A differential inequality characterizing weak lineal convexity

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Dedicated to the memory of
André Martineau (1930–1972)

Contents:

Resumo: *Diferenciala neegalaĵo karakterizanta malfortan linian konveksecon*
Behnke kaj Peschl enkondukis en 1935 la nocion de *Planarkonvexität,* nuntempe nomatan malforta linia konvekseco. Ili montris ke por regionoj kun glata rando ĝi implicas ke diferenciala neegalaĵo estas plenumita ĉe ĉiu randopunkto. Ni pruvos la inverson.

Abstract: Behnke and Peschl introduced in 1935 the notion of *Planarkonvexität*, now called weak lineal convexity. They showed that, for domains with smooth boundary, it implies that a differential inequality is satisfied at every boundary point. We shall prove the converse.

1. Introduction

In an article published in the *Mathematische Annalen* in 1935, Heinrich Behnke (1898–1979) and Ernst Peschl (1906–1986) introduced a notion of convexity called *Planarkonvexität,* nowadays known as *weak lineal convexity.* They showed that for domains in the space of two complex variables with boundary of class C^2, this property implies that a differential inequality is satisfied at every boundary point. Here we shall prove that, conversely, the differential inequality is sufficient for weak lineal convexity.

Lineal convexity is a notion of convexity in complex geometry which is intermediate between usual convexity and pseudoconvexity. By definition a set in $\mathbf{C}^n$ is *lineally convex* if its complement is a union of complex hyperplanes. An open set is called *weakly lineally convex* if there passes, through any boundary point, a complex hyperplane which does not intersect the set. If the boundary of the set is of class C^1, the only candidate for such a plane is the complex tangent plane, so then weak lineal convexity just means that no complex tangent plane shall cut the set.

Lineal convexity is not a local condition. There exist open sets with Lipschitz boundary which are not lineally convex but which are such that every point in the

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space has a neighborhood which intersects the set in a lineally convex set. This makes the study of such sets tricky and is in contrast to both pseudoconvexity and usual convexity: if a domain is such that every point on its boundary has a neighborhood which intersects the domain in a (pseudo)convex set, then the whole domain has the same property.

However, Behnke and Peschl [1935:170] proved that for sets with smooth boundary, weak lineal convexity is a local property (see Theorem 3.1 below).

Both usual convexity and pseudoconvexity can be characterized infinitesimally. The simplest example of such a result is that a C^2 function of one real variable is convex if and only if its second derivative is nonnegative. More generally, a domain in $\mathbf{R}^n$ with boundary of class C^2 is convex if and only if the Hessian of a defining function is positive semidefinite in the tangent space at every boundary point. Similarly, an open set in $\mathbf{C}^n$ with boundary of class C^2 is pseudoconvex if and only if the Levi form of a defining function is positive semidefinite in the complex tangent space at every boundary point (the *Levi condition*).

In analogy with these two classical results, we shall prove in the present paper that a connected open subset of $\mathbf{C}^n$ with boundary of class C^2 is weakly lineally convex if and only if the real Hessian of a defining function is positive semidefinite in the complex tangent space at every boundary point (the *Behnke–Peschl condition*).

It is easy to see that semidefiniteness is necessary. It is also known—indeed, this is the *Hauptsatz* of Behnke and Peschl [1935]—that the corresponding strong condition, i.e., that the real Hessian be positive definite, is sufficient. Thus what we have proved is that semidefiniteness is sufficient.

In the case of convexity and pseudoconvexity, the best way to deal with semidefiniteness is to approximate the domain by domains which satisfy the corresponding condition of definiteness. This is not how we approach the problem here, at least not directly. I do not know if a weakly lineally convex domain with smooth boundary can be approximated by domains satisfying the strong Behnke–Peschl condition. For Hartogs domains, though, this is known. The idea of proof of the main result here is to construct Hartogs domains which share a tangent plane with the given domain.

I learned about lineal convexity from André Martineau in 1967-68 when I was in Nice with him. His premature death on May 4, 1972, was a great loss to world mathematics. He introduced also the notion of strong lineal convexity [1968], which, however, is not geometrically defined. Later Znamenskij [1979] found a geometric characterization; the property is now called $\mathbf{C}$-convexity. Nowadays the most important sources for $\mathbf{C}$-convexity are the book by Hörmander [1994] and the survey article by Andersson, Passare, and Sigurdsson [1995]. My earlier contributions to the field are to be found in [1978], [1996], [1997] and [MS]. The proof of the main result here depends on that for Hartogs domains in [1996].

