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Estimates for solutions to discrete convolution equations

Christer O. Kiselman

Dedicated to Vladimir Maz'ya,
a great mathematician
and a great human being

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Abstract
We study solvability of convolution equations for functions with discrete support in $\mathbf{R}^n$, a special case being functions with support in the integer points. The more general case is of interest for several grids in Euclidean space, like the body-centred and face-centred tesselations of three-space, as well as for the non-periodic grids that appear in the study of quasicrystals. The theorem of existence of fundamental solutions by de Boor, Höllig & Riemenschneider is generalized to general discrete supports, using only elementary methods. We also study the asymptotic growth of sequences and arrays using the Fenchel transformation.

1. Introduction

Many sequences and arrays are defined recursively, like

$$f(x) = a_1 f(x-1) + a_2 f(x-2) + \cdots + a_m f(x-m), \qquad x \in \mathbf{N},\ x \geqslant x_0;$$

$$\begin{aligned} f(x,y) &= a_{1,0} f(x-1,y) + a_{0,1} f(x,y-1) + a_{1,1} f(x-1,y-1) + \cdots \\ &\quad + a_{m,m} f(x-m,y-m), \qquad (x,y) \in \mathbf{N}^2,\ x \geqslant x_0,\ y \geqslant y_0, \end{aligned}$$

typically with some initial conditions. These sequences and arrays can conveniently be described as solutions to convolution equations on $\mathbf{Z}$ and $\mathbf{Z}^2$, respectively.

The purpose of the present paper is to study convolution equations of the general form $\nu * w = \rho$, where w is an unknown function, and where ν and ρ are given functions defined on $\mathbf{R}^n$ and of finite support – sometimes we shall relax the latter condition. We thus go from functions on $\mathbf{Z}^n$ (the most studied discretization) to more general functions. This allows, for instance, discretizations corresponding to other tessellations of $\mathbf{R}^n$, like the body-centred cubic (bcc) grid and the face-centred cubic (fcc) grid in $\mathbf{R}^3$ studied by Strand (2008) and others. These are periodic, but coming to quasicrystals, we must allow for non-periodic functions. In fact, there is a scale of regularity, starting with $\mathbf{Z}^n$ as the most regular set and ending with arbitrary discrete sets. Somewhere between these are the quasicrystals. How can we measure this regularity?

Some of the results go beyond discreteness in that functions which have finite mass locally are allowed. See the algebras $\mathscr{A}_\theta(\mathbf{R}^n)$, $\mathscr{B}_\theta^\varphi(\mathbf{R}^n)$, and $\mathscr{B}_\theta(\mathbf{R}^n)$ defined in Section 3.

We shall study the solutions with the help of the Fenchel transformation. Infimal convolution and the Fenchel transformation can be viewed as tropicalizations of usual convolution and the Fourier (or Laplace) transformation, respectively – tropicalization is in itself a most interesting transformation. In my lecture on 2013 December 16 at the University of Liverpool, I also presented some results on the Fourier transforms of solutions to these equations, but they are left out here due to space limitations.

In particular we shall prove that convolution equations have fundamental solutions, a result proved by de Boor, Höllig & Riemenschneider (1989). Our method of proof is elementary, while theirs relies on a modification to the discrete case of Hörmander's proof (1958) of the division theorem for distributions, which in turn builds on the Tarski–Seidenberg theorem. See also Lojasiewicz (1958, 1959). Our result is more general, since we allow finite supports consisting of arbitrary points in $\mathbf{R}^n$, not necessarily integer points, and also some infinite discrete supports. We get a fundamental solution with support in a strict convex cone and in general with exponential growth there, while the solution of de Boor et al. is of polynomial growth but with support spread out. There is often a trade-off between estimates of the growth and information on the support; in this way our solutions are more like the solutions to hyperbolic equations and more suited to initial-value problems. See Theorems 4.1 and 4.2, and Corollary 4.5.

Notation

The boldface letters $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$ have their usual meaning according to Bourbaki; thus $\mathbf{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers, etc. We shall use

$$\mathbf{R}_! = [-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\}$$

to denote the set of extended real numbers, adding two infinities.

