\|F_D\|$

The measure dvol on M

For $f \in C(M)$, $\underline{\underline{\int_M f \, dvol := \text{"}\int_M f(x)\sqrt{g(x)}\, dx^1 \ldots dx^d\text{"}}}$.

where $\underline{g(x) = \det(g_{ij}(x))} > 0$ since $(g_{ij}(x))$ is positive definite.

More precisely, cover M by charts (U_α, x_α), $\alpha = 1, \ldots, m$, and write $\int_{U_\alpha} f \, dvol := \underbrace{\int_{U_\alpha} f(x)\sqrt{g(x)}\, dx^1 \ldots dx^d}_{\text{wrt } (U_\alpha, x_\alpha)}$

and then $\underline{\int_M \cdots := \sum_{\alpha=1}^{m} \int_{U_\alpha \setminus (U_1 \cup \ldots \cup U_{\alpha-1})} \cdots}$ or: use a partition of unity.

$\|F_D\|^2 = ?$

— this will come from a bundle metric on $\mathrm{Ad}\,E \otimes \Lambda^2 M$. Recall: $F_D \in \Omega^2(\mathrm{Ad}\,E)$. #11, Cor 1 (p. 13)

① For any E equipped with a bundle metric, Ad E has a natural bundle metric:

$$\langle A, B\rangle := -\mathrm{Tr}(\underbrace{AB}_{\in \mathrm{End}\,E_p}) \qquad \text{for any } A, B \in E_p \ (p \in M).$$

trace is well-def; independent of choice of basis for E_p!

The above $\langle A,B\rangle$ is in fact a natural symmetric bilinear form on all End E_p; however restricting to Ad E_p the form is also positive definite; indeed if $A = (A^i_j)$ w.r.t. some basis for E_p then $-\mathrm{Tr}(A^2) = -\sum_{i=1}^n \sum_{j=1}^n A^j_i A^i_j = \sum_{i=1}^n \sum_{j=1}^n (A^j_i)^2 \geq 0$.

(2) M Riemannian

$\Rightarrow$ we have a scalar product $\langle\, ,\, \rangle$ on T_pM (any $p \in M$)

$\Rightarrow$ — " — $\langle\, ,\, \rangle$ on T_p^*M

Indeed, $\langle\, ,\, \rangle$ gives us a natural linear bijection $\underline{T_pM \xrightarrow{\sim} T_p^*M; \quad v \mapsto \langle v, \cdot \rangle}$.
One uses this bijection to <u>transfer $\langle\, ,\, \rangle$ from T_pM to T_p^*M.</u> One verifies that given any ON-basis $e_1, \dots, e_d$ for T_pM, the <u>dual</u> basis $e^1, \dots, e^d$ is ON in T_p^*M!

$\Rightarrow$ we have a scalar product $\langle\, ,\, \rangle$ on $\Lambda^r(T_p^*M)$
(any $r \geq 0$, $p \in M$)

<u>Definition</u>: For $e^1, \dots, e^d$ any ON-basis for T_p^*M, declare $\underline{e^I := e^{i_1} \wedge \dots \wedge e^{i_r}}$ (for I running through all r-tuples in $\{1, \dots, d\}^r$ with $i_1 < \dots < i_r$) to be an <u>ON-basis for $\Lambda^r(T_p^*M)$</u>.

<u>Well-defined?</u> YES; indeed

$$\underline{\langle \alpha^1 \wedge \dots \wedge \alpha^r, \beta^1 \wedge \dots \wedge \beta^r \rangle = \det\left(\langle \alpha^i, \beta^j \rangle\right)}$$

$$\forall\, \alpha^1, \dots, \alpha^r, \beta^1, \dots, \beta^r \in T_p^*M.$$

Hence $\Lambda^r M$ is equipped with a bundle metric

Combining ① and ②, we now also get a scalar product on $(\mathrm{Ad}\, E_p) \otimes \Lambda^r(T_p^* M)$ (any $p \in M$):

$$\underline{\langle \mu_1 \otimes \omega_1, \mu_2 \otimes \omega_2 \rangle := \langle \mu_1, \mu_2 \rangle \cdot \langle \omega_1, \omega_2 \rangle}$$

(any $\mu_1, \mu_2 \in \mathrm{Ad}\, E_p$, $\omega_1, \omega_2 \in \Lambda^r(T_p^* M)$)

– extend to make $\mathbb{R}$-bilinear

Hence $(\mathrm{Ad}\, E) \otimes \Lambda^r M$ is now equipped with a bundle metric.

Hence Def. 1 is now explained.

(Namely $\|F_D(p)\|^2 = \langle F_D(p), F_D(p) \rangle$, any $p \in M$.)

□□

Def 2: A metric connection D on E is called a Yang-Mills connection if it is a critical point of YM

$$\left(\overset{\text{def}}{\Longleftrightarrow} \quad \frac{d}{dt} YM(D+tB)_{|t=0} = 0, \quad \forall\, B \in \Omega^1(\mathrm{Ad}\, E) \right)$$

indeed, recall #11 (p.13) Cor 1

Make explicit? For any $\sigma \in \Gamma E$:

$$F_{D+tB}(\sigma) = (D+tB)(D+tB)(\sigma) =$$

$$= F_D(\sigma) + t\, B \circ D\sigma + t\, D(B(\sigma)) + t^2 B \circ B(\sigma)$$

vector-wedge product; cf. Problem

Use $D(B(\sigma)) = (DB)(\sigma) - B(D\sigma)$; cf. Problem (c) and Problem (c); use $B \in \Omega^1(\mathrm{Ad}\, E)$

$$= \left(F_D + t\,DB + t^2 B \circ B\right)(\sigma)$$

Hence $\displaystyle \frac{d}{dt} YM(D+tB)_{|t=0} = \frac{d}{dt} \int \langle F_{D+tB}, F_{D+tB} \rangle \, d\mathrm{vol}_{|t=0}$

$$= 2 \int_M \langle F_D, DB \rangle \, d\mathrm{vol} = 2 \int_M \langle D^* F_D, B \rangle \, d\mathrm{vol}$$

$\therefore$ $\boxed{\text{Lemma: } D \text{ is YM-connection iff } D^* F_D = 0}$

<u>Def 3 (structure group)</u>: A vector bundle (E, π, M) of rank n is said to have <u>structure group G</u> (or: to be a <u>G-vector bundle</u>) (for G a closed Lie subgroup of $GL_n(\mathbb{R})$) if there is given an atlas of bundle charts on E for which <u>all transition maps are in G</u>.

{cf #10, p. 8 / Jost, p. 42}

{We call such an atlas a <u>G-atlas</u>.}

<u>Ex</u>: <u>Any</u> vector bundle E has structure group $GL_n(\mathbb{R})$.

<u>Ex</u>: If E is equipped with a <u>bundle metric</u> then E has structure group <u>$O(n)$</u>.

{Use <u>metric</u> charts to form an $O(n)$-atlas!}

<u>Def 4</u>: A vector bundle E is said to be <u>oriented</u> if E has structure group <u>$GL_n^+(\mathbb{R})$</u> $:= \{T \in GL_n(\mathbb{R}) : \det T > 0\}$

Also: A manifold M is said to be <u>oriented</u> if (TM, π, M) is oriented!

{Not quite the standard def., but equivalent!}

<u>Ex</u>: If E oriented <u>and</u> has a bundle metric then E is an <u>$SO(n)$-vector bundle</u>

{Namely use <u>oriented & metric</u> charts to form an $SO(n)$-atlas!}

Def 5: For a G-vector bundle E with a G-atlas $\mathcal{A}$:

$\mathrm{Aut}(E_p) := \{\varphi_p^{-1} \circ g \circ \varphi_p \ : g \in G\}$ for any $(U, \varphi) \in \mathcal{A}$ with $p \in U$.

Problem: Verify that this is well-defined, i.e. independent of the choice of (U, φ)!

(cf #7, p.3)

Also $\mathrm{Aut}(E) :=$ the fiber bundle with $\mathrm{Aut}(E)_p := \mathrm{Aut}(E_p)$, $\forall p \in M$.

(this is a fiber bundle with standard fiber $= G$)

Finally the gauge group of E is

$\mathcal{G} := \Gamma(\mathrm{Aut}(E))$ with group operation = pointwise composition.

Any $s \in \mathcal{G}$ is called a gauge transformation

Thus: to give a gauge transformation is to give one point in $\mathrm{Aut}(E_p)$ for each $p \in M$, depending smootly on p!

Facts: Suppose E is a vector bundle with a bundle metric (thus: E is an $O(n)$-bundle) and D is a metric connection on E. Then for every $s \in \mathcal{G}$, also $s^*(D) := s^{-1} \circ D \circ s$ is a metric connection on E.
Its curvature is $s^*F := F_{s^*D} = s^{-1} \circ F \circ s$, and $\|s^*F\| = \|F\|$. Hence:

Thm 1: (Jost Thm 4.2.1)
$YM(s^*D) = YM(D)$ for every $s \in \mathcal{G}$, i.e. the Yang-Mills functional is invariant under $\mathcal{G}$.
Hence also the set of Yang-Mills connections is invariand under $\mathcal{G}$.

12.1. **Notes.** .

p. 1: Here we define the natural volume measure "$d$vol"; this exists on an arbitrary Riemannian manifold M. Note that Jost introduces (in passing) this measure $d$vol already in [5, (1.4.2–3)]. The geometrical motivation for the factor "$\sqrt{g(x)}$" is that this equals the *volume of the parallelotope in T_pM spanned by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}$*, with respect to the natural volume measure on the vector space T_pM; namely the volume measure coming from identifying T_pM with $\mathbb{R}^d$ by any linear isomorphism carrying the Riemannian scalar product on T_pM to the standard Euclidean scalar product on $\mathbb{R}^d$. (Proof of this fact: By Problem 64(c), said volume equals $\|\frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^d}\|$, which, by Problem 64(b), is equal to $\sqrt{g(x)}$.)

To prove that $\int_M f \, d\mathrm{vol}$ is well-defined we need to prove that it does not depend on the choice of the charts (U_α, x_α). For this, it suffices to verify that if (U, x) and (V, y) are any two C^∞ charts, and the Riemannian metric is given by $(g_{ij}(x))$ wrt (U, x) and by $(h_{ij}(y))$ wrt (V, y), then

$$\int_{U \cap V} f(x) \sqrt{g(x)} \, dx^1 \cdots dx^d = \int_{U \cap V} f(y) \sqrt{h(y)} \, dy^1 \cdots dy^d \tag{7}$$

for any $f \in C^\infty(U \cap V)$. (Here of course $g(x) = \det(g_{ij}(x))$ and $h(y) = \det(h_{ij}(y))$.) To prove (7), note that both sides of (7) really stand for integrals over open subsets of $\mathbb{R}^d$ (namely the sets $x(U \cap V)$ and $y(U \cap V)$, respectively), and by the formula for changing variables in d-dimensional integrals (cf., e.g., [12, Thm. 10.9]) we see that the right hand side of (7) equals

$$\int_{U \cap V} f(x) \sqrt{h(y)} \, |\det J(x)| \, dx^1 \cdots dx^d, \tag{8}$$

where $J(x)$ is the Jacobian,

$$J(x) := \begin{pmatrix} \dfrac{\partial y^1}{\partial x^1} & \cdots & \dfrac{\partial y^1}{\partial x^d} \\ \vdots & & \vdots \\ \dfrac{\partial y^d}{\partial x^1} & \cdots & \dfrac{\partial y^d}{\partial x^d} \end{pmatrix}.$$

Next, recall from Lecture #2 that

$$g_{ij}(x) = \frac{\partial y^k}{\partial x^i} \frac{\partial y^\ell}{\partial x^j} h_{k\ell}(y) \qquad \text{in } U \cap V.$$

In terms of matrix multiplication, this means that

$$(g_{ij}(x)) = J(x) \cdot (h_{k\ell}(y)) \cdot J(x)^T.$$

Taking the determinant, this implies:

$$h(y) = (\det J(x))^{-2} \cdot g(x) \qquad \text{in } U \cap V.$$

Inserting this in (8), we conclude that the right hand side of (7) equals the left hand side, as desired! $\square$

p. 3, regarding the scalar product on $\bigwedge^r(T_p^*)$: Jost defines this in the beginning of [5, Ch. 3.3]. The same construction applies to endow $\bigwedge^r(V)$ with a scalar product, whenever V is a finite dimensional vector space over $\mathbb{R}$ equipped with a scalar product. As an aside remark, let us point out that in the special case $V = \mathbb{R}^d$ with its standard scalar product, we have the following geometrical fact: For any "pure tensor" $\beta = v_1 \wedge \cdots \wedge v_r$ (with $v_1, \ldots, v_r \in \mathbb{R}^d$), the "length"

$$\|\beta\| := \sqrt{\langle v_1 \wedge \cdots \wedge v_r, v_1 \wedge \cdots \wedge v_r \rangle}$$

equals the (r-dimensional) volume of the r-dimensional parallelotope spanned by $v_1, \ldots, v_r$! This follows from the formula which we state at the bottom of p. 3 in the lecture.

p. 5: This is the same computation as in [5, (4.2.13–14)]. At the end of the page: Note that we do not have time to introduce the operator D^* in this course (so any discussion/understanding of D^* is extracurricular); however I just wanted to *mention* the final equation, $D^* F_D = 0$...

#13. The Levi-Civita connection.

Let M be a C^∞ manifold.
Let ∇ be a connection on TM.

Def 1: The torsion of ∇ is defined by

$$T(X,Y) = T_\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \qquad (X,Y \in \Gamma(TM))$$

Lemma 1: T is a tensor (field); $T \in \Omega^2(TM)$.

proof: By definition, T is a map

$$T : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM).$$

⊛

It is clearly $\mathbb{R}$-bilinear. In fact it is even $C^\infty(M)$-bilinear. Indeed, for any $f \in C^\infty(M)$ and $X, Y \in \Gamma(TM)$ we have

$$\begin{aligned}
T(fX,Y) &= \nabla_{fX} Y - \nabla_Y(fX) - [fX,Y] \\
&= f\cdot\nabla_X Y - (f\cdot\nabla_Y X + \cancel{(Yf)\cdot X}) - \underbrace{(-\cancel{(Yf)\cdot X} + f\cdot[X,Y])}_{\text{see Problem 47(d)}} \\
&= f\cdot\nabla_X Y - f\cdot\nabla_Y X - f\cdot[X,Y] \\
&= f\cdot T(X,Y),
\end{aligned}$$

and similarly $T(X,fY) = f\cdot T(X,Y)$; this proves the $C^\infty(M)$-bilinearity.

By the defining property of tensor product, the $C^\infty(M)$-bilinear map corresponds to a unique $C^\infty(M)$-linear map

$$\Gamma(TM) \otimes \Gamma(TM) \to \Gamma(TM),$$

and by Problem 43 this map can be identified with a section in

$$\Gamma\, \mathrm{Hom}(TM \otimes TM, TM)$$
$$= \underline{\Gamma\left(TM \otimes (TM \otimes TM)^*\right)}$$

Finally note that T is antisymmetric in X, Y:

$$T(Y, X) = -T(X, Y).$$

Hence the above section (which is "$= T$") is in fact in

$$\Gamma(TM \otimes \Lambda^2(T^*M)) = \underline{\underline{\Omega^2(TM)}}.$$

$\square$

Def. 2: ∇ is said to be torsion free if $T_\nabla \equiv 0$.
(i.e., if $\nabla_X Y - \nabla_Y X \equiv [X, Y], \ \forall\, X, Y \in \Gamma(TM)$).

In local coordinates:

If (U,x) is a chart on M then $\underline{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}}$ is a basis of sections in $\Gamma(TU)$ $\left(= \Gamma(TM_{|U})\right)$, and the Christoffel symbols for ∇ are given by

$$\underline{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^{k} \frac{\partial}{\partial x^k}} \qquad \left(\Gamma_{ij}^{k} \in C^\infty(U)\right).$$

Components of T:

$$\underline{T_{ij}} := T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - 0 = \underline{\left(\Gamma_{ij}^{k} - \Gamma_{ji}^{k}\right)\frac{\partial}{\partial x^k}}.$$

$\left(T_{ij} \in C^\infty(U)\right)$. Hence:

<u>Lemma 2</u>: ∇ is torsion free iff $\underline{\Gamma_{ij}^{k} = \Gamma_{ji}^{k}}$, $\forall i,j,k$, for any chart.