I am grateful to Ragnar Sigurðsson for comments to the manuscript.

2. Definitions

To be able to characterize sets by infinitesimal conditions, we shall describe boundaries and their curvature using defining functions and the Hesse and Levi forms. In this section we give the needed definitions.

A *defining function* for an open set Ω is a real-valued function ρ of class C^1 such that its differential never vanishes when ρ vanishes, and such that Ω is the set of points where ρ is negative.

The complex tangent space at a boundary point a, denoted by $T_{\mathbf{C}}(a)$, is the set of all $t \in \mathbf{C}^n$ such that

$$\sum \rho'_{z_j}(a)t_j = 0, \tag{2.1}$$

whereas the real tangent space $T_{\mathbf{R}}(a)$ is the set of all $t \in \mathbf{C}^n$ such that

$$\operatorname{Re} \sum \rho'_{z_j}(a)t_j = 0. \tag{2.2}$$

Here and in the following we write

$$\rho'_{z_j} = \frac{\partial \rho}{\partial z_j}, \qquad \rho''_{z_j \bar{z}_k} = \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}, \quad \text{etc.}$$

for partial derivatives. The complex tangent plane is then $a + T_{\mathbf{C}}(a)$; it is contained in the real tangent plane $a + T_{\mathbf{R}}(a)$.

The complex Hessian (complex Hesse form) of a function ρ of class C^2 is defined to be the quadratic form

$$H = H_\rho(z;t) = \sum \rho''_{z_j z_k}(z) t_j t_k, \qquad z \in \mathbf{C}^n, t \in \mathbf{C}^n. \tag{2.3}$$

The Levi form of ρ is the Hermitian form

$$L = L_\rho(z;t) = \sum \rho''_{z_j \bar{z}_k}(z) t_j \bar{t}_k, \qquad z \in \mathbf{C}^n, t \in \mathbf{C}^n. \tag{2.4}$$

Finally the real Hessian of a function ρ of real variables $x_1, ..., x_m$ is the quadratic form

$$H_{\mathbf{R}} = H_{\mathbf{R},\rho}(x;s) = \sum \rho''_{x_j x_k}(x) s_j s_k, \qquad x \in \mathbf{R}^m, s \in \mathbf{R}^m. \tag{2.5}$$

When a function of n complex variables is given, its real Hessian in the $2n$ real variables $(\operatorname{Re} z_1, \operatorname{Im} z_1, ..., \operatorname{Re} z_n, \operatorname{Im} z_n)$ can be expressed using its complex Hessian and its Levi form as

$$H_{\mathbf{R}}(z;s) = 2(\operatorname{Re} H(z;t) + L(z;t)), \qquad z \in \mathbf{C}^n, s \in \mathbf{R}^{2n}, t \in \mathbf{C}^n, t_j = s_{2j-1} + is_{2j}.$$

Thus the characterization of convexity mentioned in the introduction is that $\operatorname{Re} H(a;t) + L(a;t)$ be nonnegative for all $a \in \partial\Omega$ and all $t \in T_{\mathbf{R}}(a)$. For a lineally convex set the same inequality holds for all $t \in T_{\mathbf{C}}(a)$. It is then equivalent to $L(a;t) \geqslant |H(a;t)|$ for $a \in \partial\Omega$ and $t \in T_{\mathbf{C}}(a)$. We shall say that Ω satisfies the *Behnke–Peschl condition* at a if

$$\operatorname{Re} H(a;t) + L(a;t) \geqslant 0, \qquad t \in T_{\mathbf{C}}(a). \tag{2.6}$$

We shall say that Ω satisfies the *strong Behnke–Peschl condition* at a if the form is positive definite, i.e.,

$$\operatorname{Re} H(a;t) + L(a;t) > 0, \qquad t \in T_{\mathbf{C}}(a) \smallsetminus \{0\}. \tag{2.7}$$

It is easy to prove that these conditions are invariant under complex affine mappings. They also do not depend on the choice of defining function. They were introduced for $n = 2$ by Behnke and Peschl [1935:169].

