The cultural significance of mathematics

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1. Mathematics and its applications

Mathematics is studied in every school and university. Nevertheless it is rather unknown and often misunderstood. Schoolchildren—hence also adults, since children become adults—are often afraid of mathematics or hate it. Where is the source of this problem? Is it contained in the nature of mathematics itself?

Mathematics has been successfully applied first in the natural sciences (astronomy, physics, later chemistry, meteorology, biology), and these applications are still fruitful. This is probably the reason why mathematics is often looked upon as one of the natural sciences. But it does not belong there. Mathematical applications in economics are important today, and mathematics is an indispensible tool in any technical science. Already these applications make it impossible to view mathematics as one of the natural sciences.

But there is another and fundamentally more important reason not to classify mathematics as a natural science. Its development on a superficial level is driven by needs which make themselves felt in technology and other sciences but on a deeper level by curiosity and an urge to act similar to the driving forces one finds in art. And once we realize that, this idea will have a great impact on how we plan education for different age groups, from the youngest children to postgraduate education.

To illustrate the role of mathematics in science and technology, I shall mention a few examples.

Albert Einstein (1879–1955) used Riemannian geometry and tensor calculus in his general theory of relativity. These intellectual tools were not developed for the sake of physics, but much earlier in pure mathematics. They were completely ready when Einstein began to use them. Bernhard Riemann (1826–1866) had introduced what is now called Riemannian differential geometry; on a Riemannian manifold one can compute distances and one has different concepts of curvature. Carl Friedrich Gauss (1777–1855) developed a theory for surfaces where he distinguished between intrinsic and extrinsic properties, i.e., on the one hand properties that can be studied if we live inside the surface and do not know of anything else, and on the other hand those which depend on the fact that we can look at the surface as lying in an ambient space. Gauss had to confront concrete geodetic problems concerning the inner geometry of surfaces during the triangulations of the surface of the earth that where initiated during his time. He was director of the astronomical observatory in Göttingen 1807–1855 and made degree measurements himself in 1821–1824. The geoid as a fundamental surface in geodesy was introduced by him in 1828. He was inspired by the geodetic problems but went much further in his mathematical theory than was needed for their solution. With the name of Gregorio Ricci (1853–1925) we associate the tensor calculus, which makes it possible to describe quantities of various kind and how they behave under coordinate changes. Marcel Grossman (1878–1936) explained to Einstein part of Gauss' theory of surfaces [Grattan-Guinness 1994:1239]. Tensor calculus became well-known because of the fact that Einstein used it in his general theory of relativity, published in 1916.

Another example is the theory of spectral decomposition of self-adjoint operators in Hilbert space. David Hilbert (1862–1943) published in 1912 a theory for linear integral equations. It was later extended by Torsten Carleman (1892–1949) to a more general case, called singular integral equations. Carleman, whose work appeared in 1923, expressed his results not with the help of abstract Hilbert space theory but in terms of an integral equation. It had a real and symmetric kernel, which could be so unpleasant that the corresponding operator was not continuous. It was John von Neumann (1903–1957) who put all these results into an abstract and unified theory. His work appeared in 1929. The real and symmetric kernel corresponds to a self-adjoint operator in the abstract theory. In an almost miraculous way it turned out that the results about spectral decomposition of selfadjoint operators could be used as a mathematical model in quantum mechanics. Hilbert's and Carleman's investigations did not envision that goal at all. It developed that important physical quantities correspond to discontinuous operators in the mathematical model, thus vindicating Carleman's theory: the continuous operators proved insufficient. In quantum mechanics, there exist two fundamental concepts: the states and the observable quantities. The states are equivalence classes of vectors in a complex—not real—Hilbert space, and the observable quantities are self-adjoint operators, not necessarily continuous, that act on these vectors. Nothing in everyday life leads directly to complex numbers, and they did not appear in any physical observations, but still they turned out to be essential for the formulation of the quantum-theoretical laws.