Theorem 1: On each Riemannian manifold M, there exists a unique connection ∇ on TM which is metric and torsion free.

Def. 3: This ∇ is called the Levi-Civita connection of M.

proof: First assume ∇ is metric and torsion free.

We then seek an "explicit formula" for ∇!

Now for any $X, Y, Z \in \Gamma(TM)$:

$$\left.\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \end{aligned}\right\} \text{using } \nabla \text{ metric}$$

Also, since ∇ is torsion free, we get formulas for differences, like $\langle \nabla_X Y, Z\rangle - \langle Z, \nabla_Y X\rangle = \langle [X,Y], Z\rangle$!

Playing around with this, one finds:

$$\begin{aligned} &X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle = \\ &= \langle [X,Z], Y\rangle + \langle [Y,Z], X\rangle + \langle \nabla_X Y + \nabla_Y X, Z\rangle \\ &= \langle [X,Z], Y\rangle + \langle [Y,Z], X\rangle + \langle [Y,X], Z\rangle + 2\langle \nabla_X Y, Z\rangle, \end{aligned}$$

i.e.

$$\boxed{\langle \nabla_X Y, Z\rangle = \tfrac{1}{2}\Big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle \Big)} \quad (*)$$

This determines ∇ uniquely – indeed check that the following def. of ∇ is forced on us!
From now on we focus on proving <u>existence</u> of ∇...

<u>Given</u> $X, Y \in \Gamma(TM)$, define $\underline{w : \Gamma(TM) \to \mathbb{R}}$ by $\underline{w(Z) := [\text{right hand side of } \circledast \text{ on p. 4}]}$.

Note: $\underline{\underline{w \text{ is } C^\infty(M)\text{-linear}}}$.

<u>proof</u>: Clearly w is $\mathbb{R}$-linear. Also, for any $f \in C^\infty(M)$:

$\underline{w(fZ) =}$?

Use $[Y, fZ] = f[Y, Z] + Y(f) \cdot Z$

$[fZ, X] = f[Z, X] - X(f) \cdot Z$ (Problem 47(d))

Also $\underline{X\langle Y, fZ\rangle} = X\big(f \cdot \langle Y, Z\rangle\big) = \underline{(Xf) \cdot \langle Y, Z\rangle + f \cdot X\langle Y, Z\rangle}$

and similarly $\underline{Y\langle fZ, X\rangle = (Yf) \cdot \langle Z, X\rangle + f \cdot Y\langle Z, X\rangle}$.

Hence
$$\underline{w(fZ)} = f \cdot w(Z) + \tfrac{1}{2}\Big((Xf) \cdot \langle Y, Z\rangle + (Yf) \cdot \langle Z, X\rangle - Y(f) \cdot \langle X, Z\rangle - X(f) \cdot \langle Y, Z\rangle\Big)$$
$$= \underline{f \cdot w(Z)}, \ \underline{\text{qed}}$$

Hence $w \in \Gamma(TM)^* = \Gamma(T^*M)$, and using the Riemannian metric there exists a unique $A \in \Gamma(TM)$ (cf. # 8, p. 2 !) such that $\underline{w(Z) = \langle A, Z\rangle, \ \forall Z \in \Gamma(TM)}$.

$\underline{\underline{\text{Define } \nabla \text{ by } \nabla_X Y := A}}$ $(\forall X, Y \in TM)$

{Thus $X, Y \rightsquigarrow w \rightsquigarrow A$.}

Then ∇ is a connection on TM:

Indeed, ① $\nabla_X Y$ is $C^\infty(M)$-linear in X.

This is equivalent with the right hand side in ⊛ on $\rho.Y$ being $C^\infty(M)$-linear in X, and this is proved by almost the same computation as for Z.

② Hence for given $Y \in \Gamma(TM)$,

$$X \mapsto \nabla_X Y$$

is a $C^\infty(M)$-linear map $\Gamma(TM) \to \Gamma(TM)$,

i.e. $\nabla_\cdot Y \in \mathrm{End}(\Gamma(TM)) = \Gamma(\mathrm{End}(TM)) = \Omega^1(TM)$

Therefore $Y \mapsto \nabla Y$ is a map $\Gamma(TM) \to \Omega^1(TM)$.

③ Clearly ∇ is $\mathbb{R}$-linear. Also, for any $f \in C^\infty(M)$,

$$[\text{r.h. ⊛ "for } fY\text{"}] = f \cdot [\text{r.h. ⊛}] + \tfrac{1}{2}\Big((Xf)\cdot\langle Y, Z\rangle - (Zf)\cdot\langle X, Y\rangle + (Zf)\cdot\langle X, Y\rangle + (Xf)\cdot\langle Z, Y\rangle\Big)$$

$$= f \cdot [\text{r.h. ⊛}] + (Xf)\cdot\langle Y, Z\rangle.$$

This means: $\langle \nabla_X(fY), Z\rangle = f\cdot\langle \nabla_X Y, Z\rangle + (Xf)\cdot\langle Y, Z\rangle$ $\forall X, Y, Z \in \Gamma(TM)$.

$\therefore$ $\nabla_X(fY) = f\cdot\nabla_X Y + (Xf)\cdot Y \quad \forall X, Y \in \Gamma(TM)$

$\therefore$ $\nabla(fY) = f\cdot\nabla Y + Y \otimes df, \quad \forall Y \in \Gamma(TM)$. (and $f \in C^\infty(M)$)

This completes the proof that ∇ is a connection on TM.

Also, ∇ is metric.

Indeed, for any X, Y, Z we get from ⊛:

$$\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle = \text{check...} = X\langle Y, Z \rangle$$

Hence ∇ is metric!

Finally, ∇ is torsion free

Indeed, for any X, Y, Z we get from ⊛:

$$\langle \nabla_X Y, Z \rangle - \langle \nabla_Y X, Z \rangle = \text{check...} = \langle [X, Y], Z \rangle$$

True $\forall Z \Rightarrow \nabla_X Y - \nabla_Y X = [X, Y]$. Done!

$\square$ Thm 1 proved!

Explicit formula for Γ_{ij}^k? (given a chart (U,x))

Recall $g_{ij} := \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$ and $(g^{ij}) := (g_{ij})^{-1}$;

i.e. $g^{ij} g_{jk} = \delta_{ik}$ (and $g_{ij} g^{jk} = \delta_{ik}$)

(Also $g_{ij} = g_{ji}$, $g^{ij} = g^{ji}$) Also $\frac{\partial}{\partial x^i} g_{jk} =: \underline{g_{jk,i}}$

Now p.4 $\circledast$ (and $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] \equiv 0$, $\forall\, i,j$)

$$\Rightarrow \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right\rangle = \frac{1}{2}\left(\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij} + 0 \right)$$

$$= \frac{1}{2}\left(g_{jk,i} + g_{ki,j} - g_{ij,k} \right)$$

But also $\circledast = \left\langle \Gamma_{ij}^l \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} \right\rangle = \Gamma_{ij}^l \cdot g_{lk}$

matrix times vector; multiply with the inverse matrix to get hold of Γ_{ij}^l!

Multiply the above identity with g^{km}, then add over k!

$$\Rightarrow \underbrace{\Gamma_{ij}^l g_{lk} g^{km}}_{} = \frac{1}{2} g^{km}\left(g_{jk,i} + g_{ki,j} - g_{ij,k} \right)$$

$$= \Gamma_{ij}^l \delta_{l,m} = \underline{\underline{\Gamma_{ij}^m}}$$

$$\therefore \boxed{\Gamma_{ij}^m = \frac{1}{2} g^{km}\left(g_{jk,i} + g_{ki,j} - g_{ij,k} \right)}$$

This is the same as Γ_{ij}^m in the Euler-Lagrange equation for geodesics (Lemma 5 in #3)!

In fact for any connection ∇ on TM we say that a curve ~~[illegible]~~ $\gamma : I \to M$ is autoparallel (or "geodesic") wrt ∇ if

$\nabla_{\dot\gamma}(\dot\gamma) \equiv 0.$ ← Jost, Def 4.1.6

pedantic: $(\gamma^*\nabla)_1(s)$

see #9, p. 7–10; there we discussed "$\nabla_{\dot\gamma}(s)$" for an arbitrary $s \in \Gamma_\gamma(TM)$, and called it $\dot s(t)$; applying this to $\dot\gamma \in \Gamma_\gamma(TM)$ - for any C^∞-curve $\gamma : I \to M$, we might write "$\ddot\gamma(t)$" for $\nabla_{\dot\gamma}(\dot\gamma)$!

Also from #9, p. 7–10 we see:

$\nabla_{\dot\gamma}(\dot\gamma) \equiv 0 \iff \dot\gamma \equiv P_\gamma(\dot\gamma(0))$

i.e. $\dot\gamma$ is parallel along γ!

In local coordinates (U, x) we have (by #9, p. 8)

$$\nabla_{\dot\gamma}(\dot\gamma) \equiv 0 \iff \left[\ddot\gamma^k + \Gamma_{ij}^k \dot\gamma^i \dot\gamma^j \equiv 0 \quad (\forall k)\right]$$

and so the γ is a geodesic iff γ is autoparallel w.r.t. the Levi-Civita connection!

14. Curvature of Riemannian manifolds

#14. Curvature of Riemannian manifolds

M — a Riemannian manifold.

∇ — the Levi-Civita connection on M.

$R = F = \nabla \circ \nabla \in \Omega^2(\mathrm{End}\, TM)$ — the <u>curvature</u>.

<u>Recall</u>:

$$\underline{R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z}, \qquad \forall X, Y, Z \in \Gamma(TM)$$

(#11, Thm 1). Also, for (U,x) a chart on M:

$$\underline{R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \underline{R^l_{kij}} \frac{\partial}{\partial x^l}} \qquad \boxed{\begin{array}{l} R^l_{kij} \in C^\infty(U), \\ R^l_{kij} = -R^l_{kji} \end{array}}$$

(#11, p. 3).

We now introduce the tensor field $\underline{Rm \in \Gamma(T^0_4 M)}$

by $\underline{Rm(X,Y,Z,W) := \langle R(Z,W)Y, X\rangle}$, $\qquad \forall X, Y, Z, W \in \Gamma(TM)$.

Thus w.r.t. a chart (U,x):

$$\underline{Rm = \underline{R_{ijkl}}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l}$$

with $\underline{R_{ijkl}} = Rm\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = \left\langle R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right\rangle$

$$= \underline{g_{im} R^m_{jkl}}$$

In many books the tensor field Rm is <u>also</u> called just "R". Jost gives <u>no</u> name to the tensor field Rm, but only to its coefficients R_{ijkl}.

Note that we follow Jost's notation re R_{ijkl}, and note the strange permutation we have of X, Y, Z, W in the def. of Rm! In fact many different conventions/definitions exist in the literature, however they all agree up to sign (as one sees using Lemma 1 below).

Note that with our definitions we have

$$Rm(X, Y, Z, W) = -\langle R(X, Y) Z, W\rangle \quad \text{(by Lemma 1 below)}.$$

Lemma 1: basic identities for curvature

① $R_{ijkl} = -R_{ijlk}$ (clear from $R^m_{jkl} = -R^m_{jlk}$)

② $R_{ijkl} + R_{iklj} + R_{iljk} = 0$

i.e. think... $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.

③ $\langle R(X, Y) Z, W\rangle = -\langle R(X, Y) W, Z\rangle$, i.e. $R_{ijkl} = -R_{jikl}$

④ $\langle R(X, Y) Z, W\rangle = \langle R(Z, W) X, Y\rangle$, i.e. $R_{ijkl} = R_{klij}$

proof ① done.

$C^\infty(M)$-linearity $\Rightarrow$ it suffices to check ②, ③, ④

for $X, Y, Z, W \in \{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}\}$. (thus $[X,Y]=[X,Z]=\ldots=0$)

② $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$

$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y$

use $\nabla_Y Z - \nabla_Z Y = [Y,Z] = 0$, since ∇ torsion free!
Similarly $\nabla_X Z - \nabla_Z X = 0$ and $\nabla_Y X - \nabla_X Y = 0$.

$= 0.$

③ Immediate from Cor. 2 in #11 (p. 13)!

Indeed, that Cor. 2 says that $F_D \in \Omega^2(\mathrm{Ad}\, E)$ for any metric connection D on a vector bundle E equipped with a bundle metric. This means that $F_D(X,Y) \in \Gamma(\mathrm{Ad}\, E)$, $\forall\, X, Y \in \Gamma(TM)$, and by the definition of $\mathrm{Ad}\, E$ (#11, p. 11) this means that $\langle F_D(X,Y)(\mu_1), \mu_2\rangle = -\langle F_D(X,Y)(\mu_2), \mu_1\rangle$, $\forall\, \mu_1, \mu_2 \in \Gamma E$. Now apply this with $E = TM$!

④ PLAY with ① – ③ to express $\langle R(X,Y)Z, W\rangle$ in a nice "symmetric" way:

$$\langle R(X,Y)Z, W\rangle \underset{②}{=} -\langle R(Y,Z)X, W\rangle - \langle R(Z,X)Y, W\rangle$$
$$= \langle R(Z,Y)X, W\rangle + \langle R(X,Z)Y, W\rangle$$

and

$$\langle R(X,Y)Z, W\rangle \underset{③}{=} -\langle R(X,Y)W, Z\rangle$$
$$\underset{②}{=} \langle R(Y,W)X, Z\rangle + \langle R(W,X)Y, Z\rangle$$

Adding the above gives:

$$\boxed{\langle R(X,Y)Z, W\rangle = \frac{1}{2}\Big(\langle R(Z,Y)X, W\rangle + \langle R(X,Z)Y, W\rangle + \langle R(Y,W)X, Z\rangle + \underbrace{\langle R(W,X)Y, Z\rangle}_{= \langle R(X,W)Z, Y\rangle}\Big)}$$

by ①, ③

The last expression is invariant under $\begin{bmatrix} X \leftrightarrow Z \\ Y \leftrightarrow W \end{bmatrix}$!

Hence we get ④.

□

Lemma 2 (Second Bianchi Identity)

$$⊛ \quad \underbrace{(\nabla_X R)}(Y,Z) + \underbrace{(\nabla_Y R)}(Z,X) + \underbrace{(\nabla_Z R)}(X,Y) \equiv 0, \quad \forall X,Y,Z \in \Gamma(TM)$$

Covariant derivative of $R \in \Gamma((\mathrm{End}\,TM)\otimes T^0_2M)$, **not** exterior covariant derivative!

Equivalently:

$$⊛⊛ \quad (\nabla_X Rm)(V,W,Y,Z) + (\nabla_Y Rm)(V,W,Z,X) + (\nabla_Z Rm)(V,W,X,Y) \equiv 0$$

$$\forall X,Y,Z,V,W \in \Gamma(TM)$$

Note: Jost's Lemma 4.3.2 seems incorrect! Why would his expression in (4.3.14) be a "tensor" ??

Proof: ⊛ is just a reformulation of $D(F_D) \equiv 0$ (#11, Thm 2) – indeed see Problem 62.

To see ⊛ $\Leftrightarrow$ ⊛⊛, note that (for any $X,Y,Z,V,W \in \Gamma(TM)$):

$$\underbrace{\langle \underbrace{[(\nabla_X R)(Y,Z)]}_{\text{in } \Gamma(\mathrm{End}\,TM)} W, V \rangle}_{\text{in } C^\infty(M)} \overset{\uparrow}{=}$$

∇ metric, and general formula for ∇ on contractions; Problem 59.

$$= X\langle R(Y,Z)W,V\rangle - \langle R(\nabla_X Y,Z)W,V\rangle - \langle R(Y,\nabla_X Z)W,V\rangle$$

$$- \langle R(Y,Z)(\nabla_X W),V\rangle - \langle R(Y,Z)W,\nabla_X V\rangle =$$

$= X\big(Rm(V,W,Y,Z)\big) - Rm(V,W,\nabla_X Y, Z) - Rm(V,W,Y,\nabla_X Z)$

$- Rm(\nabla_X V, W, Y, Z) - Rm(V, \nabla_X W, Y, Z) =$

$= \underline{\underline{(\nabla_X Rm)(V,W,Y,Z)}}.$

Using this (and two analogous identities), we see that ⊛ ⟺ ⊛⊛ !