3. Main result

As noted in the introduction, lineal convexity is not a local condition. Simple examples of sets which are locally lineally convex but not weakly lineally convex can be found in Kiselman [1996: section 3]. However, weak lineal convexity is a local condition for sets with smooth boundary. The precise result is as follows.

Theorem 3.1. *Let Ω be a connected open set in $\mathbf{C}^n$ with boundary of class C^1. Assume that for every boundary point a, the closure of the intersection of Ω with the complex tangent plane at a does not contain a. Then Ω is weakly lineally convex.*

For sets in $\mathbf{C}^2$ or $\mathbf{P}^2$ with boundary of class C^2, this was proved by Behnke and Peschl [1935:170]. For a proof under the hypotheses stated here, see Hörmander [1994: Proposition 5.6.4]. The assumption there that Ω be bounded is not needed for the conclusion cited here. Cf. also Andersson, Passare and Sigurdsson [1995: Proposition 2.4.7]. We shall need this result in our proof.

The following two results are well known and easy to prove. They are due to Behnke and Peschl [1935: Theorems 7 and 8]; local weak lineal convexity is called *Planarkonvexität im kleinen* by them. Cf. also Zinov′ev [1971], Hörmander [1994: Corollary 5.6.5], and Kiselman [1996: Lemmas 5.2 and 5.3].

Lemma 3.2. *Let Ω be an open set in $\mathbf{C}^n$ with boundary of class C^2. If Ω is locally weakly lineally convex, then it satisfies the Behnke–Peschl condition (2.6) at every boundary point.*

Lemma 3.3. *Let Ω be an open set in $\mathbf{C}^n$ with boundary of class C^2. If Ω satisfies the strong Behnke–Peschl condition (2.7) at a point $a \in \partial\Omega$, then the complex tangent plane $a + T_{\mathbf{C}}(a)$ at a avoids Ω in a neighborhood of a.*

Combining Lemma 3.3 and Theorem 3.1 we can deduce that the strong Behnke–Peschl condition (2.7) at all boundary points is sufficient for weak lineal convexity. This is the *Hauptsatz* of Behnke and Peschl [1935:170] (for sets in $\mathbf{C}^2$ or $\mathbf{P}^2$). We now state our main result, that in fact also the weaker condition (2.6) is sufficient:

Theorem 3.4. *Let Ω be a connected open set in $\mathbf{C}^n$ with boundary of class C^2. Then Ω is weakly lineally convex if and only if Ω satisfies the Behnke–Peschl condition (2.6) at every boundary point.*

If Ω is locally weakly lineally convex, has a C^1 boundary, and in addition is bounded, then Ω is also $\mathbf{C}$-convex and lineally convex. This follows from Andersson, Passare and Sigurdsson [1995: Proposition 2.4.7], who consider sets in projective space. I do not know how their result can be applied to unbounded domains in $\mathbf{C}^n$ with smooth boundary; such domains are not necessarily smoothly bounded in $\mathbf{P}^n$.

4. Results for Hartogs sets

A subset A of $\mathbf{C}^n$ is called a *Hartogs set* if there is a set A_1 in $\mathbf{C}^{n-1} \times \mathbf{R}$ such that $z \in A$ if and only if $(z_1, ..., z_{n-1}, |z_n|) \in A_1$; A is said to be a *complete Hartogs set* if $z \in A$ implies $(z_1, ..., z_{n-1}, t) \in A$ for all t with $|t| \leqslant |z_n|$. The *base* of A is the subset of $\mathbf{C}^{n-1}$ consisting of all points $(z_1, ..., z_{n-1})$ such that $(z_1, ..., z_n) \in A$ for some complex number z_n.

Lineal convexity for Hartogs sets is easier to handle than in the general case. The following is known.

Theorem 4.1. *Let Ω be a complete Hartogs set in $\mathbf{C}^n$ which is open and connected. Assume that its boundary is of class C^2 except perhaps where $z_n = 0$. If Ω satisfies the Behnke–Peschl condition (2.6) at all boundary points with $z_n \neq 0$, then Ω is weakly lineally convex. If in addition the base of Ω is lineally convex, then Ω is lineally convex.*

This result was proved in Kiselman [1996: Theorem 7.6] for $n = 2$ and under slightly stronger hypotheses. The proof, however, is valid with small changes under the hypotheses given here. The case $n > 1$ is proved in Kiselman [MS].