As a third example we may take the mathematical foundations of computer science. The theory of computable functions was developed in the thirties before modern computers existed, and it was seemingly without application. Modern logic programming is built on a theorem of Jacques Herbrand (1908–1931) from the early thirties. During that decade, Alonzo Church (1903–1995) created lambda calculus. It was published in 1941 and became the basis of the functional programming languages, of which LISP from 1960 is an example. The basic principles of how computers work were developed in the forties by, among others, John von Neumann. Self-correcting codes, which are now used in digital communication all over the world, are based on Galois theory, a creation by Évariste Galois (1811–1832). (That theory is otherwise most noted in that it shows that a fifth degree equation cannot be solved by radicals.)

Can string theory become a fourth example? It exploits very modern and abstract mathematics; at the same time it inspires development of even more mathematics. It implies large changes in our view of the world. Our concept of spacetime seems according to Edward Witten (b. 1951) "destined to turn out to be only an approximate, derived notion, much as classical concepts such as the position and velocity of a particle are understood as approximate concepts in the light of quantum mechanics" [1996:28]. It might be too early to say anything definite about the role of mathematics in this case, since string theory presently is in what some call the second superstring revolution (the first happened in the eighties); Witten [1996:30]. At least it is clear that classical mathematical concepts like manifolds and differential forms play a basic role, and that the latest development in mathematical fields such as topology and knot theory are highly relevant for what some with perhaps not fully developed humbleness call *the Theory of Everything*; Taubes [1995].

Let us quote Freeman Dyson (b. 1923): "One factor that has remained constant through all the twists and turns of the history of physical science is the decisive importance of mathematical imagination. Each century had its own particular preoccupations in science and its own particular style in mathematics. But in every century in which major advances were achieved the growth in physical understanding was guided by a combination of empirical observation with purely mathematical intuition. For a physicist mathematics is not just a tool by means of which phenomena can be calculated; it is the main source of concepts and principles by means of which new theories can be created" [1968:249].

But of course the mathematicians are not always successful. According to Dyson [1972] it has happened a number of times that the mathematicians have missed opportunities to develop their science. For example, the equations that James Clerk Maxwell (1831–1879) published in 1873 offered a very interesting field that did not attract as much attention by the mathematicians as it deserved. Perhaps, if the mathematicians had begun to study these new problems when they first arose, they would have had the opportunity to discover relativity theory several decades before Einstein did. Dyson bases this bold statement on the fact that Maxwell's equations are invariant under certain transformations that form a group, i.e., a set consisting of transformations that can be composed and inverted. Such a group is an important mathematical object in itself. Maxwell's equations are invariant under the Lorentz group, whereas Newtonian mechanics is invariant under another group, the so-called Galilei group. The Lorentz group is mathematically simpler and more beautiful than the Galilei group. If the mathematical properties of these groups had been studied, perhaps the special theory of relativity could have been discovered. Of course we must be aware that this reasoning is in the conditional mood. We cannot prove what would have happened if the mathematicians had done something else than they did. But Dyson's speculation still points in the same direction as the preceding positive examples: a great confidence in the possibilities of finding physically interesting theories within mathematics.

Eugene Wigner (1902–1995) wrote that "the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it" [1960:2]. And "it is just this uncanny usefulness of mathematical concepts that raises the question of the uniqueness of our physical theories"; i.e., whether other—totally different—theories could explain the phenomena as well as those that we happen to have at hand.

Reflecting on these suggestions of the power of mathematics to influence how science is formulated, we are drawn to ask: are the theories of physics just those that the mathematical theories and methods allow a certain investigator at a certain moment? If the answer is yes, why are these methods available at a given moment? If mathematics were different, would also physics be different? What are the implications of these questions for the responsibility of the mathematicians? And what are the implications for research policy?