□ □

Alternative: (following Jost, Lemma 4.3.2)

We seek to prove the formula at a fixed $p \in M$.

Choose a chart (U, x) which is normal at p.

$(x(p) = 0 \in \mathbb{R}^d)$ ← thus in the following, "at p" and "at 0" means the same thing!

Then $g_{ij}(0) = \delta_{ij}$, $\Gamma^i_{jk}(0) = 0$, $g_{ij,k}(0) = 0$ $(\forall i,j,k)$

#4, p. 7: Lemma 1

Notation: $g_{ij,k} = \dfrac{\partial g_{ij}}{\partial x^k}$.

Let us use short-hand notation: $\partial_j := \dfrac{\partial}{\partial x^j}$

Now at p: $(\nabla_{\partial_h} Rm) = (\partial_h R_{klij}) \cdot dx^k \otimes dx^l \otimes dx^i \otimes dx^j$

see Problem 66, and using $\Gamma^l_{jk}(0) = 0$, $\forall j,k,l$

Hence (using also $\mathbb{R}$-multilinearity) it now suffices to prove $\underline{\underline{\partial_h R_{kl ij} + \partial_i R_{kljh} + \partial_j R_{klhi} = 0}}$ $\underline{\underline{\text{at } p}}$ ⊛

($\forall h, k, l, i, j$.)

Now $\underline{R_{klij} = g_{km} R^m_{lij}}$; thus

$$\partial_h R_{klij} = (\partial_h g_{km}) \cdot R^m_{lij} + g_{km} (\partial_h R^m_{lij}),$$

and $\underline{\text{at } p}$ we have $\partial_h g_{km} = 0$ and $g_{km} = \delta_{k,m}$;

hence $\underline{\underline{\partial_h R_{klij} = \partial_h R^k_{lij} \quad \text{at } p}}$ $\boxed{(1)}$

Next, by #11, p.3:

$$\underline{\underline{R^k_{lij} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}}} \qquad \text{(in all } U\text{)}$$

and by #13, p.8:

$$\underline{\underline{\Gamma^k_{jl} = \frac{1}{2} g^{mk}(g_{lm,j} + g_{jm,l} - g_{jl,m})}} \qquad \text{(in all } U\text{)}$$

Hence:

$$\underline{\underline{\frac{\partial \Gamma^k_{jl}}{\partial x^i} = \frac{1}{2}\frac{\partial g^{km}}{\partial x^i}(g_{lm,j} + g_{jm,l} - g_{jl,m})}} \qquad \boxed{(2)}$$

$$\underline{\underline{+ \frac{1}{2} g^{mk}(g_{lm,ji} + g_{jm,li} - g_{jl,mi})}} \qquad \underline{\text{in all } U}$$

Notation: $\underline{\underline{g_{lm,ji} := \frac{\partial^2 g_{lm}}{\partial x^j \partial x^i}}}$, symmetric both $l \leftrightarrow m$ and $j \leftrightarrow i$!

At O we have $\dfrac{\partial g^{km}}{\partial x^i} = 0$ (- exercise; follows from $g_{ab}(O) = \delta_{a,b}$ and $\dfrac{\partial}{\partial x^c} g_{a,b} = 0$)

and $g^{km} = \delta_{k,m}$ and hence

$$\frac{\partial \Gamma^k_{j\ell}}{\partial x^i} = \frac{1}{2}\left(g_{\ell k,ji} + g_{jk,\ell i} - g_{j\ell,ki}\right) \qquad \text{at } O$$

Hence at O:

$$R_{k\ell ij} = R^k_{\ell ij} = \frac{\partial \Gamma^k_{j\ell}}{\partial x^i} - \frac{\partial \Gamma^k_{i\ell}}{\partial x^j} + 0 - 0 =$$

$$= \frac{1}{2}\left(g_{\ell k,ji} + g_{jk,\ell i} - g_{j\ell,ki}\right) - \frac{1}{2}\left(g_{\ell k,ij} + g_{ik,\ell j} - g_{i\ell,kj}\right)$$

$$= \frac{1}{2}\left(g_{jk,\ell i} + g_{i\ell,kj} - g_{j\ell,ki} - g_{ik,\ell j}\right)$$

We have thus proved:

Lemma 3: If (U,x) are normal coordinates at p, then $R_{k\ell ij} = R^k_{\ell ij} = \frac{1}{2}\left(g_{jk,\ell i} + g_{i\ell,kj} - g_{j\ell,ki} - g_{ik,\ell j}\right)$ at p.

Beautiful, symmetric formula! Note that all of Lemma 1 can be rederived from Lemma 3; namely ①, ③, ④ are immediate and ② is proved as follows:

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} =$$

$$= \frac{1}{2}\big(\cancel{g_{il,jk}} + \cancel{g_{jk,il}} - \cancel{g_{ik,jl}} - \cancel{g_{jl,ik}}$$
$$+ \cancel{g_{ij,kl}} + \cancel{g_{kl,ij}} - \cancel{g_{il,kj}} - \cancel{g_{kj,il}}$$
$$+ \cancel{g_{ik,lj}} + \cancel{g_{lj,ik}} - \cancel{g_{ij,lk}} - \cancel{g_{lk,ij}} \big) \; \underline{= 0.}$$

Continuing now with our alternative proof of Lemma 2:

From p. 7:

$$\underline{\partial_h R_{klij} = \partial_h R^k_{\ lij}} = \underline{\frac{\partial^2 \Gamma^k_{jl}}{\partial x^h \partial x^i} - \frac{\partial^2 \Gamma^k_{il}}{\partial x^h \partial x^j} + \underbrace{\frac{\partial}{\partial x^h}\left(\Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}\right)}_{\substack{= 0 \text{ at } 0 \\ \text{since } \Gamma^k_{im}(0) = \ldots = 0}}}$$

Also, from p. 7 $\boxed{(2)}$:

$$\underline{\frac{\partial^2 \Gamma^k_{jl}}{\partial x^h \partial x^i} = \frac{1}{2}\frac{\partial^2 g^{km}}{\partial x^h \partial x^i}\left(g_{lm,j} + g_{jm,l} - g_{jl,m}\right)}$$
$$\underline{+ \frac{1}{2}\frac{\partial g^{km}}{\partial x^i}\left(g_{lm,jh} + g_{jm,lh} - g_{jl,mh}\right)}$$
$$\underline{+ \frac{1}{2}\frac{\partial g^{km}}{\partial x^h}\left(g_{lm,ji} + g_{jm,li} - g_{jl,mi}\right)}$$
$$\underline{+ \frac{1}{2} g^{km}\left(g_{lm,jih} + g_{jm,lih} - g_{jl,mih}\right)},$$

in all U. At p, the first 3 lines vanish, since $g_{lm,j}(0) = 0$, $\forall\, l,m,j$ and $\frac{\partial g^{km}}{\partial x^i}(0) = 0$, $\forall\, k,m,i$.

Hence we get:

Lemma 3′: If (U, x) are normal coordinates at p, then $\partial_h R_{klij} = \partial_h R^k_{lij} = \frac{1}{2}\left(g_{jk,lih} + g_{il,kjh} - g_{jl,kih} - g_{ik,ljh}\right)$ at p.

Using Lemma 3′ we now complete our alternative proof of Lemma 2 (the Second Bianchi Identity), namely by proving p. 7 ⊛:

$$\partial_h R_{klij} + \partial_i R_{kljh} + \partial_j R_{klhi} =$$

$$= \frac{1}{2}\Big(\cancel{g_{jk,lih}} + \cancel{g_{il,kjh}} - \cancel{g_{jl,kih}} - \cancel{g_{ik,ljh}}$$

$$+ \cancel{g_{kh,lji}} + \cancel{g_{lj,khi}} - \cancel{g_{kj,lhi}} - \cancel{g_{lh,kji}}$$

$$+ \cancel{g_{ki,lhj}} + \cancel{g_{lh,kij}} - \cancel{g_{kh,lij}} - \cancel{g_{li,khj}}\Big) = 0.$$

□

14.1. **Notes.** .

pp. 6–10: Here we follow Jost's proof of [5, Lemma 4.3.2], giving more details. Key points are the beautifully symmetric formulas

$$R_{ijk\ell} = \frac{1}{2}(g_{i\ell,jk} + g_{jk,i\ell} - g_{ik,j\ell} - g_{j\ell,ik})$$

and

$$\frac{\partial}{\partial x^h} R_{ijk\ell} = \frac{1}{2}(g_{i\ell,jkh} + g_{jk,i\ell h} - g_{ik,j\ell h} - g_{j\ell,ikh}),$$

which we state as Lemmata 3 and 3'. It is of course very important to remember that these formulas hold only *at the point* $p \in M$, and under the assumption that we are using *normal coordinates around* p*!* As we point out p. 8 (bottom), the first of the above two formulas allows us to immediately (re-)prove all of Lemma 1.

p. 7(top): This formula "$\partial_h R_{k\ell ij} + \partial_i R_{k\ell jh} + \partial_j R_{k\ell hi} = 0$" is what most directly implies our Lemma 2 (via $\mathbb{R}$-multilinearity); however note that Jost's (4.3.14), "$\partial_h R_{k\ell ij} + \partial_k R_{\ell hij} + \partial_\ell R_{hkij} = 0$" is *also* true[13] indeed these two formulas are seen to be equivalent by using the fact that $R_{abcd} \equiv R_{cdab}$.

---

[13] in the same context, i.e. *at the point* p, and assuming normal coordinates around p.

Sectional / Ricci / scalar curvature — Lecture #15

Def 1: $K(X,Y) := \langle R(X,Y)Y, X\rangle = Rm(X,Y,X,Y)$ $\forall X,Y \in T_pM$.

Also, for $X, Y \in T_pM$ linearly independent,

$$K(X\wedge Y) := \frac{K(X,Y)}{|X\wedge Y|^2}$$

cf. Problem 64

$K(X\wedge Y)$ depends only on the 2-dimensional plane spanned by X, Y in T_pM (Problem 69)

and $K(X\wedge Y)$ is called the sectional curvature of that 2-plane.

also called: a space form – Jost Def. 4.3.4

Ex: A space of constant curvature means "a Riemannian manifold of constant sectional curvature".

By the Killing-Hopf Theorem, a simply connected space of constant curvature is either

① a sphere, ② Euclidean $\mathbb{R}^d$, or ③ hyperbolic d-space!

Def 2: The Ricci tensor (on a Riemannian manifold M)

is $Ric(X,Y) := g^{jl}\left\langle R\left(X, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}, Y\right\rangle$ (any $X,Y\in T_pM$).

Thus $Ric \in \Gamma(T^0_2(M))$. Also, for $X \in T_pM$ with $\|X\|=1$, the Ricci curvature in direction X $:= Ric(X,X)$.

Note symmetry: $Ric(X,Y) = Ric(Y,X)$!

__Well-defined?__ The def. should of course be understood to say that for (U, x) a C^∞ chart with $p \in U$ and $(g^{jl}(x))$ = the inverse of the matrix giving the Riemannian metric, $Ric(X,Y) = g^{jl}\left\langle R\left(X, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}, Y\right\rangle$.

Now if (V, y) is any other C^∞ chart with $p \in V$, and the matrix giving the Riemannian metric wrt (V, y) is $(h_{ij}(y))$, then

$$h_{ij} = \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} \cdot g_{kl} \quad \text{and so} \quad h^{ij} = \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l} g^{kl}$$

and $h^{jl}\left\langle R\left(X, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^l}, Y\right\rangle =$

$$= \frac{\partial y^j}{\partial x^k}\frac{\partial y^l}{\partial x^m} g^{km}\left\langle R\left(X, \frac{\partial x^r}{\partial y^j}\frac{\partial}{\partial x^r}\right) \cdot \frac{\partial x^s}{\partial y^l} \cdot \frac{\partial}{\partial x^s}, Y\right\rangle$$

$$= \delta_{k,r}\,\delta_{m,s}\, g^{km}\left\langle R\left(X, \frac{\partial}{\partial x^r}\right)\frac{\partial}{\partial x^s}, Y\right\rangle$$

$$= g^{km}\left\langle R\left(X, \frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^m}, Y\right\rangle; \quad \text{ok!}$$

__Alternative, intrinsic:__ Given $X, Y \in \Gamma(TM)$, define $A := \langle R(X, \cdot)\cdot, Y\rangle \in \Gamma(T^0_2 M)$ and then define $A^\# \in \Gamma(T^1_1 M) = \Gamma(\mathrm{End}\, M)$ by $\langle A^\#(Z), W\rangle = A(Z, W)$, $\forall Z, W \in \Gamma(TM)$

(Cf. Lecture #8, p. 2.) Then $Ric(X,Y) = tr\, A^\#$!

Def 3: The <u>scalar curvature</u> of M is

$$R := g^{ik} \cdot Ric\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) \quad \in C^\infty(M).$$

Note:

- Well-defined; just as for Ric.
- In local coordinates, Ric has coefficients $R_{ik} = g^{jl} R_{ijkl}$, and the scalar curvature is $R = g^{ik} R_{ik}$.
- The <u>Ricci curvature</u> ~~equ~~ in direction X equals $d-1$ times the <u>uniform average of the sectional curvature of all planes in T_pM containing X.</u> <u>Similarly the scalar curvature</u> ∝ <u>average of Ricci curvatures.</u> — Problem 71

Lemma 1: At each point $p \in M$, the sectional curvature determines $(Rm)_p$ uniquely!

Precise statement: For any vector space V over $\mathbb{R}$ and any multilinear forms $\mathcal{R}_1, \mathcal{R}_2 : V \times V \times V \times V \to \mathbb{R}$ subject to

$$\mathcal{R}_j(X,Y,Z,W) = -\mathcal{R}_j(Y,X,Z,W) = -\mathcal{R}_j(X,Y,W,Z) = \mathcal{R}_j(Z,W,X,Y)$$

and $\mathcal{R}_j(X,Y,Z,W) + \mathcal{R}_j(Y,Z,X,W) + \mathcal{R}_j(Z,X,Y,W) = 0$

$(\forall X,Y,Z,W \in V, \ \forall j \in \{1,2\})$,

<u>if</u> $\mathcal{R}_1(X,Y,Y,X) = \mathcal{R}_2(X,Y,Y,X)$, $\forall X,Y \in V$,

<u>then</u> $\mathcal{R}_1 \equiv \mathcal{R}_2$.

proof: Consider $\mathcal{R}_0 := \mathcal{R}_1 - \mathcal{R}_2$; then $\mathcal{R}_0(X,Y,Y,X) = 0$, $\forall\, X, Y \in V$. We want to prove $\mathcal{R}_0 \equiv 0$.

① For all $X, Y, Z \in V$ we have

$$0 = \mathcal{R}_0(X+Y, Z, Z, X+Y) = \mathcal{R}_0(X, Z, Z, X) + \mathcal{R}_0(X, Z, Z, Y) + \mathcal{R}_0(Y, Z, Z, X) + \mathcal{R}_0(Y, Z, Z, Y)$$

$$= 2\,\mathcal{R}_0(X, Z, Z, Y)$$

$$\therefore \quad \boxed{\mathcal{R}_0(X, Z, Z, Y) = 0, \quad \forall\, X, Y, Z \in V}$$

② This leads to

$$\boxed{\mathcal{R}_0(X, Y, Z, W) = -\mathcal{R}_0(X, Z, Y, W), \quad \forall\, X, Y, Z, W \in V}$$

③ Hence $\mathcal{R}_0$ is antisymmetric under all three transpositions $\mathcal{R}_0(X, Y, Z, W)$. These generate S_4; hence $\boxed{\mathcal{R}_0 \in \Lambda^4(V^*)}$

④ Now $\forall\, X, Y, Z, W \in V$:

$$0 = \mathcal{R}_0(X, Y, Z, W) + \underbrace{\mathcal{R}_0(Y, Z, X, W)}_{= \mathcal{R}_0(X, Y, Z, W)} + \underbrace{\mathcal{R}_0(Z, X, Y, W)}_{= \mathcal{R}_0(X, Y, Z, W)}$$

plus sign since a 3-cycle is an even permutation.

$$\Rightarrow \quad 3\,\mathcal{R}_0(X, Y, Z, W) = 0; \quad \text{done!}$$

□

From the above proof it is a standard exercise to obtain an explicit formula for R in terms of K. – Problem 72

Theorem 1 (Schur): Assume $d = \dim M \geq 3$.

