UPPSALA UNIVERSITET
Matematiska institutionen
Christer Kiselman, 0708-870708

Prov i matematik
Teknisk fysik, B
Transformer för beräkningar
2001 06 07

Skrivtid: 08:00–13:00.
Tillåtna hjälpmedel: Skrivdon. Ordlista 2001 04 05. Formelsamling 2001 05 06. Räknare. Ej mobiltelefon.
Svara på svenska eller annat språk.

1. Let a and b be two vectors in $l^2(\mathbf{Z}_5)$, a being given by its Fourier transform $\hat{a} = (3, 2, 1, 0, -1)$, and $b = (1, \omega^2, \omega^4, \omega, \omega^3)$, where $\omega = e^{-2\pi i/5}$.

(a) Calculate the Fourier transform $\hat{b}$. (3)

(b) Calculate the convolution product $a * b$. (2)

2. We consider two black boxes B and C, defined by vectors $b, c \in l^1(\mathbf{Z})$ and the relations $u = b * z$ and $v = c * z$ between the common input signal $z \in l^2(\mathbf{Z})$ and the output signals $u, v \in l^2(\mathbf{Z})$.

(a) Prove that the output signals satisfy the relation $\hat{c}(\tau)\hat{u}(\tau) = \hat{b}(\tau)\hat{v}(\tau)$ for all frequencies $\tau \in \mathbf{R}$. (3)

(b) Suppose now that the black boxes satisfy $|\hat{b}(\tau)| + |\hat{c}(\tau)| \geqslant \varepsilon > 0$ for all τ. Prove that it is possible to reconstruct z from knowledge of u and v. More precisely, prove that there is a formula $\hat{z}(\tau) = \varphi(\tau)\hat{u}(\tau) + \psi(\tau)\hat{v}(\tau)$ with functions φ and ψ such that z becomes a continuous function of u and v in the sense that there is an estimate $\|z\|_2 \leqslant C(\|u\|_2 + \|v\|_2)$. (3)

3. Let $u = (8, 0, 0)$, $v = (7, 7, 8)$, and $z = (8, 7, 0, 7, 0, 8)$. Compute $\hat{u}$, $\hat{v}$ and then $\hat{z}$ using the fast Fourier transformation. (5)

4. Let $u \in l^2(\mathbf{Z}_N)$ and define $v \in l^2(\mathbf{Z}_{N/2})$ by the formula $v(n) = u(n) + u(n + N/2)$, assuming N to be even. Prove that $\hat{v}(n) = \hat{u}(2n)$. (Note that $\hat{u}$ and $\hat{v}$ are actually defined on different groups, $\mathbf{Z}_N$ and $\mathbf{Z}_{N/2}$, respectively, but that we may consider them as periodic functions defined on $\mathbf{Z}$, albeit with different periods, just to allow for a brief notation.) (6)

5. We consider a convolution equation of the fourth degree, $z * z * z * z = \delta$, on (a) the finite cyclic group $\mathbf{Z}_5$ and (b) the infinite cyclic group $\mathbf{Z}$.

(a) Prove that the equation has more than a thousand solutions $z \in l^2(\mathbf{Z}_5)$. Give an explicit formula for $\hat{z}(k)$, $k = 0, 1, 2, 3, 4$. (3)

(b) Prove that the equation has only four solutions $z \in l^1(\mathbf{Z})$. Give all solutions explicitly. (3)

6. Show that the discrete sine transform of $z = (z(1), ..., z(N-1))$,

$$\mathrm{DST}(z)(j) = \sum_{k=1}^{N-1} z(k) \sin \frac{jk\pi}{N}, \qquad j = 1, ..., N-1,$$

can be written on the form

$$\mathrm{DST}(z)(j) = \frac{i}{2} \sum_{k=0}^{2N-1} z(k)\, e^{-2\pi i jk/(2N)},$$

where

$$z(0) = z(N) = 0,$$

and

$$z(k) = -z(2N-k), \quad k = N+1, ..., 2N-1,$$

thus as a Fourier transform of a vector with period $2N$. (This means that the discrete sine transform can be computed using the FFT-algorithm.) (6)

7. It is known that we can define the angle α formed by two nonzero vectors x, y in a real inner-product space by the formula

$$\langle x, y \rangle = \|x\| \, \|y\| \cos \alpha.$$

Here $0 \leqslant \alpha \leqslant \pi$. Calculate the angle between the functions $f(x) = 1 + \sqrt{2/3}\, x$ and $g(x) = 1$, $x \in \mathbf{R}$, in the space of real-valued Hermite polynomials. (The fact that $\|1\|^2 = \langle 1, 1 \rangle = \sqrt{\pi}$ in that space may be used without proof.) (6)

Svar till tentamen i Transformer för beräkningar 2001 06 07

1. (a) We find that $\hat{b} = (0, 0, 0, 5, 0)$, so that $\hat{a}\hat{b} = 0$.
(b) The convolution product is $a * b = 0$, since its Fourier transform is $\hat{a}\hat{b} = 0$.

2. (a) We get $\hat{u} = \hat{b}\hat{z}$ and $\hat{v} = \hat{c}\hat{z}$ so that $\hat{c}\hat{u} = \hat{c}\hat{b}\hat{z} = \hat{b}\hat{v}$.
(b) The reconstruction formula follows with $C = 2/\varepsilon$ if we choose for example $\varphi(\tau) = 1/\hat{b}(\tau)$ and $\psi(\tau) = 0$ when $|\hat{b}(\tau)| \geqslant \varepsilon/2$; and $\varphi(\tau) = 0$ and $\psi(\tau) = 1/\hat{c}(\tau)$ when $|\hat{b}(\tau)| < \varepsilon/2$. Then $\|\varphi\|_\infty, \|\psi\|_\infty \leqslant 2/\varepsilon$.