Theorem 4.2. *Let Ω be a complete Hartogs set in $\mathbf{C}^2$ defined by a function R as*

$$\Omega = \{(z, w) \in \omega \times \mathbf{C};\ |w| < R(z)\},$$

where ω is an open disk in $\mathbf{C}$ and $R \in C^2(\omega)$ has positive values in all of ω. If Ω satisfies the Behnke–Peschl condition (2.6) at all points $(z, R(z))$ with $z \in \omega$, then Ω is lineally convex.

This theorem was proved in Kiselman [1996: Theorem 9.7]. Note that the boundary of Ω is not necessarily of class C^1 at a boundary point (z, w) with $z \in \partial\omega$ and $w \neq 0$. The important step in the proof is to approximate Ω by domains with smooth boundary satisfying condition (2.6). This can be done when ω is a disk, but for no other domain which is equal to the interior of its closure [1996: Theorems 8.3 and 8.4].

The Behnke–Peschl condition for a domain of the type described in Theorem 4.2 takes the form

$$\left|R'_z\right|^2 \geqslant \left|\left(R'_z\right)^2 + RR''_{zz}\right| + RR''_{z\bar z};$$

the strong condition corresponds to strict inequality here.

Proposition 4.3. *Let Ω be an open set in $\mathbf{C}^n$ and define*

$$\widetilde\Omega = \{z \in \mathbf{C}^n;\ (z_1, ..., z_{n-1}, \lambda z_n) \in \Omega \text{ for all } \lambda \in \mathbf{C} \text{ with } |\lambda| \leqslant 1\}. \tag{4.1}$$

This is the largest complete Hartogs set contained in Ω. If Ω is lineally convex, then $\widetilde\Omega$ is lineally convex; similarly for weak lineal convexity. If $\partial\Omega$ is of class C^2 except perhaps where $z_n = 0$, then so is the boundary of $\widetilde\Omega$ at all points z with $z_n \neq 0$ and satisfying the condition

$$2M|z_n| < \left|\rho'_{z_n}(z)\right|, \tag{4.2}$$

where M is a bound for the second derivatives $\rho''_{z_n z_n}$ and $\rho''_{z_n \bar z_n}$. If, in addition, Ω satisfies the Behnke–Peschl condition (2.6) at all boundary points with $z_n \neq 0$, then so does $\widetilde\Omega$ at all boundary points with $z_n \neq 0$ satisfying (4.2).

Proof. If Ω is lineally convex, then also $\widetilde\Omega$, as an intersection of lineally convex sets, has this property:

$$\widetilde\Omega = \bigcap_{|\lambda|\leqslant 1} \Omega_\lambda, \quad \text{where} \quad \Omega_\lambda = \{z \in \mathbf{C}^n;\ (z_1, ..., z_{n-1}, \lambda z_n) \in \Omega\}.$$

Assume now that Ω is only weakly lineally convex, and let a point a on the boundary of $\widetilde\Omega$ be given. Then for some λ with $|\lambda| = 1$, a is on the boundary of Ω_λ defined above, and a hyperplane through a which does not intersect Ω_λ does not intersect $\widetilde\Omega$ either. (The argument is valid for all a; if $a_n = 0$ we even have $a \in \partial\Omega_\lambda$ for all λ.)

If ρ defines Ω, then

$$\widetilde{\rho}(z) = \sup_{\theta} \rho(z_1, ..., z_{n-1}, e^{i\theta} z_n) \tag{4.3}$$

defines $\widetilde{\Omega}$ in a neighborhood of its closure. Define

$$\varphi(z_1, ..., z_n, \theta) = \rho(z_1, ..., z_{n-1}, e^{i\theta} z_n), \qquad (z, \theta) \in \mathbf{C}^n \times \mathbf{R}.$$

We can calculate

$$\begin{aligned} \varphi'_\theta &= -2\,\mathrm{Im}(\rho'_{z_n} e^{i\theta} z_n); \\ \varphi''_{\theta\theta} &= -2\,\mathrm{Re}\left(\rho'_{z_n} e^{i\theta} z_n\right) - 2\,\mathrm{Re}\left(\rho''_{z_n z_n} e^{2i\theta} z_n^2\right) + 2\rho''_{z_n \overline{z}_n} |z_n|^2. \end{aligned}$$