2. Immobility and mobility of mathematics

Many people believe that mathematics is a collection of fixed truths and unchangeable laws. It is not hard to see the roots of such a belief. We learn that two plus two is four, and we cannot imagine that this truth one day should be untrue. A stone that we see on the ground can be several billions of years old, and it might be dust within a few million years, but at that time it will still be true that two plus two is four. Or don't you believe that? Mathematics appears to be much more stable than the most stable parts of our physical reality. Even general knowledge has changed more in other fields. According to the theory that Alfred Wegener (1880–1930) published in 1912, the continents are moving relative to each other. When I went to school I learned that his theory was naive and false. We schoolchildren nevertheless thought that Africa and South America fit rather well together. Nowadays it is an established fact that these continents once were together. I also learned that humans have 48 chromosomes. Now the children learn that a human has 46 chromosomes. (The number 48 is said to come from a miscalculation that was done on a photo where everyone today sees only 46.) In this way my knowledge about the world has changed. On the other hand, in mathematics I learned more than forty years ago that the derivative of the function $x \mapsto x^4$ is $x \mapsto 4x^3$, and so far I have not heard anything else. These facts give an inevitable impression that the geosciences develop, biology develops, but not mathematics. Or is that impression really inevitable?

I claim that mathematics, like a living creature, consists of immobile and mobile parts. Does a human need rigid bones or soft muscles? To be able to run, it seems that a human needs both. The skeleton alone cannot move, and without it the muscles have nothing to work against. Similarly, while some parts of mathematics appear to be very immobile, others are in a state of fast development and very mobile. The parts which have been immobile for a long time are what we teach in schools; the mobile parts are less known. Thus it may not be so surprising that people think of mathematics more as a skeleton rather than as muscles.

Every year tens of thousands of articles are published about new results in mathematics. Lots of new facts become known and old ones become understood in a new light. (And... it might be added here that mathematics is free of the often very hampering difficulties of experiments or observations that hold up experimental sciences...)

But it is not only that mathematics develops: mathematics also contains a lot of arbitrariness. In the same way that the rigid mathematics is extremely stable, the mobile mathematics is extremely mobile in its unlimited arbitrariness. This can be very disturbing for those who resort to mathematics in a desire for security and stable values; the arbitrariness makes them disappointed and even can seem scary.

One example of this arbitrariness comes from the history of the parallel postulate. According to this axiom posed by Euclid (ca. 303 – ca. 275 B.C.), there is exactly one straight line through a given point that is parallel to a given straight line. Is it possible to prove this axiom using the other axioms? This question occupied mathematicians for two millenia. Finally three mathematicians in the nineteenth century proved that this is impossible. They were Janos Bolyai (1802–1860), Nikolaj Ivanovich Lobachevskij (b. 1792 or 1793, d. 1856) and Gauss. They proved this by constructing geometries where through a given point there is either none or more than one straight line. And these geometries are as valid and as true as Euclid's. Through the existence of these new geometries, in which all other axioms are valid, one understands that Euclid's parallel postulate is not possible to prove by means of the other axioms. Because if that were the case, the parallel postulate would also be true in the new geometries. Elementary! Why did the solution of this problem take two thousand years? Such a question can hardly be answered, but a possible reason is that it was very shocking for people to accept the fact of the arbitrariness of the axioms, these so-called "self-evident" starting-points for the human mind. Such an explanation is supported by the fact that Gauss did not publish his discovery—despite the fact that he was highly respected and would not have risked his career by publishing it.

Another example of arbitrariness is perhaps even more dramatic when it comes to the limits of our reasoning power: the independence of the continuum hypothesis. This hypothesis says that every infinite subset of the field $\mathbf{R}$ of real numbers (in this context called the *continuum*) either has as many elements as the natural numbers $\mathbf{N}$ or as the whole continuum $\mathbf{R}$. To express this with mathematical symbols we shall denote the number of element in a set A by card A; we say that card A is the *cardinal number* of the set A. It is a number, finite or infinite. (As an example we mention that prime numbers have the same cardinal number or cardinality as $\mathbf{N}$, as do the rational numbers, whereas the positive numbers have the same cardinality as the whole continuum.) The continuum hypothesis says that it cannot happen that $\mathrm{card}\,\mathbf{N} < \mathrm{card}\,A < \mathrm{card}\,\mathbf{R}$. The proof of this was the first of twenty-three problems posed by Hilbert in Paris in 1900 as "the future problems of mathematics." He thought it was very plausible that this conjecture was true [1902:70].