(a) If there is a function $f: M \to \mathbb{R}$ such that for all $x \in M$ and all $X, Y \in T_x M$ (linearly independent) one has $K(X \wedge Y) = f(x)$; then $f(x) \equiv \text{const}$.

That is: If the sectional curvature of M is constant at each point then the constant is the same everywhere, i.e. M is a space form!

(b) If there is a function $c: M \to \mathbb{R}$ such that for all $x \in M$, the Ricci curvature in every direction in $T_x M$ equals $c(x)$, then $c(x) \equiv \text{const}$, i.e. M is an Einstein manifold.

Def!

Note: The assumption in (b) is: $Ric(X, X) = c(x)$ for every $X \in T_x M$ with $\|X\| = 1$.

$\Leftrightarrow$ $Ric(X, X) = c(x) \|X\|^2$ for all $X \in T_x M$

$\Leftrightarrow$ $Ric(X, Y) = c(x) \langle X, Y \rangle$ for all $X, Y \in T_x M$

By polarization; namely expand $Ric(X+Y, X+Y) = c(x) \|X+Y\|^2$

In local coordinates, this is

$\Leftrightarrow$ $R_{ik} = c(x) \cdot g_{ik}$

Similarly M Einstein $\Leftrightarrow$ $R_{ik} \equiv c \cdot g_{ik}$

Proof, part a: The assumption implies

$$K(X,Y) = f(x)\cdot |X\wedge Y|^2 \qquad \forall\, X, Y \in T_xM$$

(also if X, Y are linearly dependent, trivially)

Write (in local coordinates): $X = \alpha^j \frac{\partial}{\partial x^j}$, $Y = \beta^j \frac{\partial}{\partial x^j}$; then

$$|X\wedge Y|^2 = \langle X\wedge Y, X\wedge Y\rangle = \left\langle \alpha^i \frac{\partial}{\partial x^i} \wedge \beta^k \frac{\partial}{\partial x^k}, \alpha^j \frac{\partial}{\partial x^j} \wedge \beta^l \frac{\partial}{\partial x^l} \right\rangle$$

$$= \alpha^i \beta^k \alpha^j \beta^l \left\langle \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^l} \right\rangle$$

$$= \alpha^i \alpha^j \beta^k \beta^l \cdot \begin{vmatrix} g_{ij} & g_{il} \\ g_{kj} & g_{kl} \end{vmatrix} \qquad \text{See Problem 64}$$

$$= \alpha^i \alpha^j \beta^k \beta^l (g_{ij} g_{kl} - g_{il} g_{kj}),$$

and thus

$$R_{ijkl}\, \alpha^i \beta^j \alpha^k \beta^l = Rm\left(\alpha^i \frac{\partial}{\partial x^i}, \beta^j \frac{\partial}{\partial x^j}, \alpha^k \frac{\partial}{\partial x^k}, \beta^l \frac{\partial}{\partial x^l} \right) =$$

$$= K\left(\alpha^i \frac{\partial}{\partial x^i}, \beta^j \frac{\partial}{\partial x^j} \right) = f(x) \cdot (g_{ij} g_{kl} - g_{il} g_{kj})\, \alpha^i \alpha^j \beta^k \beta^l$$

$$= f(x) \cdot (g_{ik} g_{jl} - g_{il} g_{kj})\, \alpha^i \beta^j \alpha^k \beta^l$$

Now by Lemma 1, this determines Rm (i.e. all R_{ijkl}) uniquely! In fact, using Problem 72 we could compute (by a fairly long computation) all R_{ijkl}; however it is easier to just guess!

We claim: $\underline{\underline{R_{ijkl} = f(x)\cdot(g_{ik}g_{jl} - g_{il}g_{jk})}}$ $\circledast$

proof: At any fixed $x \in M$, $\circledast$ makes Rm have all the required symmetries, i.e. $Rm(X,Y,Z,W) = -Rm(X,Y,W,Z) = -Rm(Y,X,Z,W) = Rm(Z,W,X,Y)$ and $Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0$ (verify!) and also $Rm(X,Y,X,Y) = f(x)\cdot |X \wedge Y|^2$.
Hence by Lemma 1, Rm must be given by $\circledast$ at x!

Now fix a point $p \in M$ and let (U,x) be normal coordinates around p. Then by the second Bianchi identity (cf #14; Lemma 2 and p.7 $\circledast$):

$$\partial_h R_{ijkl} + \partial_k R_{ijlh} + \partial_l R_{ijhk} = 0 \quad \text{at } p$$

$\partial_h := \dfrac{\partial}{\partial x^h}$

Using this together with $\circledast$ and the fact that all $\partial_h g_{ij} = 0$ at p, we get:

$$(\partial_h f)\cdot(g_{ik}g_{jl} - g_{il}g_{jk}) + (\partial_k f)\cdot(g_{il}g_{jh} - g_{ih}g_{jl}) + (\partial_l f)\cdot(g_{ih}g_{jk} - g_{ik}g_{jh}) = 0$$

at p.

Also $g_{ik} = \delta_{i,k}$ at p!

Now for every $h \in \{1, \dots, d\}$ we can find $i, j, k, l \in \{1, \dots, d\}$ such that $(i = k) \neq (j = l)$

$$\neq \; h \; \neq$$

Possible since $d \geq 3$!

Then the previous equation reads:

$$(\partial f_h) \cdot (1-0) + (\partial_k f)(0-0) + (\partial_l f) \cdot (0-0) = 0,$$

i.e. $\partial f_h = 0$ at p.

This is true for all $h \in \{1, \dots, d\}$; hence $df_p = 0$.

This is true for all $p \in M$ and M is connected; hence $f \equiv$ constant.

$\square$

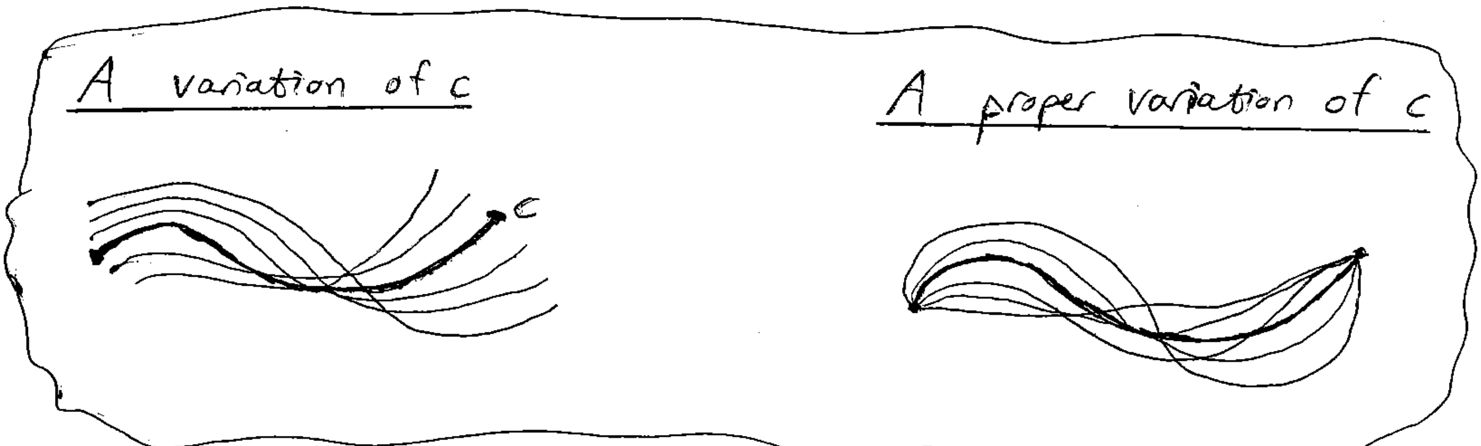
15.1. **Notes.** .

p. 5, Theorem 1: This is Jost, [5, Thm. 4.3.2].

A correction in Jost's book: The formula [5, p. 165, (4.3.20)] is incorrect; it should be "$R_{ijk\ell} = K(g_{ik}g_{j\ell} - g_{i\ell}g_{jk})$"; indeed cf. "(*)" on p. 7 in the lecture. Also the formula on [5, p. 166 (line 3)] is incorrect, it is corrected by negating one of the sides.

#16. 1st & 2nd variation of arc length and energy

Let M be a Riemannian manifold with Levi-Civita connection ∇. Let $c : [a,b] \to M$ be a C^∞ curve. Recall from #3, p 8 that a variation of c is a C^∞ map $F : [a,b] \times (-\varepsilon, \varepsilon) \to M$ with $F(t,0) = c(t)$, $\forall\, t \in [a,b]$. The variation is called proper if $F(a,s) = c(a)$, $F(b,s) = c(b)$, $\forall\, s \in (-\varepsilon, \varepsilon)$.



We also write $c(t,s) = c_s(t) = F(t,s)$.

Also:
$$\dot{c}(t,s) = \frac{\partial}{\partial t} c(t,s) = dF\left(\frac{\partial}{\partial t}\Big|_{(t,s)}\right) \in T_{c(t,s)}(M)$$

$$c'(t,s) = \frac{\partial}{\partial s} c(t,s) = dF\left(\frac{\partial}{\partial s}\Big|_{(t,s)}\right) \in T_{c(t,s)}(M)$$

We are going to study (for a given variation of c):
$E(s) := E(c_s)$ and $L(s) := L(c_s)$.

Lemma 1: Jost Lemma 5.1.1

$E, L \in C^1((-\varepsilon, \varepsilon))$ (in fact even C^∞),

and $$E'(s) = \langle c', \dot{c} \rangle \Big]_{t=a}^{t=b} - \int_a^b \langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c} \rangle \, dt$$

and $$L'(s) = \int_a^b \frac{\frac{\partial}{\partial t} \langle c', \dot{c} \rangle - \langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c} \rangle}{\| \dot{c} \|} \, dt$$

Explanations

① The formulas hold $\forall s \in (-\varepsilon, \varepsilon)$. The expressions should be evaluated "at s".

② $\langle c', \dot{c} \rangle \Big]_{t=a}^{t=b} = \langle c'(b,s), \dot{c}(b,s) \rangle - \langle c'(a,s), \dot{c}(a,s) \rangle$

③ $\nabla_{\frac{\partial}{\partial t}} \dot{c} = ?$

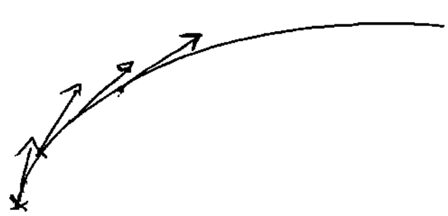
Here "∇" is short-hand for the connection $F^*(\nabla)$ on $F^*(TM)$.

$$\begin{tikzcd} F^*(TM) \arrow[d] & TM \arrow[d, "\pi"] \\ {[a,b] \times (-\varepsilon, \varepsilon)} \arrow[r, "F"] \arrow[ur, bend left, "\dot{c}"] \arrow[ur, "c'"'] & M \end{tikzcd}$$

$\dot{c}, c' \in \Gamma(F^*(TM))$ - Problem 44(a); hence

$\nabla_{\frac{\partial}{\partial t}} \dot{c} \in \Gamma(F^*(TM))$

"rate of change of $\dot{c}_s$ along c_s"!



In fact: $\nabla_{\frac{\partial}{\partial t}} \dot{c} = \nabla_{\dot{c}(t,s)} \mu$ for any $\mu \in \Gamma(TM)$

with $\mu(c(t_1,s)) = \dot{c}(t_1,s)$ for all t_1 near t.

(This makes $\nabla_{\frac{\partial}{\partial t}} \dot{c}$ well-def at any (t,s) where $\dot{c}(t,s) \neq 0$.)

– see Problem 57(b) (also #9, p. 7).

Notes about Lemma 1

- If $\|\dot{c}(t,0)\| = \underline{\text{const}}$ (independent of t), then get

$$L'(0) = \frac{1}{\|\dot{c}_0\|} \left(\langle c', \dot{c} \rangle \Big]_{t=a}^{t=b} - \int_a^b \langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c} \rangle \, dt \right)$$

constant!

at $s=0$

- Lemma 1 implies that

[c is stationary for E w.r.t. proper variations]

{that is, $E'(0) = 0$ for all such variations}

iff. $\nabla_{\frac{\partial}{\partial t}} \dot{c}(t,0) = 0, \quad \forall\, t \in [a,b]$.

In other words, we have proved Lemma 5 in #3 again!

proof sketch: Proper variation implies $c'(a,0) = c'(b,0) = 0$; thus $\langle c', \dot{c} \rangle \Big]_a^b = 0$.

Also for any $X \in \Gamma(c_0^*(TM))$ there exists a variation of c with $c'(t,0) \equiv X$...

Next: 2nd variation!

Theorem 1 (= Jost Thm 5.1.1)

Let $c:[a,b]\to M$ be a geodesic. Then for any variation of c,

$$E''(0)=\int_a^b\left\|\nabla_{\frac{\partial}{\partial t}}c'\right\|^2dt-\int_a^b\langle R(\dot c,c')c',\dot c\rangle\,dt+\left\langle\nabla_{\frac{\partial}{\partial s}}c',\dot c\right\rangle\Big|_{t=a}^{t=b}$$

at $s=0$

$L''(0)=\frac{1}{\|\dot c\|}$ [The above expression except every c' in the two integrals is replaced by $c'^{\perp}:=c'-\left\langle\frac{\dot c}{\|\dot c\|},c'\right\rangle\frac{\dot c}{\|\dot c\|}$]

= the component of c' orthogonal to $\dot c$

Note: For proper variations, $\left\langle\nabla_{\frac{\partial}{\partial s}}c',\dot c\right\rangle\Big|_{t=a}^{t=b}=0$, and thus $E''(0)$ depends only on the first derivative of the variation, c'.

Def 1: Given a C^∞ curve $c:[a,b]\to M$, set $\mathcal{V}_c:=\Gamma(c^*(TM))$ and $\mathring{\mathcal{V}}_c:=\{X\in\mathcal{V}_c : X(a)=X(b)=0\}$.

For c a geodesic, and any $X\in\mathcal{V}_c$, we set

$$I(X,X):=\int_a^b\left\|\nabla_{\frac{\partial}{\partial t}}X\right\|^2dt-\int_a^b\langle R(\dot c,X)X,\dot c\rangle\,dt$$

The index form of c; extend by polarization to a symmetric $\mathbb{R}$-bilinear form $\mathcal{V}_c\times\mathcal{V}_c\to\mathbb{R}$.

Then Thm. 1 says that if the variation is proper ($\Leftrightarrow c'\in\mathring{\mathcal{V}}_c$):

$$\boxed{E''(0)=I(c',c')}$$

If also the variation is normal to c ($\Leftrightarrow\langle c',\dot c\rangle\equiv 0$) and $\|\dot c_0\|\equiv 1$ then $\boxed{L''(0)=I(c',c')}$

Proof of Lemma 1: Recall $E(s) = \frac{1}{2}\int_a^b \langle \dot{c}, \dot{c}\rangle\, dt$ at s

Hence $E'(s) = \frac{1}{2}\int_a^b \frac{\partial}{\partial s}\langle \dot{c}, \dot{c}\rangle\, dt$

$F^*(\nabla)$ is metric! – see Problem 74.

$$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial s}} \dot{c}, \dot{c}\right\rangle dt$$

∇ is torsion free $\Rightarrow \nabla_{\frac{\partial}{\partial s}}(\dot{c}) = \nabla_{\frac{\partial}{\partial t}}(c')$ – see Problem 75.

$$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} c', \dot{c}\right\rangle dt$$

Again use "$F^*(\nabla)$ is metric"!

$$= \int_a^b \left(\frac{\partial}{\partial t}\langle c', \dot{c}\rangle - \left\langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c}\right\rangle\right) dt$$

$$= \langle c', \dot{c}\rangle\Big]_a^b - \int_a^b \left\langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c}\right\rangle dt,$$ as claimed!