Addition $\mathbf{R}^2 \ni (x, y) \mapsto x + y \in \mathbf{R}$ can be extended in two different ways to operations $(\mathbf{R}_!)^2 \to \mathbf{R}_!$: the *upper sum* $x \dotplus y$ is defined as $+\infty$ if one of the terms is equal to $+\infty$, and the *lower sum* $x \mathbin{\underset{\cdot}{+}} y$ is defined as $-\infty$ if one of the terms is equal to $-\infty$. We use $x \wedge y$ for the minimum of x and y; $x \vee y$ for the maximum. Under these operations $\mathbf{Z}$ and $\mathbf{R}$ are lattices, and $\mathbf{Z}_!$ and $\mathbf{R}_!$ complete lattices.

The *indicator function* of a set A takes the value 0 in A and $+\infty$ in the complement. It will be denoted by $\mathbf{ind}_A$ and is equal to $-\log \chi_A$, where χ_A is the characteristic function of A.

Following Bourbaki (1954: ch. II, § 5, N^o 1, p. 101) the set of all mappings of a set X into a set Y will be denoted by $\mathscr{F}(X,Y)$. If Y is a vector space or an abelian group, the *support* of a function $f \in \mathscr{F}(X,Y)$, denoted by $\operatorname{supp} f$, is the subset of X where it is nonzero. (If we use the discrete topology on X, every set is closed, so the definition agrees with the usual one. The support of a function defined in $\mathbf{R}^n$ can for instance be equal to $\mathbf{Q}^n$.) We shall write $\mathscr{F}_{\text{finite}}(X,Y)$ and $\mathscr{F}_{\text{discr}}(X,Y)$ for the set of mappings with finite and discrete support, respectively, where the latter has a sense if X is a topological space.

We shall use the l^p-norm $\|x\|_p = (\sum_j |x_j|^p)^{1/p}$, $1 \leqslant p < +\infty$, and the l^∞-norm $\|x\|_\infty = \sup_j |x_j|$ for $x \in \mathbf{R}^n$. We shall use these norms also for functions; see Section 3. When any norm can serve, we write just $\|x\|$.

The inner product is written $\xi \cdot x = \xi_1 x_1 + \dots + \xi_n x_n$, $(\xi, x) \in \mathbf{R}^n \times \mathbf{R}^n$.

2. Discrete, dense, and asymptotically dense sets

Following Bourbaki (1961:16) we shall say that a subset A of a topological space is *discrete*, if for each point $a \in A$, a is the only point of A in some neighbourhood of a.

A subset A of a set B in a topological space is said to be *dense in* B if the closure of A contains B.

In $\mathbf{R}^n$ we shall say that a subset A is *asymptotically dense* if for every nonzero $x \in \mathbf{R}^n$ there is a sequence $(a^{(j)})_{j\in\mathbf{N}}$ of points in A such that $\|a^{(j)}\| \to +\infty$ and $a^{(j)}/\|a^{(j)}\| \to x/\|x\|$ as $j \to +\infty$.

3. Convolution

Let G be an abelian group – most of the time we shall take $G = \mathbf{Z}^n$ or $G = \mathbf{R}^n$. We define the *convolution product* $h = f * g$ of two functions $f, g \colon G \to \mathbf{C}$ by

$$h(x) = \sum_{y+z=x} f(y)g(z), \qquad x \in G, \tag{3.1}$$

provided the sum is convergent in a suitable sense. We can define several kinds of algebras satisfying this provision.

The *Kronecker delta* δ_a, defined by $\delta_a(a) = 1$ and $\delta_a(x) = 0$ for $x \neq a$, satisfies $\delta_a * \delta_b = \delta_{a+b}$. Placed at the origin, this function is a neutral element for convolution: $f * \delta_0 = f$ for all functions f.

We shall use the function spaces $l^p(G)$, $1 \leqslant p \leqslant \infty$, defined as the set of all functions $f \colon G \to \mathbf{C}$ such that the l^p-norm is finite, where we define the *l^p-norm* by

$$\|f\|_p = \left(\sum_{x\in G} |f(x)|^p\right)^{1/p} \text{ if } 1 \leqslant p < \infty \text{ and } \|f\|_\infty = \sup_{x\in G} |f(x)|.$$

The l^1-norm of f is also called the *mass* of f.