A little more symmetry is possible: we can take

$$\varphi = \frac{|\hat{b}|}{(|\hat{b}| + |\hat{c}|)\hat{b}}, \qquad \psi = \frac{|\hat{c}|}{(|\hat{b}| + |\hat{c}|)\hat{c}},$$

where φ is defined as zero when $\hat{b}$ is zero, and similarly ψ is defined as zero when $\hat{c}$ vanishes. We have $|\varphi|, |\psi| \leqslant 2/\varepsilon$ everywhere.

3. We find $\hat{u} = (8, 8, 8)$ and $\hat{v} = (22, 7 + 7\omega + 8\omega^2, 7 + 7\omega^2 + 8\omega) = (22, -1 - \omega, \omega)$, where $\omega = e^{-2\pi i/3} = -\frac{1}{2} - \frac{i}{2}\sqrt{3}$. With the fast Fourier transformation we the obtain

$$\begin{aligned}
\hat{z}(0) &= \hat{u}(0) + \hat{v}(0) = 30,\\
\hat{z}(1) &= \hat{u}(1) + \theta\hat{v}(1) = 8 - \omega = \tfrac{17}{2} + \tfrac{1}{2}\sqrt{3}\, i,\\
\hat{z}(2) &= \hat{u}(2) + \theta^2\hat{v}(2) = 7 - \omega = \tfrac{15}{2} + \tfrac{1}{2}\sqrt{3}\, i,\\
\hat{z}(3) &= \hat{u}(0) - \hat{v}(0) = -14,\\
\hat{z}(4) &= \hat{u}(1) - \theta\hat{v}(1) = 8 + \omega = \tfrac{15}{2} - \tfrac{1}{2}\sqrt{3}\, i,\\
\hat{z}(5) &= \hat{u}(2) - \theta^2\hat{v}(2) = 9 + \omega = \tfrac{17}{2} - \tfrac{1}{2}\sqrt{3}\, i,
\end{aligned}$$

where $\theta = e^{-\pi i/3} = -\omega^2 = 1 + \omega$, $\theta^2 = \omega$, $\theta^3 = -1$, $\theta^4 = \omega^2 = -1 - \omega$, $\theta^5 = -\omega$. So $\hat{z} = (30, 8-\omega, 7-\omega, -14, 8+\omega, 9+\omega) = (30, \frac{17}{2}+\frac{1}{2}\sqrt{3}\, i, \frac{15}{2}+\frac{1}{2}\sqrt{3}\, i, -14, \frac{15}{2}-\frac{1}{2}\sqrt{3}\, i, \frac{17}{2}-\frac{1}{2}\sqrt{3}\, i)$.

4. Let $\omega = e^{-2\pi i/N}$, $\theta = e^{-2\pi i/M}$, where $M = N/2$. Then $\omega^2 = \theta$, and we get

$$\hat{v}(n) = \sum_{k=0}^{M-1} \theta^{nk} u(k) + \sum_{k=0}^{M-1} \theta^{nk} u(k + M).$$

Substituting ω^2 for θ and introducing $j = k + M$ as a new summation index going from M to $2M - 1$ in the second sum, we see that the two sums together form the sum defining $\hat{u}(2n)$. Indeed, the factor θ^{nk} there is equal to $\theta^{nk} = \omega^{2n(j-M)} = \omega^{2nj}\omega^{-nN} = \omega^{2nj}$.

5. (a) Taking the Fourier transform we see that $\hat{z}(t)^4 = 1$ so that $\hat{z}(k) = i^{m_k}$ for some numbers $m_k \in \{0, 1, 2, 3\}$, $k = 0, 1, 2, 3, 4$. This gives $4^5 = 2^{10} = 1024$ solutions. And $1024 > 1000$.
(b) We get $\hat{z}(\tau) = i^{m(\tau)}$ for every τ, but since $\hat{z}(\tau)$ is a continuous function of the real variable τ, the number m must be the same for all τ; hence $\hat{z}(\tau) = 1, i, -1, -i$ are the only possibilities, corresponding to $z = i^m\delta$, $m = 0, 1, 2, 3$.

6. We start by writing the sine function as a sum of two exponential functions

$$\mathrm{DST}(z)(j) = \frac{1}{2i}\left(\sum_{k=1}^{N-1} z(k)\, e^{ijk\pi/N} - \sum_{k=1}^{N-1} z(k)\, e^{-ijk\pi/N}\right).$$

Replacing k by $2N-k$ and using the periodicity of the exponential function, we can write the first sum above as

$$\sum_{k=1}^{N-1} z(k)\, e^{ijk\pi/N} = \sum_{k=N+1}^{2N-1} z(2N-k)\, e^{-ijk\pi/N}.$$

Extending z as suggested we obtain the result.

7. We need to calculate also $\|x\|$. This can be done using partial integration:

$$\|x\|^2 = \langle x^2, 1\rangle = \langle x, x\rangle = \tfrac{1}{2}\sqrt{\pi}.$$

Thus for affine functions $f(x) = a + bx$, $g(x) = c + dx$,

$$\langle f, g\rangle = \langle a + bx, c + dx\rangle = ac\langle 1, 1\rangle + bd\langle x, x\rangle = \sqrt{\pi}(ac + \tfrac{1}{2}bd),$$

so that

$$\cos\alpha = \frac{ac + \frac{1}{2}bd}{\sqrt{a^2 + \frac{1}{2}b^2}\sqrt{c^2 + \frac{1}{2}d^2}}.$$

In particular, taking $a = 1$, $b = \sqrt{2/3}$, $c = 1$, $d = 0$, we get $\cos\alpha = \frac{1}{2}\sqrt{3}$, so that the angle is $\pi/6$ radians or 30 degrees.