Next recall $L(s) = \int_a^b \sqrt{\langle \dot{c}, \dot{c}\rangle}\, dt$.

Hence $L'(s) = \int_a^b \frac{\partial}{\partial s}\sqrt{\langle \dot{c}, \dot{c}\rangle}\, dt = \int_a^b \frac{\frac{\partial}{\partial s}\langle \dot{c}, \dot{c}\rangle}{2\sqrt{\langle \dot{c}, \dot{c}\rangle}}\, dt$

$$= \int_a^b \frac{\frac{\partial}{\partial t}\langle c', \dot{c}\rangle - \left\langle c', \nabla_{\frac{\partial}{\partial t}} \dot{c}\right\rangle}{\|\dot{c}\|}\, dt$$

Exactly same manipulations as for $E'(s)$!

□□

Proof of Theorem 1: From above, $E'(s) = \int_a^b \langle \nabla_{\frac{\partial}{\partial t}} c', \dot{c} \rangle \, dt$.

Hence $E''(s) = \int_a^b \frac{\partial}{\partial s} \langle \nabla_{\frac{\partial}{\partial t}} c', \dot{c} \rangle \, dt$

Bring $\frac{\partial}{\partial s}$ inside $\langle \cdot, \cdot \rangle$; then use $\nabla_{\frac{\partial}{\partial s}} \dot{c} = \nabla_{\frac{\partial}{\partial t}} c'$.

$$= \int_a^b \Big(\langle \underbrace{\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}}}_{\neq} c', \dot{c} \rangle + \langle \nabla_{\frac{\partial}{\partial t}} c', \nabla_{\frac{\partial}{\partial t}} c' \rangle \Big) \, dt$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} + \underbrace{\nabla_{[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]}}_{=0} + \tilde{R}\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)$$

By Thm. 1. in #11. Here $\tilde{R}$ is the curvature of $F^*\nabla$

Next use $\tilde{R}$ = F-pullback of R, where R is the curvature tensor on M; $R \in \Omega^2(\mathrm{End}\, TM)$; Problem 76.

$$= \int_a^b \Big(\underbrace{\langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} c', \dot{c} \rangle}_{} + \underbrace{\langle R(c', \dot{c}) c', \dot{c} \rangle}_{\text{switch to "standard format for sectional curv.}} + \| \nabla_{\frac{\partial}{\partial t}} c' \|^2 \Big) \, dt$$

Use again $\langle \nabla_{\frac{\partial}{\partial t}} X, Y \rangle = \underbrace{\frac{\partial}{\partial t} \langle X, Y \rangle}_{\text{integrate out!}} - \langle X, \nabla_{\frac{\partial}{\partial t}} Y \rangle$, $\forall X, Y \in \Gamma(F^*(TM))$

Use also $\nabla_{\frac{\partial}{\partial t}} \dot{c} = 0$ at $s=0$, since c_0 geodesic!

$$\therefore \; E''(0) = \left\langle \nabla_{\frac{\partial}{\partial s}} c', \dot{c} \right\rangle \Big|_{t=a}^{t=b} - \int_a^b \langle R(\dot{c}, c') c', \dot{c} \rangle \, dt + \int_a^b \| \nabla_{\frac{\partial}{\partial t}} c' \|^2 \, dt$$

☐ ☐

Theorem 2 (Bonnet's Theorem) An application of Thm 1

Let M be a complete Riemannian manifold with sectional curvature everywhere $\geq \mu > 0$.

Then the diameter of M is $\leq \pi/\sqrt{\mu}$.

Also M is compact and $\pi_1(M)$ is finite.

Def: [diameter of M] $:= \sup\{d(p,q) : p,q \in M\}$

Proof: Assume the diameter of M is $> \dfrac{\pi}{\sqrt{\mu}}$.

Take $p, q \in M$ with $L := d(p,q) > \dfrac{\pi}{\sqrt{\mu}}$.

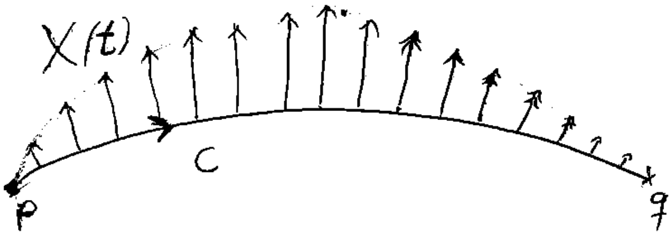
Let $c : [0,L] \to M$ be a geodesic from p to q with $L(c) = L = d(p,q)$ and $\|\dot{c}\| \equiv 1$. (Exists by the Hopf–Rinow Theorem.)

Let $X_1(t)$ be a parallel normal unit vector field along c. (Viz., $\overset{\circ}{X}_1(t) \equiv 0$, $\langle X_1(t), \dot{c}(t)\rangle \equiv 0$, $\|X_1(t)\| \equiv 1$.

Such a vector field exists by Lemma 2 in #9 and since parallel transport gives isometries – since ∇ is metric.)

Set $X(t) = \left(\sin \dfrac{\pi t}{L}\right) \cdot X_1(t)$ $\qquad t \in [0,L]$.

A Jacobi field if M had const. curvature $(\pi/L)^2$ – see #17, p.5-6



Then

$$\underline{\underline{I(X,X)}} = \int_0^L \|\dot X\|^2\,dt - \int_0^L \underbrace{\langle R(\dot c, X)X, \dot c\rangle}_{\substack{= K(\dot c, X) = \\ = K(\dot c\wedge X)\cdot\|\dot c\wedge X\|^2 \\ = K(\dot c\wedge X)\cdot \sin^2\left(\frac{\pi t}{L}\right) \\ \geqslant \mu\cdot\sin^2\left(\frac{\pi t}{L}\right)}}\,dt$$

$$\dot X(t) = \frac{\pi}{L}\cdot\left(\cos\left(\frac{\pi t}{L}\right)\right)X_1(t)$$

$$\leqslant \left(\frac{\pi}{L}\right)^2 \underbrace{\int_0^L \cos^2\left(\frac{\pi t}{L}\right)dt}_{=L/2} - \mu\cdot\underbrace{\int_0^L \sin^2\left(\frac{\pi t}{L}\right)dt}_{=L/2}$$

$$= \frac{L}{2}\cdot\left(\frac{\pi^2}{L^2} - \mu\right) \underline{\underline{< 0}}, \quad \text{since} \quad L > \frac{\pi}{\sqrt{\mu}}.$$

Hence if $c(t,s)$ is any proper variation of c with $c' \equiv X$ then by Lemma 1 and Theorem 1, $\underline{L'(0) = 0}$ and $\underline{\underline{L''(0) < 0}}$, so there exists an $s \neq 0$ with $L(c_s) = L(s) < L(0)$. Now:

$\underline{d(p,q) \leqslant L(c_s) < L(0) = d(p,q)}$, $\underline{\underline{\text{contradiction}}}$!

Hence $\underline{\underline{\mathrm{Diam}(M) \leqslant \frac{\pi}{\sqrt{\mu}}}}$.

thus: = !

It follows that for any $p\in M$, $M \subset \overline{B_{\pi/\sqrt{\mu}}(p)}$ and hence $\underline{\underline{M \text{ is compact}}}$, by the Hopf–Rinow Theorem.

Finally, all of the above applies also to the universal covering space $\tilde{M}$ of M; in particular $\tilde{M}$ is compact.

Cf. ~~notes~~ to #6 !

Hence for any $p \in M$, $\pi^{-1}(p)$ is finite since it is a closed discrete set in a compact space.

The inverse image of p by the covering map $\pi : \tilde{M} \to M$

This implies that the fundamental group $\pi_1(M)$ is finite ! $\square$

Note also Myers' Theorem (= Jost Cor. 5.3.1): In Theorem 2 the assumption can be weakened to saying that the Ricci curvature is everywhere $\geq (d-1)\mu > 0$

(Since the Ricci curvature in direction X equals the uniform average of the sectional curvature of all planes through X, Myers' Theorem is clearly stronger than Bonnet's.)

16.1. **Notes.** .

p. 4, Def. 1: The notation $\mathcal{V}_c$ for the space of vector fields along c (and also the notation $\mathring{\mathcal{V}}_c$) is introduced in Jost, [5, p. 222]; however it seems convenient to introduce it already in the present lecture. The index form is defined in [5, pp. 210(bot)–211(top)]; note that polarization gives the symmetric $\mathbb{R}$-bilinear form explicitly given by [5, (5.1.8)].

17. Jacobi Fields

#17. Jacobi Fields

Let $c:[a,b] \to M$ be a geodesic.

Recall $\mathcal{V}_c := \Gamma(c^*(TM))$, (#16, Def 1)

$\mathring{\mathcal{V}}_c := \{X \in \mathcal{V}_c : X(a) = X(b) = 0\}$,

and for $X, Y \in \mathcal{V}_c$:

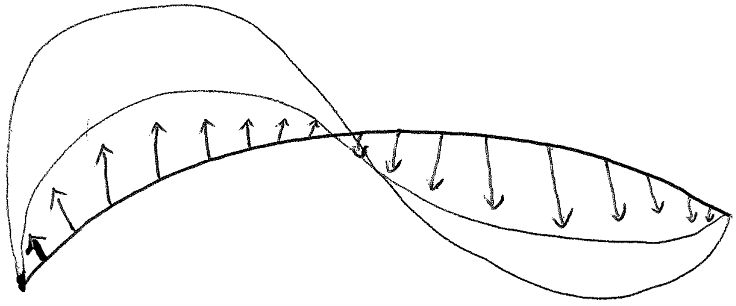
$$I(X,Y) := \int_a^b \left(\left\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} Y \right\rangle - \langle R(\dot{c}, X)Y, \dot{c} \rangle \right) dt$$

The index form on $\mathcal{V}_c$. Motivation: If c_s is a proper variation of $c = c_0$ (thus $c' \in \mathring{\mathcal{V}}_c$) then

$$E''(0) = \frac{d^2}{ds^2} E(s)_{|s=0} = I(c', c').$$

at $s=0$

A (proper) variation of c:



Also shown: $X = c' \in \mathring{\mathcal{V}}_c$

Given $X \in \mathcal{V}_c$, when is <u>X a critical point of $I(X,X)$</u>
wrt <u>all variations in $X + \mathring{\mathcal{V}}_c$</u> ??
(Viz., wrt all variations of X keeping $X(a), X(b)$ fixed)

Equivalent:

$$\forall\, Y \in \mathring{\mathcal{V}}_c : \quad \frac{d}{ds} \underbrace{I(X+sY,\, X+sY)}_{= I(X,X) + 2I(X,Y)\cdot s + I(Y,Y)\cdot s^2}\Big|_{s=0} = 0$$

$$\Leftrightarrow \boxed{\forall\, Y \in \mathring{\mathcal{V}}_c : \quad I(X,Y) = 0} \circledast$$

Let's give an equivalent reformulation of ⊛!

Rewrite $I(X,Y)$ using

$$\left\langle \nabla_{\frac{\partial}{\partial t}} X, \nabla_{\frac{\partial}{\partial t}} Y \right\rangle = \underbrace{\frac{\partial}{\partial t} \left\langle \nabla_{\frac{\partial}{\partial t}} X, Y \right\rangle}_{\text{integrate out, use } Y(a)=Y(b)=0} - \left\langle \underbrace{\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} X}_{=:\ddot{X}}, Y \right\rangle$$

and $\langle R(\dot{c}, X)Y, \dot{c}\rangle = \langle R(X,\dot{c})\dot{c}, Y\rangle$.

$$\therefore\ I(X,Y) = -\int_a^b \langle \ddot{X} + R(X,\dot{c})\dot{c}, Y\rangle\, dt$$

Using this, ⊛ is seen to be equivalent with

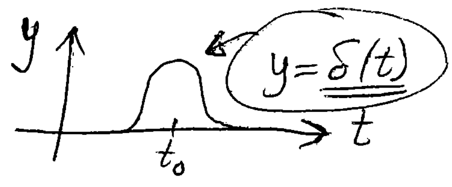
⊛⊛ $\boxed{\ddot{X} + R(X,\dot{c})\dot{c} \equiv 0}$

This is by "the basic principle of calculus of variation".

Proof: If $\ddot{X} + R(X,\dot{c})\dot{c} \neq 0$ at some $t_0 \in [a,b]$ then if $Y_1, \dots, Y_d \in \mathcal{V}_c$ is a basis of sections along c (exists!), there is at least one j with $\langle \ddot{X} + R(X,\dot{c})\dot{c}, Y_j\rangle \neq 0$ at t_0; say > 0.

Then this > 0 in some neighborhood of t_0;

and taking $Y = \delta(t)\cdot Y_j$ with



get $I(X,Y) > 0$ with $Y \in \mathring{\mathcal{V}}_c$!

Def 1: Any $X \in \mathcal{V}_c$ satisfying ⊛ ($\Leftrightarrow$ ⊛⊛) is called a **Jacobi field** along c.

Lemma 1: For any $t_0 \in [a,b]$ and $v, w \in T_{c(t_0)}M$, $\exists!$ Jacobi field along c with $X(t_0)=v$, $\dot{X}(t_0)=w$.

proof: This will follow from a basic ODE result!

Take $X_1, \ldots, X_d \in \mathcal{V}_c$ to be parallel vector fields along c which form an ON-basis in each $T_{c(t)}M$.

This exists; namely take ON-basis $v_1, \ldots, v_d \in T_{c(a)}M$, then parallel transport!

Define $\rho_i^k \in C^\infty([a,b])$ by $R(X_i, \dot{c})\dot{c} = \rho_i^k X_k$.

Now an arbitrary $X \in \mathcal{V}_c$ takes the form $X = \xi^i X_i$ with $\xi_1, \ldots, \xi_d \in C^\infty([a,b])$,

and $\ddot{X} + R(X, \dot{c})\dot{c} = 0$

$$\Leftrightarrow \underbrace{\frac{d^2 \xi^k}{dt^2} X_k}_{} + \xi^i \rho_i^k X_k = 0$$

Since $\nabla_{\frac{\partial}{\partial t}} X^k = 0$, $\forall k$

$$\Leftrightarrow \frac{d^2}{dt^2} \xi^k(t) + \rho_i^k(t)\, \xi^i(t) = 0 \quad \forall k \in \{1, \ldots, d\}.$$

This is a linear ODE of order 2; can be reduced to a linear ODE of order 1 (in 2k variables) in standard way. Now the statement follows from basic ODE result; cf. #9, p.9, and notes in Sec. 9.1.

□

Ex: Assume that M has constant (sectional) curvature $= \rho$.