We note the Minkowski inequality

$$\|f * g\|_p \leqslant \|f\|_1 \|g\|_p, \qquad 1 \leqslant p \leqslant \infty,$$

which shows that $l^p(G)$ is a module over the ring $l^1(G)$ (see (3.4) below). The closure of $\mathscr{F}_{\mathrm{finite}}(G, \mathbf{C})$ in $l^\infty(G)$ is denoted by $c_0(G)$ and is also a module over $l^1(G)$, since

$$f \in l^1(G),\ g \in c_0(G) \text{ implies } f * g \in c_0(G).$$

When $1 < p < \infty$, we have $l^1(G) \subset l^p(G) \subset c_0(G) \subset l^\infty(G)$, with strict inclusions for all infinite groups G.

We also have Hölder's inequality,

$$\|f * g\|_\infty \leqslant \|f\|_p \|g\|_q, \qquad 1 < p < \infty,\ \ q = p/(p-1),$$

valid also for $(p, q) = (1, \infty)$ and $(p, q) = (\infty, 1)$.

We write $l_0^p(\mathbf{R}^n)$ for the set of functions in $l^p(\mathbf{R}^n)$ which have bounded support, and $l_{\mathrm{loc}}^p(\mathbf{R}^n)$ for the set of functions whose restriction to any bounded subset A of $\mathbf{R}^n$ is in $l^p(A)$.

For $1 \leqslant p \leqslant \infty$ we have the inclusions

$$\mathscr{F}_{\mathrm{finite}}(\mathbf{R}^n, \mathbf{C}) = l_0^p(\mathbf{R}^n) \cap \mathscr{F}_{\mathrm{discr}}(\mathbf{R}^n, \mathbf{C}) \subset \mathscr{F}_{\mathrm{discr}}(\mathbf{R}^n, \mathbf{C}) \subset l_{\mathrm{loc}}^p(\mathbf{R}^n), \tag{3.2}$$

$$\mathscr{F}_{\mathrm{finite}}(\mathbf{R}^n, \mathbf{C}) \subset l_0^p(\mathbf{R}^n) \subset l^p(\mathbf{R}^n) \subset l_{\mathrm{loc}}^p(\mathbf{R}^n). \tag{3.3}$$

Case α': *Functions of finite mass, $l^1(G)$.* In view of the inequalities

$$\|f + g\|_1 \leqslant \|f\|_1 + \|g\|_1 \text{ and } \|f * g\|_1 \leqslant \|f\|_1 \|g\|_1, \tag{3.4}$$

$l^1(G)$ is an algebra under addition and convolution. An important subalgebra is the space $\mathscr{F}_{\mathrm{finite}}(G, \mathbf{C})$ of all functions which are nonzero only at finitely many points. Another subalgebra in case $G = \mathbf{R}^n$ is the space $l_0^1(\mathbf{R}^n)$.

We always have

$$\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g, \qquad f, g \in l^1(G), \tag{3.5}$$

in general with a strict inclusion. However, if f, g are nonnegative, or more generally if the set $\{f(y)g(z);\ y, z \in G\}$ of all products of values of f and g is contained in a strict convex cone in the complex plane, then we have equality:

$$\operatorname{supp}(f * g) = \operatorname{supp} f + \operatorname{supp} g, \qquad f, g \in l^1(G). \tag{3.6}$$

Every function $f \in l_{\mathrm{loc}}^1(\mathbf{R}^n)$ defines a Radon measure, simply by interpreting the Kronecker deltas in the representation $f = \sum_{x \in \mathbf{R}^n} f(x)\delta_x$ as Dirac measures. This simple fact opens up ways to apply distribution theory.

Let us denote by $\mathbf{cvxh}(A)$ the *convex hull* of a subset A of a vector space; it is the smallest convex set containing A. When $G = \mathbf{R}^n$, we have

$$\mathbf{cvxh}\left(\overline{\operatorname{supp}(f * g)}\right) = \mathbf{cvxh}\left(\overline{\operatorname{supp} f}\right) + \mathbf{cvxh}\left(\overline{\operatorname{supp} g}\right), \qquad f, g \in l_0^1(\mathbf{R}^n), \tag{3.7}$$

where we this time take the closure for the usual topology of $\mathbf{R}^n$. This is Titchmarsh's convolution theorem, and it seems that it is as difficult to prove in this setting as it is in the classical case when f and g are continuous functions or distributions with compact support. However, if $f, g \in \mathscr{F}_{\text{finite}}(\mathbf{R}^n, \mathbf{C})$, then (3.7) is easy to prove by induction over the dimension.

Weiss (1968) studied the validity of Titchmarsh's convolution theorem for other groups, and Domar (1989) extended the theorem from functions in $l^1_0(\mathbf{R}^n)$ to functions in $l^2_0(\mathbf{R}^n)$.