The value of θ which defines the supremum in (4.3) solves the equation $\varphi'_\theta = 0$, and the implicit function theorem can be applied if $\varphi''_{\theta\theta} \neq 0$ there. This condition is fulfilled if

$$|\,\mathrm{Re}(\rho'_{z_n} e^{i\theta} z_n)| > 2M |z_n|^2, \tag{4.4}$$

where M is a bound for the second derivatives of ρ as defined in the statement of the proposition. However, when $\varphi'_\theta = 0$, the expression $\rho'_{z_n} e^{i\theta} z_n$ is real, so that (4.4) simplifies to (4.2). The implicit function theorem then says that the boundary of $\widetilde{\Omega}$ is as smooth as that of Ω where the condition is satisfied.

Now assume that Ω satisfies the Behnke–Peschl condition at a boundary point a of $\widetilde{\Omega}$ with $a_n \neq 0$. Then a is on the boundary of some Ω_λ, $|\lambda| = 1$, as already noted above. Consider the functions

$$\varphi_\lambda(s) = \rho_\lambda(a + st), \qquad \widetilde{\varphi}(s) = \widetilde{\rho}(a + st), \qquad s \in \mathbf{R}, t \in T_{\mathbf{C}}(a),$$

where $\rho_\lambda(z) = \rho(z_1, ..., z_{n-1}, \lambda z_n)$, the defining function for Ω_λ obtained by rotating ρ in the last coordinate.

The Behnke–Peschl condition holds for Ω_λ, which means that $(\varphi_\lambda)''(0) \geqslant 0$. Now $\widetilde{\varphi} \geqslant \varphi_\lambda$ and both functions vanish at the origin, which implies $\widetilde{\varphi}''(0) \geqslant (\varphi_\lambda)''(0)$. Thus the condition holds for $\widetilde{\Omega}$. This completes the proof.

In an application of this proposition in the next section we shall let Ω be defined near an arbitrarily given point by an inequality $y_n < f(z', x_n)$ for some real-valued function f of $n - 1$ complex and one real variable. Then $\rho(z) = y_n - f(z', x_n)$ is a defining function for Ω near the given point. (Here $x_n = \mathrm{Re}\, z_n$, $y_n = \mathrm{Im}\, z_n$, and $z' = (z_1, ..., z_{n-1})$.) We see that $\rho'_{z_n} = -\frac{1}{2}(f'_{x_n} + i)$, so that $|\rho'_{z_n}| \geqslant \frac{1}{2}$. Moreover

$$\rho''_{z_n z_n} = \rho''_{z_n \overline{z}_n} = -\tfrac{1}{4} f''_{x_n x_n}.$$

This implies that a sufficient condition for (4.2) to hold is

$$C|z_n| < 1, \tag{4.5}$$

where C is a bound for $f''_{x_n x_n}$.

Remark 4.4. Condition (4.2) has a simple geometric meaning. With the defining function $\rho(z) = y_n - f(z', x_n)$ it says that the intersection of the boundary of Ω with the subspace $z' =$ constant has smaller curvature than the intersection of the boundary of $\widetilde{\Omega}$ with the same subspace where the two boundaries meet. For simplicity we shall use the stronger condition (4.5) instead.

5. Proof of the main result

We shall now prove Theorem 3.4. In view of Theorem 3.1 it is enough to prove that the complex tangent plane $a + T_{\mathbf{C}}(a)$ does not cut Ω near a. We shall assume that $a + T_{\mathbf{C}}(a)$ cuts Ω in a point b and then show that this leads to a contradiction if b is close to a.

First of all we may assume that $n = 2$ by looking at the two-dimensional affine complex subspace which contains a, b and a third point on the normal to $\partial\Omega$ through a. We may also assume that the coordinate system is chosen so that $a = 0$ and the real tangent plane $a + T_{\mathbf{R}}(a)$ has the equation $\operatorname{Im} z_2 = 0$. We recall that both weak lineal convexity and the Behnke–Peschl condition (2.6) are invariant under complex affine mappings. The complex tangent plane at a then has the equation $z_2 = 0$, so that $b_2 = 0$. We shall consider a neighborhood W of a such that three conditions are satisfied. Let

$$W = \{z \in \mathbf{C}^2;\ |z_1| < R_1, |z_2| < R_2\},$$

and let V be its intersection with $\mathbf{C} \times \mathbf{R}$:

$$V = \{(z_1, x_2) \in \mathbf{C} \times \mathbf{R};\ |z_1| < R_1, |x_2| < R_2\}.$$

The three conditions are:

(A) First of all the set Ω shall be defined in W by an inequality $\operatorname{Im} z_2 < f(z_1, \operatorname{Re} z_2)$ for some function f which is of class C^2 in a neighborhood of the closure of V.