Then for any $p \in M$ and any orthonormal $X, Y, Z \in T_pM$ (viz., $\|X\| = \|Y\| = \|Z\| = 1$, $\langle X, Y\rangle = \langle X, Z\rangle = \langle Y, Z\rangle = 0$):

$$\underline{\langle R(X,Y)Y, X\rangle} = K(X,Y) = \underbrace{K(X\wedge Y)}_{\rho} \cdot \underbrace{\|X\wedge Y\|^2}_{1} = \underline{\rho}$$

$$\underline{\langle R(X,Y)Y, Z\rangle} = \tfrac{1}{2}\big(K(X+Z, Y) - K(X,Y) - K(Z,Y)\big)$$

$$= \tfrac{1}{2}(2\rho - \rho - \rho) = \underline{0}$$

since $\|(X+Z)\wedge Y\| = \sqrt{2}$

$$\underline{\langle R(X,Y)Y, Y\rangle = 0}$$

Hence $\underline{R(X,Y)Y = \rho X}$. This is true for any orthonormal X, Y; hence:

$$\boxed{R(X,Y)Y = \rho \|Y\|^2 X \quad \text{for any orthogonal } X, Y \in T_pM.}$$

Hence if we take parallel vector fields $X_1, \ldots, X_d \in V_c$ as in the above proof of Lemma 1, and also assume $\underline{X_1(t) = \dot{c}(t) \quad \forall\, t \in [a,b]}$ (thus $\|\dot{c}\| \equiv 1$), then we get

$$\boxed{\begin{aligned} &\rho_1^{\,k} = 0 \quad \forall\, k \\ &\underline{\rho_i^{\,k} = \rho\cdot\delta_{i,k}} \quad \forall\, i \geq 2 \end{aligned}}$$

in $R(X_i, \dot{c})\dot{c} = \rho_i^{\,k} X_k$

Hence the "Jacobi equation" becomes, for $k \geq 2$:

$$\underline{\underline{\ddot{\xi}^k(t) + \rho\, \xi^k(t) = 0}}$$

This equation has the standard solutions

$$\xi^k(t) = \quad c_\rho(t) = \begin{cases} \cos(\sqrt{\rho}\, t) & (\rho > 0) \\ 1 & (\rho = 0) \\ \cosh(\sqrt{-\rho}\, t) & (\rho < 0) \end{cases} \qquad \boxed{c_\rho(0)=1,\ \dot{c}_\rho(0)=0}$$

and

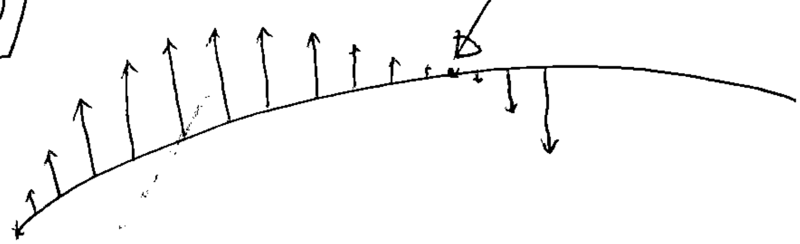
$$\xi^k(t) = \quad s_\rho(t) = \begin{cases} \frac{1}{\sqrt{\rho}} \sin(\sqrt{\rho}\, t) & (\rho > 0) \\ t & (\rho = 0) \\ \frac{1}{\sqrt{-\rho}} \sinh(\sqrt{-\rho}\, t) & (\rho < 0) \end{cases} \qquad \boxed{s_\rho(0) = 0,\ \dot{s}_\rho(0) = 1}$$

The unique solutions with these initial values.

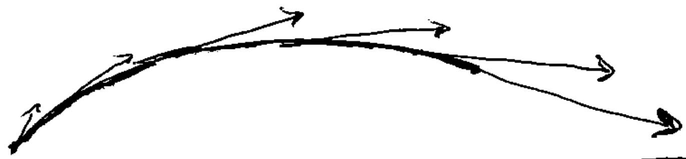
Ex, if $\rho > 0$:

$\boxed{s_\rho(t)}$

$t = \frac{\pi}{\sqrt{\rho}}$



Lemma 2: For any $\lambda, \mu \in \mathbb{R}$, the Jacobi field along c with $X(a) = \lambda \cdot \dot{c}(a)$, $\dot{X}(a) = \mu \cdot \dot{c}(a)$ is: $X(t) = (\lambda + (t-a)\mu) \cdot \dot{c}(t)$



tangential Jacobi fields — trivial and uninteresting, since they don't depend on the geometry of M.

proof: The given $X(t)$ satisfies $\ddot{X}(t) \equiv 0$ (since $\nabla_{\frac{\partial}{\partial t}} \dot{c}(t) = 0$ etc) and $R(X, \dot{c}) \equiv 0$ (since $R(\dot{c}, \dot{c}) = 0$). Hence $\ddot{X} + R(X, \dot{c})\dot{c} \equiv 0$. □

Lemma 3: For any Jacobi field X along c, split X as $X = X^{tan} + X^{nor}$ where $X^{tan}(t) = \langle X(t), \dot{c}(t) \rangle \cdot \frac{\dot{c}(t)}{\|\dot{c}(t)\|^2}$ and $X^{nor}(t) = X(t) - X^{tan}(t)$. (Thus $\langle X^{nor}(t), \dot{c}(t) \rangle \equiv 0$.)

Then both X^{tan} and X^{nor} are Jacobi fields.

proof: Since the "Jacobi equation" p. 3 (**) is linear in X, it suffices to verify that X^{tan} is a Jacobi field. Now one computes:

$$\ddot{X}^{tan}(t) = \left(\frac{d^2}{dt^2} \langle X(t), \dot{c}(t) \rangle \right) \cdot \frac{\dot{c}(t)}{\|\dot{c}(t)\|^2}$$

since $\nabla_{\frac{\partial}{\partial t}} \dot{c}(t) = 0$ and $\|\dot{c}(t)\|$ constant!

$$= \left\langle \ddot{X}(t), \dot{c}(t) \right\rangle \cdot \frac{\dot{c}(t)}{\|\dot{c}(t)\|^2}$$

since $F^*(\nabla)$ is metric – Problem

$$= \left\langle -R(X, \dot{c})\,\dot{c}, \dot{c} \right\rangle \cdot \frac{\dot{c}(t)}{\|\dot{c}(t)\|^2} = 0.$$

! use antisymmetry of R !

Also: $R(X^{tan}, \dot{c})\,\dot{c} \equiv 0$, since $X^{tan}(t) \in \mathrm{Span}_{\mathbb{R}}\{\dot{c}(t)\}$.

Hence X^{tan} satisfies p. 3 (**) i.e. X^{tan} is a Jacobi field. $\square$

Theorem 1: Let $c:[a,b] \to M$ be a geodesic. Let $c(t,s)$ be a variation of c such that $c(\cdot, s)$ is a geodesic, $\forall s \in (-\varepsilon, \varepsilon)$. – that is, a "variation through geodesics"

Then $X = c'$ is a Jacobi field along c.

Conversely, every Jacobi field along c may be obtained in this way.

Proof: Assume $c(t,s)$ is a variation of c through geodesics. Let $X = c'$. Then

$$\ddot{X}(t) = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} (c') = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} (\dot{c}) =$$

∇ torsion free

$$= \nabla_{\frac{\partial}{\partial s}} \underbrace{\nabla_{\frac{\partial}{\partial t}} (\dot{c})}_{\equiv 0 \ \forall s!} + R(\dot{c}, c')\dot{c} =$$

$$= R(\dot{c}, X)\dot{c} = -R(X, \dot{c})\dot{c}.$$

Hence X is a Jacobi field!

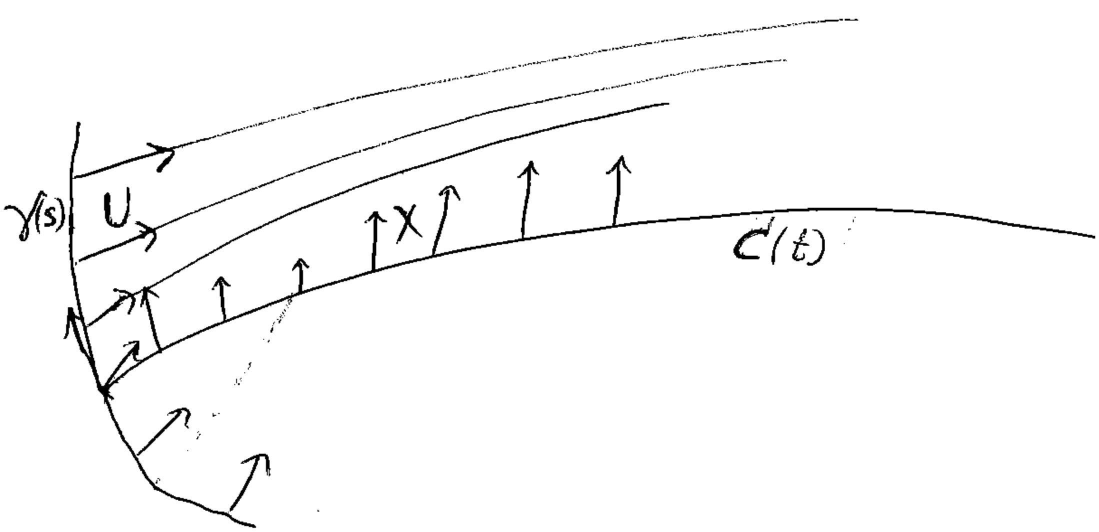
Key point above: $\nabla_{\frac{\partial}{\partial t}} \dot{c} \equiv 0$, for all t and all s, exactly since $c(t,s)$ is a variation through geodesics! Thus the Jacobi equation is a linearization of the equation for geodesic curves!

Conversely, assume that X is a Jacobi field along c. We will now construct a variation of c through geodesics such that $X = c'$!

Assume $\boxed{a=0}$, for simplicity!

Take any curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = c(0)$ and $\dot\gamma(0) = X(0)$. Then take any $U \in \Gamma_\gamma(TM)$ (a vector field along γ) with $U(0) = \dot c(0)$ and $\dot U(0) = \dot X(0) \in T_{c(0)}M$. Possible! – see notes!

Set $\boxed{c(t,s) = \exp_{\gamma(s)}(t \cdot U(s))}$ $\forall t \in [a,b]$, $s \in (-\varepsilon, \varepsilon)$. well-defined after possibly shrinking ε – see notes!



Then $c(t,s)$ is a variation of c through geodesics – by construction!

Set $\underline{\underline{Y = c'}}$. This is a Jacobi field along c by the first half of the present proof!

Also $\underline{\underline{Y(0)}} = \underbrace{\frac{\partial}{\partial s} c(t,s)\Big|_{t=s=0}}_{c(0,s)=\gamma(s)} = \dot{\gamma}(0) = \underline{\underline{X(0)}}$

and

$$\underline{\underline{\dot{Y}(0)}} = \nabla_{\frac{\partial}{\partial t}}\left(\frac{\partial}{\partial s} c(t,s)\right)\Big|_{t=s=0}$$

again use: $\underline{\nabla \text{ torsion}}$ $\underline{\text{free}}$!

$$= \nabla_{\frac{\partial}{\partial s}}\left(\frac{\partial}{\partial t} c(t,s)\right)\Big|_{t=s=0}$$

$$= \nabla_{\frac{\partial}{\partial s}} U(s)\Big|_{s=0} = \dot{U}(0) = \underline{\underline{\dot{X}(0)}}.$$

Thus both X and Y are Jacobi fields along c, with $Y(0) = X(0)$ and $\dot{Y}(0) = \dot{X}(0)$; hence by the uniqueness part of Lemma 1, $\underline{\underline{X \equiv Y \equiv c'}}$. Done!

$\square$

(Jost, Cor. 5.2.2)

Cor 1: Let $c : [0,T] \to M$ be a geodesic and set $p = c(0)$. (Note $c(t) = \exp_p(t \cdot \dot{c}(0))$, $\forall\, t \in [0,T]$.)
If X is a Jacobi field along c with $X(0) = 0$, then $X(t) = (d\exp_p)_{t\cdot\dot{c}(0)}(t \cdot \dot{X}(0))$, $\forall\, t \in [0,T]$.

$\exp_p : T_pM \to M$, thus

$(d\exp_p)_{t\cdot\dot{c}(0)} : T_{t\cdot\dot{c}(0)}(T_pM) = T_pM \to T_{c(t)}M$

Proof: Set $w = \dot{X}(0) \in T_pM$ and

$$\boxed{c(t,s) = \exp_p\big(t(\dot{c}(0) + sw)\big)} \quad \begin{pmatrix} t \in [0,T] \\ s \in (-\varepsilon, \varepsilon) \end{pmatrix}$$

This is a special case of the $c(t,s)$ we had on p. 8, namely with $\gamma \equiv$ [the constant curve $c(0)$] and $U(s) = \dot{c}(0) + sw$. Note $U(0) = \dot{c}(0)$ and $\dot{U}(0) = w = \dot{X}(0)$, hence as in the proof of Thm. 1, $c' \equiv X$. (at $s=0$)

$$\because\ X(t) = \frac{\partial}{\partial s} c(t,s)\Big|_{s=0} =$$

$$= \frac{\partial}{\partial s} \exp_p\big(t\cdot\dot{c}(0) + s\cdot tw\big)\Big|_{s=0}$$

$$= (d\exp_p)_{t\cdot\dot{c}(0)}(t\,w) \quad (w = \dot{X}(0))$$

Done! □

Cor. 2 (Jost Cor. 5.2.3) "Gauss' Lemma"

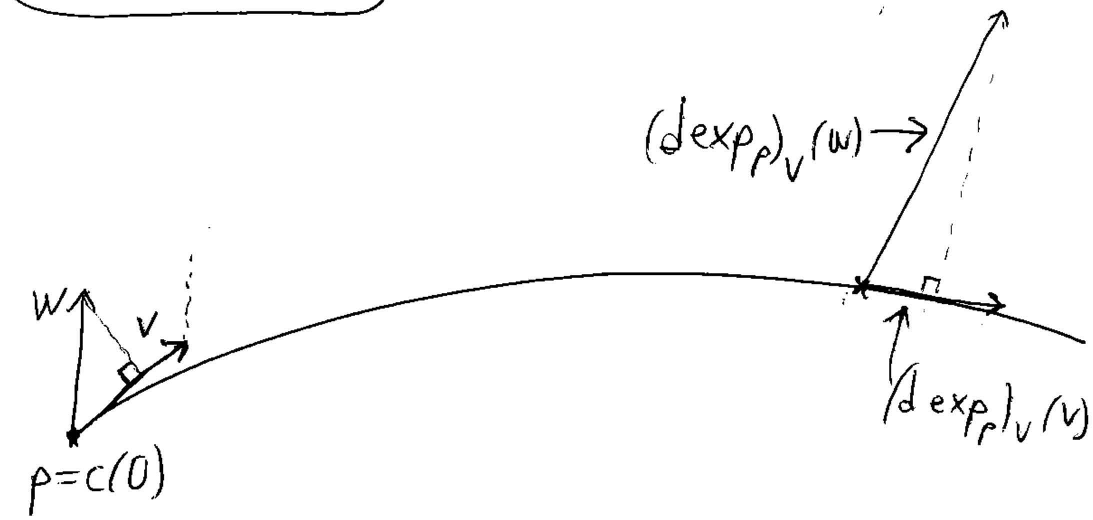
Let $c:[0,1] \to M$ be a geodesic and set $p = c(0)$, $v = \dot{c}(0)$ (thus $c(t) = \exp_p(tv)$).

Then for all $w \in T_pM$:

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w) \rangle = \langle v, w \rangle.$$

$\langle , \rangle$ in $T_{c(1)}(M)$

View as $v, w \in T_v(T_pM)$



proof: Consider Jacobi fields X, Y along c with $X(0) = Y(0) = 0$, $\dot{X}(0) = v$, $\dot{Y}(0) = w$. By Cor. 1:

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w) \rangle = \langle X(1), Y(1) \rangle \underset{\text{Lemma 2}}{=} \langle \dot{c}(1), Y(1) \rangle =$$

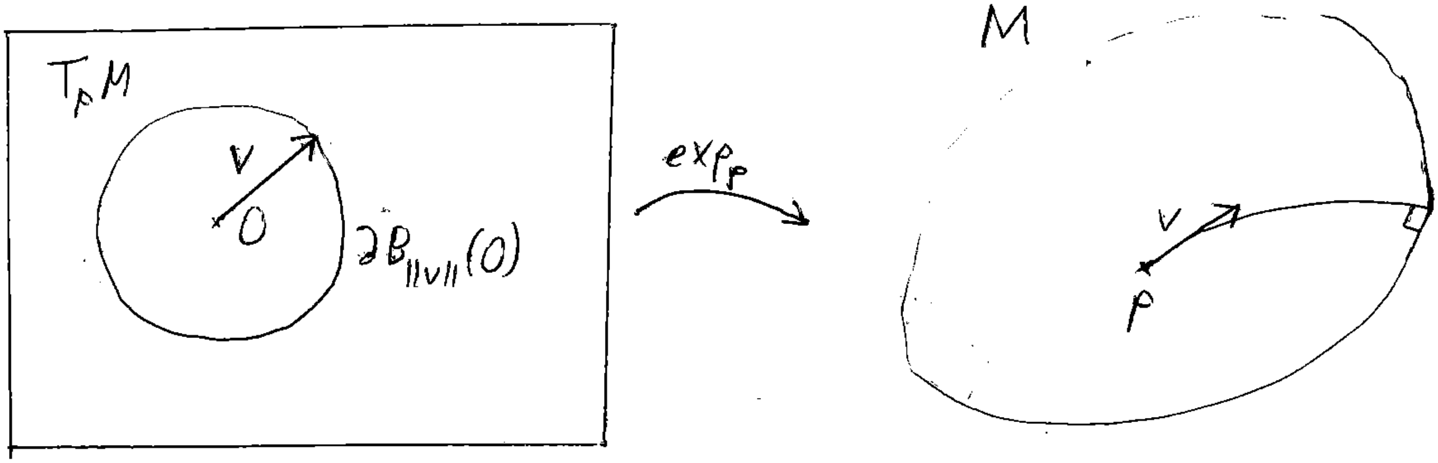
$$= \langle \dot{c}(1), Y^{tan}(1) \rangle$$

(Note $Y^{tan}(0) = 0$ since $Y(0) = 0$. Take $\mu \in \mathbb{R}$ so that $\dot{Y}^{tan}(0) = \mu \cdot v$. Then $Y^{tan}(1) = \mu \cdot \dot{c}(1)$ by Lemma 2.)

$$= \|\dot{c}\|^2 \cdot \mu = \langle v, \dot{Y}^{tan}(0) \rangle = \langle v, \dot{Y}(0) \rangle = \langle v, w \rangle.$$

$\square$

Note: Cor. 2 implies that $\dot{c}(1)$ is orthogonal to the $\exp_p$-image of the sphere $\partial B_{\|v\|}(0)$!