To Hilbert, like probably to any mathematician of his generation, either there existed a subset A of $\mathbf{R}$ such that $\mathrm{card}\,\mathbf{N} < \mathrm{card}\,A < \mathrm{card}\,\mathbf{R}$, or there did not exist such a set. Research should make it clear to us which alternative was the right one. But it later turned out that the continuum hypothesis is independent of the other axioms. According to Kurt Gödel (1906–1978) one can add the continuum hypothesis to the other axioms of set theory without introducing (new) contradictions, and according to Paul Cohen (b. 1934) one can do the same with the negation of the hypothesis. This means that a set theory where there exists a set A with $\mathrm{card}\,\mathbf{N} < \mathrm{card}\,A < \mathrm{card}\,\mathbf{R}$ is as valid and as true as a set theory where the continuum hypothesis is valid.

To sum up we can say that mathematics does not help us to verify whether, in the real world, there is no, one or many straight lines through a given point parallel with a given straight line. Also mathematics does not help us to verify whether there exist or does not exist certain infinite sets. Here the arbitrariness of mathematics manifests itself, and it leaves us in the lurch. But at the same time, paradoxically, we should remember that mathematics is the main or even only source of concepts and principles in the natural sciences, and the only language in which the natural sciences can express derivations and results.

3. Mathematics as subculture and as a cultural element

Since the role of mathematics, as we just saw, is so paradoxical in relation to the other sciences, it is permissible, and perhaps also desirable, to look for other perspectives that can explain its function. One such alternative perspective is to accept that mathematics is part of the human culture and to compare it in general with other cultural phenomena. White [1956] and Wilder [1981] have written from this point of view.

First of all we should make clear that human culture can be of two kinds:

a *cultural element* is a part of the culture common to a certain group of people; a *subculture* is a culture that is specific to a certain subgroup of that group (the subgroup is too small or too spread out to carry a culture itself).

Mathematics plays a role both as cultural element and as subculture. As part of culture, mathematics consists of all the mathematical knowledge, views and skills that a certain people own collectively. To keep these alive and perhaps expand them is a goal for general education. As an example we mention that most people are not familiar with the concepts of differentiation and integration of functions, but nevertheless have an idea of speed (in kilometers per hour), acceleration (increase of speed), interest on a mortgage, summing of monthly payments to an annual salary, as well as other things that are concrete manifestation of the abstract concepts of differentiation and integration of functions. As we see, the exact delimitation of this part of culture is not an easy task, but at least we can observe that it consists solely of parts of mathematics that were completed a long time ago.

On the other hand, mathematics as subculture is the culture that is specific for people who have had training in mathematics as a science. Although this group is certainly not homogeneous, it is an interesting observation that it is more alike between one country and another than many other cultural phenomena, and in particular more alike than in school mathematics. Long ago one could talk about Chinese, Arabic, Greek and South American mathematics, but hardly any longer.

4. If mathematics is culture...

Why would we view mathematics as culture? Normally we look upon a phenomenon as culture in order to understand it and forecast its development in that framework. I do not dare to forecast very much, but in my opinion this point of view of this paradoxical science is fruitful in order to formulate and understand many difficult problems. Dyson writes that "science is a human activity, and the best way to understand it is to understand the individual human beings who practise it. Science is an art form and not a philosophical method" [1996:805].

Between the two concepts, mathematics as a cultural element in the culture of a whole nation and mathematics as subculture, there exists a certain tension which is visible for instance in education. Indeed, education is an introduction to the cultural element as well as to the subculture, in varying degrees from the early years to postgraduate studies. I certainly cannot scrutinize mathematics education on the whole planet, but I cannot avoid noticing that mathematics education in many countries is not successful. It is often too formal and too much concentrated on transferring routine skills. This gives the schoolchildren an impression of mathematics being the dryest and least interesting field of knowledge in the world.