Equation (3.7) is a precise quantitative form of the fact that the algebra $l^1_0(\mathbf{R}^n)$ does not have zero divisors. For some groups, in fact for all groups G that possess a finite subgroup with at least two elements, the algebra $\mathscr{F}_{\text{finite}}(G, \mathbf{C})$ does have zero divisors, and (3.7) fails conspicuously.

The algebra $\mathscr{F}_{\text{finite}}(\mathbf{Z}^n, \mathbf{C})$ is isomorphic to the algebra of Laurent polynomials:

$$P_f(z) = \sum_{x\in\mathbf{Z}^n} f(x)z^x = \sum_{x\in\mathbf{Z}^n} f(x)z_1^{x_1}\cdots z_n^{x_n}, \qquad z \in \mathbf{C}^n,$$

and also to the algebra of trigonometric polynomials, putting $z_j = e^{i\zeta_j}$:

$$\hat{f}(\zeta) = \sum_{x\in\mathbf{Z}^n} f(x)e^{i\zeta\cdot x}, \qquad \zeta \in \mathbf{C}^n.$$

Case β'. *Algebras containing $l^1_0(\mathbf{R}^n)$ and contained in $l^1_{\text{loc}}(\mathbf{R}^n)$.*

Definition 3.1. Given any nonzero vector $\theta \in \mathbf{R}^n$, we define an algebra $\mathscr{A}_\theta(\mathbf{R}^n)$ as the set of all functions $f\colon \mathbf{R}^n \to \mathbf{C}$ such that $f(x) = 0$ when $\theta\cdot x \leqslant 0$ and which have finite mass in each slab $S^r_\theta = \{x \in \mathbf{R}^n;\ 0 < \theta\cdot x \leqslant r\}$, $r > 0$. □

That $\mathscr{A}_\theta(\mathbf{R}^n)$ is an algebra follows from the estimate

$$\begin{aligned}\sum_{x\in S^r_\theta} |(f*g)(x)| &\leqslant \sum_{x\in S^r_\theta} (|f|*|g|)(x) = \sum_{y+z\in S^r_\theta} |f(y)||g(z)| \\ &= \sum_{y\in S^r_\theta} |f(y)| \sum_{z\in S^{r-\theta\cdot y}_\theta} |g(z)| \leqslant \sum_{y\in S^r_\theta} |f(y)| \sum_{z\in S^r_\theta} |g(z)|.\end{aligned}$$

For any subset A of G such that $A + A \subset A$, the set of functions in $l^1(G)$ with support in A is a subalgebra. It is, however, more interesting to study functions in $l^1_{\text{loc}}(\mathbf{R}^n)$ and to consider sets $A \subset \mathbf{R}^n$ such that $A + A \subset A$ and $A \cap (x - A)$ is bounded for every $x \in \mathbf{R}^n$.

To make this tractable and to get control over the support of the functions, we introduce the following definition.

Definition 3.2. We define Φ as the set of all functions $\varphi\colon\ [0, +\infty[\ \to [0, +\infty[$ such that $\varphi(0) = 0$ and $]0, +\infty[\ \ni t \mapsto \varphi(t)/t$ is increasing. We define

$$V^\varphi_\theta = \{x \in \mathbf{R}^n;\ \theta\cdot x \geqslant 0,\ \|x\| \leqslant \varphi(\theta\cdot x)\}, \qquad \theta \in \mathbf{R}^n \smallsetminus \{0\},\ \varphi \in \Phi, \tag{3.8}$$

and, when $\varphi(t) = t$,

$$V^{\text{id}}_\theta = V_\theta = \{x \in \mathbf{R}^n;\ \|x\| \leqslant \theta\cdot x\}, \qquad \theta \in \mathbf{R}^n \smallsetminus \{0\}, \tag{3.9}$$

a strict convex cone. □

The definition of Φ implies that the smallest cone of the form

$$C(t) = \{x \in \mathbf{R}^n;\ \|x\| \leqslant R(t)(\theta \cdot x)\}, \qquad t \geqslant 0,$$

which contains all points $x \in V_\theta^\varphi$ with $\theta \cdot x = t$ is an increasing function of t.