Also, Cor. 2 implies that if we put polar coordinates on T_pM then throughout $\mathcal{D}_p$ the Riemannian metric on M (pullbacked by $\exp_p$) is represented by a matrix of the form

$$\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{\begin{matrix}\text{pos.}\\ \text{semi-definite}\end{matrix}} \\ 0 & \end{pmatrix}$$

– see Problem (b).

strictly definite whenever $d\exp_p$ is non-singular

Using this, we obtain by the same method as in #4, Thm 4:

Cor 3 (= Jost Cor 5.2.4 corrected) (essentially stronger than #4, Thm 4)

Let $c:[0,1]\to M$ be a geodesic and set $p=c(0)$, $v=\dot{c}(0)$ (thus $c(t)=\exp_p(tv)$). As in Cor. 2

Let $\gamma:[0,1]\to \mathcal{D}_p\subset T_pM$ be a pw C^∞ curve with $\gamma(0)=0$, $\gamma(1)=v$. Then $L(\exp_p\circ\gamma)\geq\|v\|$.

If equality holds, and if $(d\exp_p)_{\gamma(t)}$ is nonsingular, $\forall t\in[0,1]$, then γ must be a reparametrization of the curve $t\mapsto tv$ $t\in[0,1]$.

17.1. **Notes.** .

pp. 2–4: Here we follow the presentation in the beginning of [5, Sec. 5.2], except that we introduce Jacobi fields by studying a question, leading to Definition 1 on p. 4(top). Note that the equivalence (*)⇔(**) is [5, Lemma 5.2.1] and the equivalence proved on p. 2 between (*) and X being a critical point of $I(X,X)$ within $X + \overset{\circ}{\mathcal{V}}_c$ is [5, Lemma 5.2.2].

p. 7, Lemma 3: Jost mentions this fact on [5, p. 214(mid)]. It seems to me that his "proof" of the fact is a little bit too short. (But note that our proof of the fact is by a computation which is rather similar to the computation in the proof of Lemma 2 = Jost's Lemma 5.2.4.)

p. 9: The computation here is very similar to what we did in the proofs of Lemma 1 and Theorem 1 in Lecture #16. In particular see there for detailed justification of the manipulations.

p. 10: Here we write "take any vector field U along γ such that $U(0) = \dot{c}(0)$ and $\dot{U}(0) = \dot{X}(0)$ (in $T_{c(0)}(M)$)". To see that this is *possible,* one may (following Jost [5, p. 215(mid)]) let V and W be the unique *parallel* vector fields along γ subject to $V(0) = \dot{c}(0)$ and $W(0) = \dot{X}(0)$ (cf. Lemma 2 in Lecture #9), and then set

$$U(s) = V(s) + sW(s) \qquad \forall s \in (-\varepsilon, \varepsilon).$$

Then $U(0) = V(0) = \dot{c}(0)$ and (using $\dot{V}(0) = \dot{W}(0) = 0$): $\dot{U}(0) = W(0) = \dot{X}(0)$, as desired.

(p. 10: We here define

$$c(t,s) = \exp_{\gamma(s)}(t \cdot U(s)), \qquad \forall t \in [0,b],\ s \in (-\varepsilon, \varepsilon). \tag{9}$$

We have to prove that this is actually possible, i.e. that after perhaps shrinking ε, we have $t \cdot U(s) \in \mathcal{D}$ for all $t \in [0,b]$ and $s \in (-\varepsilon,\varepsilon)$, where $\mathcal{D}$ is the maximal domain for exp, as in Problem 21. This follows from the fact that $\mathcal{D}$ *is open* (cf. Problem 21(c)) and a standard compactness argument. Indeed, assume that this is *not* possible. Then there exists a sequence $s_1, s_2, \ldots \in (-\varepsilon, \varepsilon)$ [14] with $s_j \to 0$ and a corresponding sequence $t_1, t_2, \ldots \in [0,b]$ such that $t_j \cdot U(s_j) \notin \mathcal{D}$ for all j. Since $[0,b]$ is compact, after passing to a subsequence we may assume that the limit $t_\infty := \lim_{j\to\infty} t_j$ exists, in $[0,b]$. But then $\lim_{j\to\infty} t_j \cdot U(s_j) = t_\infty \cdot U(0) = t_\infty \cdot \dot{c}(0)$ in TM and $t_\infty \cdot \dot{c}(0) \in \mathcal{D}$ since $c(t)$ for $t \in [0,b]$ by assumption is a geodesic. This gives a contradiction against the fact that $t_j \cdot U(s_j) \notin \mathcal{D}$ for all j and $\mathcal{D}$ is open! Done!)

p. 13: In this proof we find

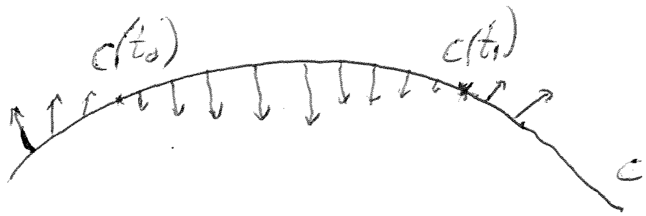
$$(d\exp_p)_v(v) = \dot{c}(1) \tag{10}$$

[14] for our original choice of $\varepsilon > 0$

by applying Cor. 1 to the tangential Jacobi field X. However note that the relation (10) is also *immediate from the definition of* exp. Indeed, we have $(d\exp_p)_v(v) = \frac{d}{dh}\exp_p(v+hv)_{|h=0} = \frac{d}{dh}c(1+h)_{|h=0} = \dot{c}(1)$.

#18. Conjugate points

Def 1: Let $c:[a,b]\to M$ be a geodesic. For $t_0 \neq t_1$ in $[a,b]$, the points $c(t_0), c(t_1)$ are said to be <u>conjugate along c</u> if there exists a Jacobi field $X \not\equiv 0$ along c with $X(t_0) = 0 = X(t_1)$.



Remark 1: For any $t_0 \neq t_1 \in [a,b]$,

[$c(t_0), c(t_1)$ are <u>not</u> conjugate along c]

<u>iff</u> $\left[\begin{array}{l} \forall v \in T_{c(t_0)}M,\ w \in T_{c(t_1)}M: \\ \exists!\ \text{Jacobi field } Y \text{ along } c \text{ such that } Y(t_0)=v,\ Y(t_1)=w \end{array}\right]$

<u>proof:</u> Consider the map

$$\{\text{Jacobi fields along } c\} \to T_{c(t_0)}M \times T_{c(t_1)}M$$

$$Y \mapsto \big(Y(t_0),\ Y(t_1)\big)$$

This map is <u>linear</u>, and both the domain and the range has dimension $2d$ (by Lemma 1 in #17). Note also that <u>$c(t_0), c(t_1)$ are conjugate along c</u> iff the <u>kernel of the above map is $\neq 0$</u>! ... Done!

Remark 2 [$c(t_0), c(t_1)$ are <u>not</u> conjugate along c]

$\Leftrightarrow \left(d\exp_{c(t_0)}\right)_{(t_1-t_0)\cdot\dot c(t_0)}$ is non-singular

"clear from #17, Cor. 1" – see Problem 86

Theorem 1: Let $c : [a,b] \to M$ be a geodesic.

If there $\left\{ \begin{matrix} \text{does not exist} \\ \text{exists} \end{matrix} \right\}$ a point $\left\{ \begin{matrix} \text{---} \\ \underline{\text{before } c(b)} \end{matrix} \right\}$

conjugate to $c(a)$ along c then c $\left\{ \begin{matrix} \text{is a } \underline{\text{strict}} \\ \text{is } \underline{\text{not}} \end{matrix} \right\}$

(see below!)

local minimum for L among pw C^∞ curves with fixed endpoints.

Remarks: ① If $c(a), c(b)$ are conjugate along c but there is <u>no other</u> point conjugate to $c(a)$ along c, then <u>Theorem 1 says nothing</u>. ② In the "2nd case", we'll even show that there exists <u>a proper variation</u> c_s of c with <u>$L(c_s) < L(c)$, $\forall s \in (-\varepsilon, \varepsilon) \setminus \{0\}$!</u>

Regarding "local minimum"

We say that c is a <u>local minimum for L among pw C^∞ curves with fixed endpoints</u> if

$\exists \, \varepsilon > 0$ such that <u>for every</u> pw C^∞ curve

$\gamma : [a,b] \to M$ with $\gamma(a) = c(a)$, $\gamma(b) = c(b)$

and $d(\gamma(t), c(t)) < \varepsilon$, $\forall \, t \in [a,b]$,

One has: <u>$L(\gamma) \geq L(c)$</u>. We say c is a <u>strict</u> such local minimum if one can even choose $\varepsilon > 0$ so that for all γ as above, $L(\gamma) \geq L(c)$ <u>with equality only if γ is a reparametrization of c.</u>

Ex; application of Thm 1:

If $p \in M$ and $r > 0$ is such that $\exp_p$ is defined and injective on $B_r(0) \subset T_p(M)$, then $\exp_{p|B_r(0)}$ is in fact a diffeomorphism onto an open subset of M.

(Thus: Normal coordinates at p with radius r are ok!)

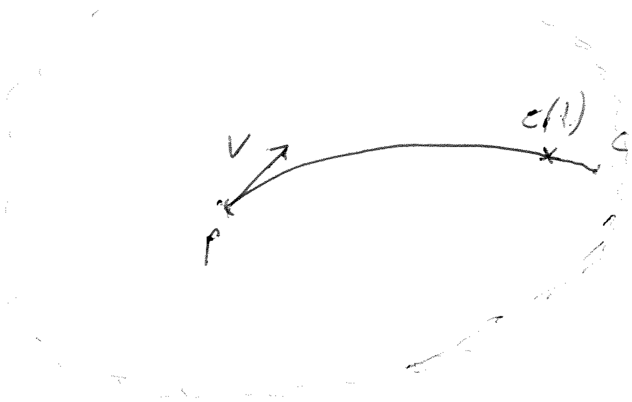
The above implies that the two definitions of the injectivity radius $i(p)$ which we have discussed are indeed equivalent.

proof – see Problem 89.

It suffices to prove $(d\exp_p)_v$ non-singular, $\forall v \in B_r(0)$.

If not, then $c(0)$, $c(1)$ are conjugate along the geodesic

$$c : [0, b] \to M, \qquad c(t) = \exp_p(tv)$$



Fix b, $1 < b < \frac{r}{\|v\|}$.

Then Theorem 1 $\Rightarrow$ $\exists$ shorter curve from p to $q := c(1)$

$\Rightarrow$ find $w \in T_p(M)$ with $\|w\| < \|v\|$, $\exp_p(w) = c(1) = \exp_p(v)$

This contradicts the injectivity of $\exp_{p|B_r(0)}$. $\square$

Detour; spaces of curves

$I = [0,1]$

Set $\quad C_M = C^{\infty}(I, M) := \{ c : I \to M \ : \ c \text{ is pw } C^{\infty} \}$

$$\Omega_{p,q} = \Omega_{p,q}(M) := \{ c \in C_M \ : \ c(0) = p, \ c(1) = q \} \quad \begin{pmatrix} \text{any} \\ p, q \in M \end{pmatrix}$$

$$\Lambda M = \{ c \in C_M : \ c(0) = c(1) \}$$

We also define a metric on C_M by

$$d(c_1, c_2) := \max \{ d(c_1(t), c_2(t)) \ : \ t \in I \}$$ — see Problem 87

Now $c : I \to M$ is a "local minimum for L among pw C^{∞} curves with fixed endpoints" iff

$$\exists \varepsilon > 0 : \ \forall \gamma \in B_\varepsilon(c) \cap \Omega_{p,q} : \ L(\gamma) \geq L(c)$$ Natural!

$p = c(0), \ q = c(1)$

On the other hand, the notion of strict local min is not the natural one for $(\Omega_{p,q}, d)$...

The metric space (C_M, d) is not complete and has other bad properties (again see Problem 87). However one can consider certain completions of C_M which can be endowed with "∞-dim manifold" structures! For example one can consider a Sobolev H^1 (L^2) completion and obtain a Hilbert manifold called $H^1(I, M)$ consisting of all H^1-curves $c : I \to M$; the tangent space of $H^1(I, M)$ at any such c can be naturally identified with the Hilbert space of H^1-vector fields along c. Also $H^1(I,M)$ can be equipped with a natural Riemannian metric...

Proof of Theorem 1:

First assume that there does not exist any point conjugate to $c(a)$ along c.

WLOG assume $[a,b] = I = [0,1]$, set $p = c(0)$, $v = \dot{c}(0)$.

Thus $c(t) = \exp_p(tv)$, $t \in I$.

We want to apply Cor. 3 in #17.

Key input: $(d\exp_p)_{tv(t)}$ is non-singular, $\forall t \in I$. ⊛

In order to apply Cor 3 in #17, we need to prove that every pw C^∞ curve near c can be expressed as "$\exp_p$" of a curve from 0 to v.

Note: ⊛ $\Rightarrow$ [$\forall t \in I$: $\exists$ open $\Omega \subset T_pM$ such that $tv \in \Omega$ and $\exp_{p|\Omega}$ is diffeomorphism

Cover $\{tv : t \in I\}$ by a finite set of such "Ω's":

$\Omega_1, \ldots, \Omega_k \subset T_pM$. Set $U_i := \exp_p(\Omega_i) \overset{\text{open}}{\subset} M$.

We can arrange this so that there exist

$0 = t_0 < t_1 < \ldots < t_k = 1$

with $\{tv : t \in [t_{i-1}, t_i]\} \subset \Omega_i$, $\forall i$.

We can now take $\varepsilon > 0$ so small that $B_\varepsilon(tv) \in U_i$ for every $t \in [t_{i-1}, t_i]$. Then every $g \in B_\varepsilon(c) \cap \Omega_{p,q}(M)$ satisfies $g(t) \in U_i$, $\forall t \in [t_{i-1}, t_i]$ $(\forall i)$.

and so we may define $\gamma : I \to T_p(M)$ by

$$\gamma(t) = \left(\exp_{p|\Omega_i}\right)^{-1}(g(t)) \quad \text{for } t \in [t_{i-1}, t_i]$$

Then $g = \exp_p \circ \gamma : I \to M$

Hence by Cor. 3 in #17, $L(g) \geq L(c)$ with equality iff g is a reparametrization of c. Done!

Next, assume that there is $\tau \in (a,b)$ such that $c(\tau)$ is conjugate to $c(a)$ along c.