(B) Next we shall assume that condition (4.5) is satisfied for all $z \in W$ with some margin:
$$R_2 \sup_V |f''_{x_2 x_2}| < \tfrac{2}{3} < 1.$$
(This is to allow a change of coordinates later.)

(C) Third, R_1 shall be so small that $MR_1 + C(1+M^2)R_1^2 < \frac{1}{4}R_2$, where $M = \frac{1}{2}CR_2$ and C is defined below.

To satisfy these conditions we have to specify the numbers R_1, R_2 and C. We first choose R_1 and R_2 so that (A) and (B) hold, and then define a constant C as follows. Since f is a function of class C^2 defined in a neighborhood of the closure of V and with vanishing derivatives of order up to one at the origin, there exists a constant C such that

$$\begin{aligned}
|f(z_1, x_2)| &\leqslant C(|z_1|^2 + x_2^2),\\
|f'_{x_2}(z_1, x_2)| &\leqslant C(|z_1| + |x_2|), \text{ and}\\
|f''_{x_2 x_2}(z_1, x_2)| &\leqslant C
\end{aligned}$$

for all $(z_1, x_2) \in V$. We finally shrink R_1 if necessary to make (C) hold.

With the choice of coordinate system we have made, the normal at a is the y_2-axis. Let c be a point on that axis with $\operatorname{Im} c_2 < 0$; it is convenient to take $c = -\frac{1}{4}iR_2$. Thus $c = (0, c_2)$ and $|c| = -\operatorname{Im} c_2 = \frac{1}{4}R_2$. The circle in the plane $z_1 = 0$ with center at c and radius $|c|$ passes through a and is tangent to the x_2-axis at that point.

We shall prove that $f(b) \leqslant 0$ (hence that $b \notin \Omega$) for all b with $|b_1| < R_1$. Assume the contrary: $f(b) > 0$. Consider the plane $z_1 = b_1$ and the graph of f restricted to that plane. Draw the normal to the graph of $f(b_1, \cdot\,)$ through the point $z_2 = if(b_1, 0)$ in the z_2-plane. This normal intersects the line $y_2 = \operatorname{Im} c_2$ at a point which we call p_2. Define $p_1 = b_1$, so that $p = (p_1, p_2)$ is a point in $\mathbf{C}^2$. The slope of the normal is determined by the slope of the graph at $z_1 = b_1$, $x_2 = 0$, i.e., by $f'_{x_2}(b_1, 0)$. This derivative can however be controlled: we know that $f'_{x_2}(b_1, 0)$ is not more than $C|b_1|$ in modulus. The distance between p and c is

$$|p - c| = |f'_{x_2}(b_1, 0)|(|c| + f(b_1, 0)) \leqslant C|b_1|(\tfrac{1}{4}R_2 + C|b_1|^2) \leqslant \tfrac{1}{2}CR_2|b_1|,$$

where the last estimate is a consequence of (C). Thus $|p - c| \leqslant M|b_1|$ with $M = \frac{1}{2}CR_2$.

We have constructed a disk D_0 in the plane $z_1 = 0$ with center at c_2 and with $z_2 = 0$ on its boundary, and now let D_1 be the disk in the plane $z_1 = b_1$ with center at p_2 and $if(b_1, 0)$ on its boundary (and therefore containing $z_2 = 0$):

$$\begin{aligned} D_0 &= \{z \in \mathbf{C}^2;\ z_1 = 0, |z_2 - c_2| < |c|\}; \\ D_1 &= \{z \in \mathbf{C}^2;\ z_1 = b_1, |z_2 - p_2| < |if(b_1, 0) - p_2|\}. \end{aligned}$$

Both disks are moreover contained in $\Omega \cap W$. For D_0 this is obvious from the construction; for D_1 this can be seen as follows. The center of D_1 is p_2 and its radius r_1 is $|if(b_1) - p_2|$. The disk is contained in W if $|p_2| + r_1 \leqslant R_2$. This inequality follows from the estimates we already have:

$$|p_2| + r_1 \leqslant 2|p_2| + C|b_1|^2 \leqslant 2|c_2| + 2|p_2 - c_2| + C|b_1|^2 \leqslant \tfrac{1}{2}R_2 + 2MR_1 + CR_1^2 \leqslant R_2,$$

where the last inequality follows from (C). Thus $D_1 \subset W$. That $D_1 \subset \Omega$ now follows from (B); cf. Remark 4.4.