Every function φ in Φ is superadditive: $\varphi(s+t) \geqslant \varphi(s) + \varphi(t)$, $s, t \geqslant 0$. If $\varphi, \psi \in \Phi$, then also $\varphi \vee \psi \in \Phi$. If φ and ψ are superadditive, then $\varphi \vee \psi$ is not necessarily superadditive, even if one of them belongs to Φ. This is the reason why we have preferred to define Φ using a slightly stronger property; it is convenient to have a class of superadditive functions which is closed under the formation of maxima.

Example 3.3. Functions in Φ are for instance $\varphi(t) = t^\alpha$ for $t \geqslant 0$ with $\alpha \geqslant 1$; and $\varphi(t) = 0$ for $0 \leqslant t \leqslant c$, $\varphi(t) = \alpha t + \beta$ for $c < t$, where $c \geqslant 0$, $\alpha \geqslant 0$, $\beta \leqslant 0$, and $\alpha c + \beta \geqslant 0$.

An example in the other direction is the floor function $t \mapsto \lfloor t \rfloor$, which is superadditive but does not belong to Φ. Here $\lfloor t \rfloor$ is the *integer part* of a real number t; i.e., the integer satisfying $t - 1 < \lfloor t \rfloor \leqslant t$. □

We note that $V_\theta^\varphi + V_\theta^\varphi = V_\theta^\varphi$ and, more generally, that $V_\theta^\varphi \cup V_\theta^\psi \subset V_\theta^\varphi + V_\theta^\psi \subset V_\theta^{\varphi \vee \psi}$ for $\varphi, \psi \in \Phi$.

If f has its support in V_θ^φ, $\varphi \in \Phi$, the product $f(y)g(x-y)$ is nonzero only when both y and $x - y$ belong to V_θ^φ, which implies that $y \in V_\theta^\varphi \cap (x - V_\theta^\varphi)$, a bounded set for any given x, thus, if $f, g \in l^1_{\mathrm{loc}}(\mathbf{R}^n)$, in particular if $f, g \in \mathscr{A}_\theta(\mathbf{R}^n)$, then the convolution $(f * g)(x)$ is defined for each x as the convolution of two functions in $l^1(\mathbf{R}^n)$.

The convolution product of two functions with support in V_θ^φ has its support in V_θ^φ.

Definition 3.4. For $\varphi \in \Phi$ and $\theta \in \mathbf{R}^n \smallsetminus \{0\}$ we define a convolution algebra $\mathscr{B}_\theta^\varphi(\mathbf{R}^n)$ as the set of functions $f \in l^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that $\operatorname{supp} f$ is contained in V_θ^φ. We define $\mathscr{B}_\theta(\mathbf{R}^n)$ as the union of all the $\mathscr{B}_\theta^\varphi(\mathbf{R}^n)$ when φ varies. □

Clearly $\mathscr{B}_\theta(\mathbf{R}^n)$ is a subalgebra of $\mathscr{A}_\theta(\mathbf{R}^n)$. In the former, we have some control of the size of the supports; in the latter not.

Case γ'. *Translates.* For $G = \mathbf{R}^n$, the functions f such that $f * \delta_b$ is in $\mathscr{A}_\theta(\mathbf{R}^n)$ for some b depending on f also form an algebra, as is the set of functions with support in a translate of some V_θ^φ. We can of course also take the union of V_θ^φ over all φ. When $n = 1$, the subalgebra of functions with discrete support is even a field.

However, sometimes we need to define a convolution product in other situations, thus not in an algebra.

Case δ'. We can define a convolution product $f_1 * \cdots * f_k$ when all factors except one have finite support in G, or, in the case $G = \mathbf{R}^n$, all factors are in $l^1_{\mathrm{loc}}(\mathbf{R}^n)$ and all except one are in $l^1_0(\mathbf{R}^n)$.

Case ε'. For $G = \mathbf{R}^n$, we can define a convolution product $f_1 * \cdots * f_k$ when all factors are in $l^1_{\mathrm{loc}}(\mathbf{R}^n)$ and all except one have their support contained in translates of a set V_θ^φ and the remaining one has its support contained in a half space

$$\{x \in \mathbf{R}^n;\ \theta \cdot x \geqslant s\}$$

for some real number s and with the same vector θ. Also here we can take the union over all $\varphi \in \Phi$.

In the five cases α'–ε', the associative law holds. However, associativity is a subtle property and can easily be lost:

Example 3.5. Take $f(x) = 1$ for all $x \in \mathbf{Z}$; $g = \delta_{-1} - \delta_0$ (a difference operator); and $h(x) = 1$ for all $x \in \mathbf{N}$, $h(x) = 0$ for $x \leqslant -1$.