In psychology, one sometimes differentiates between two types of intelligence, the so-called *convergent intelligence* and the so-called *divergent intelligence*; see, e.g., Massarenti [1980]. The former is the ability to start from given conditions and reach a solution that is uniquely determined or at least the only acceptable one. The latter starts from the given situation and, along different routes, searches solutions that work, and none of which is the only acceptable one. The risk with school mathematics is that it tends to stimulate only convergent thinking, and that the given problems are so stereotyped and well prepared for treatment by routine methods that divergent intelligence seems unnecessary. It is clear that convergent intelligence is only a special case of divergent intelligence, and convergent intelligence probably has to be trained first in order to develop work methods that later can be applied to more complicated situations, where an intelligence of divergent type is needed. Of course divergent intelligence is indispensible on the research level in any science—otherwise we would not be talking about research.

We can schematically—perhaps too schematically—divide mathematics according to three criteria: cultural status, ability to change, and intelligence type required. Let us make out the divisions:

Mathematics as cultural element vs. Mathematics as subculture
Immobile, "skeleton–like" mathematics vs. Mobile, arbitrary, "muscular" mathematics
Requires convergent intelligence vs. Requires divergent intelligence

Could it be that these three divisions coincide, more or less? If the answer is yes, we must make an effort to change mathematics as a cultural element. I believe that general mathematics education would improve if it became more movable, less routine, and if more divergent thinking was required to solve its exercises. Why? The applications of mathematics would that way gain in quality and become more credible and more realistic. That would influence in a positive way all fields of knowledge where mathematics is used. But to change education is not an easy task, partly because people who like convergent thinking already are attracted by school mathematics and are disinclined to make it less “skeleton-like.”

5. The cultural significance of mathematics

Finally, what is the cultural significance of mathematics? The answer definitely depends on ones own values. Here I limit myself to four properties which I think mathematics holds compared to other cultural phenomena:

◇ *Internationality*
◇ *Beauty*
◇ *Influence on our view of the world*
◇ *Influence on our own thought processes and our confidence in them*

When it comes to internationality it has to be said that there does not exist anything absolutely international. But a cultural phenomenon can be more or less varying within mankind. And mathematics as a subculture is certainly more international than many other cultural phenomena, and also more than many other sciences, in particular the social sciences. Mathematics as subculture can influence education in mathematics and make it more international; often that would be a good thing. But we must note also that scientific mathematics is not completely international. There are a number of national characteristics in it. We should distinguish internationality from the crossing of frontiers that is made possible by superior means of communication.

As with any cultural phenomenon we can ask: Which are the “laws” according to which culture develops? What is most important? Who decides what is most important? To decide what is most important is real power.

The beauty of mathematics is an essential property, and it is important from a number of viewpoints. As in the arts beauty is a value. But not only that: it is also the fastest guide in the continuous choice between different paths that a developing theory can take.

Mathematics influences our view of the world; in the most mathematized sciences, no other language even seems possible. So far mathematics has mainly had an ordering function: it assures us that the world is not arbitrary and chaotic but possible to order and predict. It is a fact that the desire to be able to predict (eclipses, the weather) has been an important source for the tendency towards mathematization. But also chaos has its mathematics! Mathematics certainly governs the picture of the world that we make for ourselves—but to what extent?

Mathematics also influences our mental abilities. The human brain is influenced and changed by the work it executes, at least during the first years—like a human computer that builds itself while it works. The use of language and all theoretical work influence the young brain's development. Even single speech sounds shape the brain; Näätänen et al. [1997]. This certainly makes it important to choose a good occupation! If we can solve problems and avoid difficulties, then personality profits. That way mathematics can build our self-confidence (if we succeed)—or destroy it (if we fail).

All this points to the importance of creating a mathematical environment which is as good as possible, especially during childhood.

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