Again WLOG assume $[a,b] = I = [0,1]$. (Thus $0 < \tau < 1$)

We want to find $Y \in \mathring{\mathcal{V}}_c$ with $I(Y,Y) < 0$; then Thm 1 in #16 will imply that there is a proper variation c_s of c with $L(c_s) < L(c)$, $\forall s \in (-\varepsilon, \varepsilon) \setminus \{0\}$!

Let $X \not\equiv 0$ be a Jacobi field along c with $X(0) = 0 = X(\tau)$.

First attempt: Set

$$Y(t) = \begin{cases} X(t) & \text{for } t \in [0,\tau] \\ 0 & \text{for } t \in [\tau, 1] \end{cases}$$

Note: This is only pw C^∞ vector field, i.e. $Y \notin \mathcal{V}_c$ in general.

Write Y^1, Y^2 for restr. of Y to $[0,\tau]$, $[\tau,1]$, resp. (Similarly: X^1, X^2, etc.)

Now $I(Y,Y) = I(Y^1, Y^1) = 0$

since $Y^1 = X^1$ Jacobi field $\Rightarrow I(X^1, Z) = 0$ for every C^∞ Z with $Z(0) = Z(\tau) = 0$.

Perturb Y? For any $Z \in \mathring{\mathcal{V}}_c$, we have

$$\underline{\underline{I(Y+Z,\, Y+Z)}} = I(Y'+Z',\, Y'+Z') + I(Z^2, Z^2)$$

$$= \underbrace{I(Y', Y')}_{0} + 2\underset{\underset{X'}{\parallel}}{I(Y', Z')} + I(Z', Z') + I(Z^2, Z^2)$$

$$= \underline{\underline{2\, I(X', Z') + I(Z, Z)}}$$

Here $\underline{\underline{I(X', Z')}}$ can be understood as in the proof of "$\circledast \Leftrightarrow \circledast\circledast$" in Lecture #17, p. 3, even though perhaps $Z'(\tau) \neq 0$. Indeed:

$$\underline{\langle \dot{X}, \dot{Z} \rangle} = \frac{\partial}{\partial t}\langle \dot{X}, Z\rangle - \langle \ddot{X}, Z\rangle \underset{\uparrow}{=} \underline{\frac{\partial}{\partial t}\langle \dot{X}, Z\rangle + \langle R(X,\dot{c})\dot{c},\, Z\rangle}$$

X Jacobi!

and thus

$$\underline{\underline{I(X', Z')}} = \int_0^\tau \Big(\langle \dot{X}, \dot{Z}\rangle - \langle R(X,\dot{c})\dot{c}, Z\rangle\Big)\, dt =$$

$Z(0)=0$

$$= \int_0^\tau \frac{\partial}{\partial t}\langle \dot{X}, Z\rangle\, dt = \langle \dot{X}, Z\rangle\Big|_{t=0}^{t=\tau} = \underline{\underline{\langle \dot{X}(\tau), Z(\tau)\rangle}}.$$

Note $\underline{\underline{\dot{X}(\tau) \neq 0}}$ since $X \not\equiv 0$ and $X(\tau) = 0$. Lemma 1 in #17

Fix now $Z \in \mathring{\mathcal{V}}_c$ with $\underline{\underline{Z(\tau) = -\dot{X}(\tau)}}$. The above formula with $\eta \cdot Z$ ($\eta \in \mathbb{R}$) in place of Z then says:

$$\underline{\underline{I(Y+\eta Z,\, Y+\eta Z) = -2\|\dot{X}(\tau)\|^2 \cdot \eta + I(Z,Z)\,\eta^2.}}$$

This is $\underline{< 0}$ for $\underline{\text{small } \eta > 0}$.

Finally approximate $Y+\eta Z$ by C^∞ vector field in $\mathring{\mathcal{V}}_c$ $\to$ done!

- See Problem 88

$\square$

18.1. **Notes.** .

p. 4, regarding the *Hilbert manifold* $H^1(I, M)$ of all H^1-curves $I \to M$, see Klingenberg [6, Ch. 2]. There also exist other natural (∞-dim) manifold structures on spaces of curves: For example, an often considered space is the "smooth loop space" of M, which is the space of all C^∞ *closed* curves on M equipped with a natural structure as a *Fréchet manifold.*

p. 6(mid): Here we write that if we can find $Y \in \overset{\circ}{\mathcal{V}}_c$ with $I(Y, Y) < 0$ then by Theorem 1 in #16 there is a proper variation c_s of c with $L(c_s) < L(c)$ for all $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$. To see this, note that, given such a $Y \in \overset{\circ}{\mathcal{V}}_c$, by Problem 83 there is a proper variation c_s of c with $c_0' = Y$, and then Theorem 1 in #16 implies that $E''(s) < 0$. Also $E'(s) = 0$ by Lemma 1 in #16; hence after possibly shrinking ε we have $E(s) < E(0)$ for all $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$. But also $L(s) \leq \sqrt{2E(s)}$ and $L(0) = \sqrt{2E(0)}$; hence also $L(s) < L(0)$ for all $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$. $\square$

19. Comparison theorems

#19. Comparison theorems & consequences

Theorem 1 (Rauch comparison Theorem) (Jost Thm 5.5.1)

Let $\mu \in \mathbb{R}$ and let M be a Riemannian manifold whose sectional curvature is everywhere $\leq \mu$.
Let $c : [0,T] \to M$ be a geodesic with $\|\dot{c}\| = 1$ and let J be a Jacobi field along c.

Set $f_\mu := \|J(0)\| \cdot c_\mu + \|J\|^{\cdot}(0) \cdot s_\mu$

and take $\tau \in (0,T)$ so that $f_\mu(t) > 0,\ \forall t \in (0,\tau)$.

Assume $[\mu \geqslant 0$ or $J^{\tan} \equiv 0]$

Then $\|J(t)\| \geqslant f_\mu(t),\quad \forall t \in [0,\tau]$,

and $\dfrac{\|J(t)\|}{f_\mu(t)}$ is increasing on $(0,\tau)$.

Here
$$c_\mu(t) = \begin{cases} \cos(\sqrt{\mu}\, t) & (\mu > 0) \\ 1 & (\mu = 0) \\ \cosh(\sqrt{-\mu}\, t) & (\mu < 0) \end{cases} \qquad \boxed{\begin{matrix} c_\mu(0) = 1 \\ \dot{c}_\mu(0) = 0 \end{matrix}}$$

$$s_\mu(t) = \begin{cases} \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}\, t) & (\mu > 0) \\ t & (\mu = 0) \\ \frac{1}{\sqrt{-\mu}} \sinh(\sqrt{-\mu}\, t) & (\mu < 0) \end{cases} \qquad \boxed{\begin{matrix} s_\mu(0) = 0 \\ \dot{s}_\mu(0) = 1 \end{matrix}}$$

— these describe a normal Jacobi field for M with constant curvature μ; cf pp. 5–6 in #17.

proof: Let $\tau_1 = \inf\{t>0 : J(t)=0\} > 0$

$$\tau_2 = \min(\tau, \tau_1)$$

Possibly $\tau_1 = +\infty$

We will work on $[0, \tau_2)$; there we know $J(t) \neq 0$ and $f_\mu(t) > 0$. At the end we'll see $\tau_2 = \tau$!

For $t \in [0, \tau_2)$: $\|J(t)\|^{\boldsymbol{\cdot}} = \frac{d}{dt}\sqrt{\langle J, J\rangle} = \frac{\langle J, J\rangle^{\boldsymbol{\cdot}}}{2\|J\|} = \frac{\langle \dot{J}, J\rangle}{\|J\|}$

Here "$\cdot$" means $\frac{d}{dt}$

Here "$\cdot$" means $\nabla_{\frac{\partial}{\partial t}}$

In a similar way:

$$\|J\|^{\boldsymbol{\cdot\cdot}} = \frac{d}{dt}\left(\frac{\langle \dot{J}, J\rangle}{\|J\|}\right) = \dots = \frac{\langle \ddot{J}, J\rangle}{\|J\|} + \frac{\|\dot{J}\|^2}{\|J\|} - \frac{\langle \dot{J}, J\rangle^2}{\|J\|^3}$$

Using the Jacobi equation

$$= -\frac{\langle R(J, \dot{c})\dot{c}, J\rangle}{\|J\|} + \frac{\|\dot{J}\|^2}{\|J\|} - \frac{\langle \dot{J}, J\rangle^2}{\|J\|^3}$$

Here

$$\langle R(J, \dot{c})\dot{c}, J\rangle = \underbrace{K(J \wedge \dot{c})}_{\leq \mu} \cdot \underbrace{\|J \wedge \dot{c}\|^2}_{\leq \|J\|^2,\ \text{equality iff } J^{tan}(t)=0} \overset{*}{\leq} \mu \cdot \|J\|^2$$

$*$ Here we use the assumption that [$\mu \geq 0$ or $J^{tan} \equiv 0$]!

Hence

$$\|J\|^{\cdot\cdot} + \mu\|J\| = \underbrace{\frac{\mu\|J\|^2 - \langle R(J,\dot c)\dot c, J\rangle}{\|J\|}}_{\geq 0,\ \text{seen above.}} + \underbrace{\frac{\|J\|^2\|\dot J\|^2 - \langle \dot J, J\rangle^2}{\|J\|^3}}_{\geq 0\ \text{by Cauchy-Schwarz}}$$

i.e. $\boxed{\|J\|^{\cdot\cdot} + \mu\|J\| \geq 0}$

From this, Theorem 1 follows by "standard ODE technique"! We are comparing J with f_μ, which satisfies

$$\boxed{\begin{array}{l} \ddot f_\mu + \mu f_\mu = 0 \\ f_\mu(0) = \|J(0)\|, \quad \dot f_\mu(0) = \|J\|^\cdot(0) \end{array}}$$

Idea: Consider $\|J\|/f_\mu$!

Note:

$$\left(\|J\|^\cdot f_\mu - \|J\|\dot f_\mu\right)^\cdot = \|J\|^{\cdot\cdot} f_\mu - \|J\|\ddot f_\mu \geq -\mu\|J\|f_\mu + \|J\|\mu f_\mu = 0$$

we're keeping $t\in(0,\tau_2)$; thus $f_\mu(t)>0$

Also $\left(\|J\|^\cdot f_\mu - \|J\|\dot f_\mu\right)(0) = 0$.

Hence $\|J\|^\cdot f_\mu - \|J\|\dot f_\mu \geq 0$ for $t\in[0,\tau_2]$

and so $\left(\dfrac{\|J\|}{f_\mu}\right)^\cdot = \dfrac{\|J\|^\cdot f_\mu - \|J\|\dot f_\mu}{f_\mu^2} \geq 0$ for $t\in(0,\tau_2)$

Also $\left(\frac{\|J\|}{f_\mu}\right)(0) = 1$

If $f_\mu(0) = 0$ this must be understood as a limit statement!

Hence $\boxed{\frac{\|J\|}{f_\mu} \text{ is increasing and } \geqslant 1 \text{ on } (0, \tau_2)\,!}$

Finally now we also obtain $\underline{\underline{\tau_2 = \tau}}$!

proof: Otherwise $0 < \tau_2 = \tau_1 < \tau$ and so $\underline{J(\tau_1) = 0}$ by the definition of τ_1.

But $\|J(t)\| \geqslant f_\mu(t) \quad \forall\, t \in (0, \tau_1) \Rightarrow$

$\Rightarrow \underline{\|J(\tau_1)\| \geqslant f_\mu(\tau_1) > 0}$. Contradiction!

since $\tau_1 < \tau$

$\square\ \square$

Cor 1: Assume M is complete and has sectional curvature everywhere $\leq \mu$. Then for all $p \in M$, $v \in T_pM$:

$$\left[\mu \leq 0 \text{ or } \|v\| < \frac{\pi}{\sqrt{\mu}}\right] \Rightarrow (d\exp_p)_v \text{ non-singular!}$$

Thus for $\mu \leq 0$: exp is everywhere non-singular, and there are no conjugate points along any geodesic!

proof: We know $(d\exp_p)_0 = I_{T_pM}$; hence may assume $v \neq 0$.

Consider the geodesic $c(t) = \exp(t \cdot \frac{v}{\|v\|})$; $\|\dot{c}(t)\| = 1$.

Consider any Jacobi field J along c with $J(0) = 0$, $\dot{J}(0) \neq 0$.

By Cor 1 in #17, our task is to prove $J(\|v\|) \neq 0$, if $\left[\mu \leq 0 \text{ or } \|v\| < \frac{\pi}{\sqrt{\mu}}\right]$.

We may assume $J^{tan} \equiv 0$ (cf. Lemma 3 in #17).

Note $\|J\|^{\bullet}(0) = \|\dot{J}(0)\| \neq 0$,

indeed, $\|\dot{J}(0)\| = \left\|\lim_{h \to 0} \frac{J(h)}{h}\right\| = \lim_{h \to 0} \left\|\frac{J(h)}{h}\right\| = \|J\|^{\bullet}(0)$

and after scaling we may assume $\|J\|^{\bullet}(0) = 1$.

Then Thm. 1 applies with $f_\mu = s_\mu$ and $\tau = \begin{cases} \pi/\sqrt{\mu} & (\mu > 0) \\ +\infty & (\mu \leq 0). \end{cases}$

Thm 1 implies $\|J(t)\| \geq f_\mu(t) > 0$ for all $t \in (0, \tau)$;

in particular $\|J(\|v\|)\| > 0$, if $\left[\mu \leq 0 \text{ or } \|v\| < \frac{\pi}{\sqrt{\mu}}\right]$.

□

Theorem 2 (The "Cartan–Hadamard Theorem"): (Jost Cor 5.8.1)

Assume M is complete and has sectional curvature everywhere ≤ 0. Then for any $p \in M$,

$\exp_p : T_pM \to M$ is a covering map. $\circledast$

Hence $T_pM \ (\cong \mathbb{R}^d)$ is the universal covering space of M (via $\exp_p$). In particular if M is simply connected then $\exp_p : T_pM \to M$ is a (surjective) diffeomorphism.

See Sec. 6.1 regarding "covering map (/space)"! Note in particular that all statements after $\circledast$ follow from $\circledast$ via basic properties of (universal) covering spaces.
We recall here that $\circledast$ means: [Each point $q \in M$ has an open neighborhood $U \subset M$ such that $\exp_p^{-1}(U)$ is a union of disjoint open sets in T_pM each of which is mapped diffeomorphically onto U by $\exp_p$.]

proof: Cor 1 (and the Inverse Function Theorem) implies that $\exp_p$ is a local diffeomorphism on all T_pM. Hence we can endow T_pM with a unique Riemannian metric such that $\exp_p$ becomes a local isometry. (Cf. Problem 18.) Now $\circledast$ follows by Problem 94. $\square$

T_pM is complete by the Hopf–Rinow Theorem, since every geodesic starting at $0 \in T_pM$ extends indefinitely!

19.1. **Notes.** .

p. 5: Note that Jost's [5, Cor. 5.1.1] leads to another proof of our Corollary 1 in the case $\mu \leq 0$, not using Rauch's Comparison Theorem. Indeed, if M has sectional curvature everywhere ≤ 0 then [5, Cor. 5.1.1] implies via Theorem 1 in #18 that there are no conjugate points along any geodesic, and this implies that $(d \exp_p)_v$ is non-singular for all $p \in M$ and $v \in T_pM$. (Jost proves [5, Cor. 5.1.1] by using the formula

$$E''(s) = \int_a^b \left(\langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} c', \dot{c} \rangle - \langle R(\dot{c}, c')c', \dot{c} \rangle + \left\| \nabla_{\frac{\partial}{\partial t}} c' \right\|^2 \right) dt,$$

which appears in the *proof* of Theorem 1 in #16 (= Jost's [5, Thm. 5.1.1]). For the special choice of variation $c(t, s)$ which Jost considers in the proof of [5, Cor. 5.1.1], all terms in the above integral are seen to be non-negative; therefore $E''(s) \geq 0$ for all s.)

p. 6: Theorem 2 is = Jost's [5, Cor. 5.8.1]; however we follow Lee [9, Thm. 11.5] in our presentation.

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