Then $f * g = 0$ (Case δ') and $(f * g) * h = 0$, while $g * h = \delta_{-1}$ (Case δ') and $f * (g * h) = 1 \neq 0$.

Note that neither $f * h$ nor $f * g * h$ here can be defined in accordance with any of the cases α'–ε'.

This example is, via the Fourier transformation, the same as Laurent Schwartz's example showing that distributions cannot be multiplied while keeping the associative law. □

We shall study in particular convolution equations of the form $\nu * w = \rho$, where ν and ρ have finite support (Case δ'), and also when ν has its support in some V_θ^φ.

In the theory of distributions, convolutions can be defined even more generally, but this is outside the scope of the present paper.

4. Solving convolution equations

Theorem 4.1. *Let $\nu, \rho\colon \mathbf{R}^n \to \mathbf{C}$ be functions with finite support, $\nu \neq 0$. Then there is a function $w\colon \mathbf{R}^n \to \mathbf{C}$ with discrete support which solves the equation $\nu * w = \rho$.*

Since $\rho = \sum_a \rho(a)\delta_a$, it is enough to solve $\nu * w_a = \delta_a$ for $a \in \operatorname{supp}\rho$ and then form the finite linear combination $w = \sum_a \rho(a) w_a$.

We can generalize the last theorem as follows.

Theorem 4.2. *Let $\rho\colon \mathbf{R}^n \to \mathbf{C}$ be a function with finite support and ν a function such that $(\nu - \nu(b)\delta_b) * \delta_{-b}$ belongs to $\mathscr{A}_\theta(\mathbf{R}^n)$ (see Definition 3.1) for some nonzero vector θ and some point $b \in \operatorname{supp}\nu$. Then there is a function $w \in l^1_{\mathrm{loc}}(\mathbf{R}^n)$ which solves the equation $\nu * w = \rho$. If $w(x) = 0$ when $\theta \cdot x$ is negative and sufficiently large, then* $\operatorname{supp} w \subset \{x \in \mathbf{R}^n;\ \theta \cdot b \leqslant \theta \cdot x\}$.

In particular there is a fundamental solution, i.e., a function w such that $\nu * w = \delta_0$ for certain choices of ν as indicated.

We first find a normal form for these equations.

Lemma 4.3. *Given $\nu, \rho\colon \mathbf{R}^n \to \mathbf{C}$ and any point $b \in \operatorname{supp}\nu$, define*

$$u = (\nu(b)w) * \delta_{b-a} \text{ and } \mu = -\sum_{x \neq b} \nu(x)\nu(b)^{-1}\delta_{x-b}.$$

Then w solves the equation $\nu * w = \delta_a$ if and only if u solves the equation $(\delta_0 - \mu) * u = \delta_0$.

Proof. The equation $\nu * w = \delta_a$ can be written

$$\delta_b * (\nu(b)w) + \sum_{x \neq b} (\nu(x)\nu(b)^{-1}\delta_x) * (\nu(b)w) = \delta_a,$$

equivalently

$$\delta_0 * (\nu(b)w * \delta_{b-a}) + \sum_{x \neq b} (\nu(x)\nu(b)^{-1}\delta_{x-b}) * (\nu(b)w * \delta_{b-a}) = \delta_0.$$

We can now introduce u and μ as indicated. □

In general this lemma is not so useful, viz. when b is in the convex hull of $\operatorname{supp}\mu$. However, if ν has finite support and if b is a vertex of $\mathbf{cvxh}(\operatorname{supp}\nu)$, then 0 does not belong to the convex hull of $\operatorname{supp}\mu$, a very useful fact as we shall see. In this situation, we note that the support of μ is contained in a strict convex cone:

Lemma 4.4. *Given any finite set A in $\mathbf{R}^n$ such that the origin does not belong to its convex hull, there exists a vector $\theta \in \mathbf{R}^n \smallsetminus \{0\}$ and a strict convex cone K such that*

$$A \subset K \cap \{x \in \mathbf{R}^n;\ \theta \cdot x \geqslant 1\} \subset V^{\varphi}_{s\theta} \smallsetminus \{0\},$$

where $s = \sup_{a \in A} \|a\|(\theta \cdot a)^{-1}$, provided $A \neq \varnothing$, and where $\varphi(t) = 0$ for $0 \leqslant t < 1$ and $\varphi(t) = t$ for $t \geqslant 1$; $\varphi \in \Phi$.