If we construct a Hartogs domain by rotating Ω around an axis which passes through c and p, then this Hartogs domain will have a on its boundary and contain b. This is precisely what we shall do.

We introduce new coordinates (w_1, w_2) so that the w_1-axis, i.e., the plane $w_2 = 0$, passes through c and p. The w_2-axis need not be changed. This means that the new coordinates shall be defined as

$$w_1 = z_1, \qquad w_2 = z_2 - c_2 - (p_2 - c_2)z_1/b_1.$$

Indeed $z = c$ gives $w = 0$ and $z = p$ yields $w = b = (b_1, 0)$. We now define $\widetilde{\Omega}$ in the w-coordinates. The tangent plane with equation $z_2 = 0$ has the equation $w_2 = -c_2 - (p_2 - c_2)w_1/b_1$ and is also the tangent plane to $\partial\widetilde{\Omega}$ at the point $w = (0, -c_2)$. It intersects $\widetilde{\Omega}$ at the point $z = b$, i.e., $w = (b_1, -p_2)$. That this point is an element of $\widetilde{\Omega}$ follows from the construction of D_1.

We shall now apply Theorem 4.2 to $\widetilde{\Omega}$ over the disk $|w_1| < R_1$ in the w_1-plane. To be able to do so we have to check that there is a point of $\widetilde{\Omega}$ over every point w_1 with $|w_1| < R_1$, or equivalently that $(w_1, 0) \in \widetilde{\Omega}$ for all w with $|w_1| < R_1$.

In the new coordinate system, the inequality defining Ω becomes

$$\operatorname{Im} w_2 < -\operatorname{Im} c_2 - \operatorname{Im}(p_2 - c_2)w_1/b_1 + f(w_1, \operatorname{Re} w_2 + \operatorname{Re}(p_2 - c_2)w_1/b_1).$$

Denote the right-hand side by $g(w_1, \operatorname{Re} w_2)$. In particular

$$g(w_1, 0) = -\operatorname{Im} c_2 - \operatorname{Im}(p_2 - c_2)w_1/b_1 + f(w_1, \operatorname{Re}(p_2 - c_2)w_1/b_1).$$

Recalling the estimate $|p_2 - c_2| \leqslant M|b_1|$ above, we get

$$g(w_1, 0) \geqslant \tfrac{1}{4}R_2 - M|w_1| - C(1 + M^2)|w_1|^2 \geqslant \tfrac{1}{4}R_2 - MR_1 - C(1 + M^2)R_1^2 > 0,$$

the last inequality coming from (C). This ensures that every point $(w_1, 0)$ with $|w_1| < R_1$ lies in Ω and therefore also in $\widetilde{\Omega}$.

We know that $\widetilde{\Omega}$ satisfies the Behnke–Peschl condition at all boundary points if the condition in the w-coordinates corresponding to (4.5) is valid. Note that $|w_2 - z_2 + c_2| \leqslant M|z_1|$ independently of the choice of $b \in W$, from which we deduce

$$|w_2| \leqslant |z_2| + \tfrac{1}{4}R_2 + MR_1 \leqslant \tfrac{3}{2}R_2.$$

The second derivative of g with respect to $\operatorname{Re} w_2$ is the same as the second derivative of f with respect to $x_2 = \operatorname{Re} z_2$, so from (B) we can conclude that the condition (4.5) is satisfied also in the w-coordinates for all points $w \in \partial\widetilde{\Omega}$ with $|w_1| < R_1$.

It now follows from Theorem 4.2 that $\widetilde{\Omega}$ is lineally convex, which contradicts the fact that the tangent plane at the point $w = -c$ intersects $\widetilde{\Omega}$ in $w = (b_1, -p_2)$. This completes the proof.

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