Proof. In view of the Hahn–Banach theorem there exists a vector θ such that every $x \in A$ satisfies $\theta \cdot x \geqslant 1$. We can then define the smallest cone of the form (3.9) as $V_{s\theta}$, where s is as defined in the statement of the lemma. (This strict convex cone depends on the choice of norm and need not be the smallest convex cone containing A.) □

Proof of Theorems 4.1 and 4.2. In view of Lemma 4.3 it suffices to study the equation $(\delta_0 - \mu) * u = \delta_0$, where $\mu \in \mathscr{A}_\theta$. Define

$$v = \delta_0 + \mu + \mu * \mu + \mu * \mu * \mu + \cdots = \sum_{k=0}^{\infty} \mu^{*k}. \tag{4.1}$$

There are two situations where convergence of (4.1) is easily established. The first is when there are no points of $\operatorname{supp}\mu$ with $\theta \cdot x < r$ for some positive r. Then

$$\operatorname{supp}\mu^{*k} \subset \{x \in \mathbf{R}^n;\ \theta \cdot x \geqslant kr\}, \qquad k \in \mathbf{N},$$

which implies that, in each bounded set, the sum $\sum \mu^{*k}$ is finite.

The second situation is when $\|\mu\|_1$ is less than 1. Then $\|\mu^{*k}\|_1 \leqslant \|\mu\|_1^k$ and the series (4.1) converges uniformly in the whole space.

We shall now combine these two simple situations.

In the general case we first determine a positive number r such that the mass of μ in the slab S^r_θ (see Definition 3.1) is at most $\frac{1}{2}$. Then, given any number $s \in \mathbf{N}$, as large as we wish, we denote by M_s the l^1-norm of μ restricted to the slab defined by $r < \theta \cdot x \leqslant (s+1)r$. Let us write $\mu = \mu_0 + \mu_1$, where μ_0 is the restriction of μ to the slab S_θ^r, and μ_1 is the restriction of μ to the half space determined by $r < \theta \cdot x$.

We then have, in the half space defined by $\theta \cdot x \leqslant (s+1)r$,

$$\mu^{*k} = (\mu_0 + \mu_1)^{*k} = \sum_{j=0}^{k} \binom{k}{j} \mu_0^{*(k-j)} \mu_1^{*j} = \sum_{j=0}^{s} \binom{k}{j} \mu_0^{*(k-j)} \mu_1^{*j},$$

since, for $j \geqslant s+1$, the support of the function μ_1^{*j} does not meet the half space we consider. The mass of μ^{*k} in $S_\theta^{(s+1)r}$ can therefore be estimated, when $k \geqslant 2s$, by

$$\sum_{j=0}^{s} \binom{k}{j} 2^{-k+j} M_s^j \leqslant \binom{k}{s} 2^{-k} \sum_{j=0}^{s} (2M_s)^j \leqslant k^s 2^{-k} C_s,$$

where C_s is a constant which is independent of k. We see that this norm tends to zero as $k \to +\infty$, and even so rapidly that (4.1) converges uniformly in the half plane $\theta \cdot x \leqslant (s+1)r$.

As to uniqueness, if $\mu \in \mathscr{A}_\theta(\mathbf{R}^n)$ and u solves $(\delta_0 - \mu) * u = \delta_0$, we have $(\delta_0 - \mu) * v = \delta_0 = (\delta_0 - \mu) * u$. Here $v(x)$ vanishes when $\theta \cdot x < 0$. If also $u(x)$ vanishes when $\theta \cdot x$ is negative and sufficiently large, then it follows that $v = u$.

If $\mu \in \mathscr{B}_\theta^\varphi(\mathbf{R}^n)$, the support of μ^{*k} is included in $V_\theta^\varphi + V_\theta^\varphi + \cdots + V_\theta^\varphi$ (j terms), which is equal to V_θ^φ, since φ is superadditive. Therefore also the support of v is contained in V_θ^φ. □

Corollary 4.5. *With ν, b, θ, and φ as in Theorem 4.2, let ρ now be any function with discrete (possibly infinite) support. Then there is a solution w to the equation $\nu * w = \rho$ which vanishes for $\theta \cdot x$ negative and sufficiently large in each of the following cases.*
1. If $\operatorname{supp} \rho \subset c + V_\theta^\varphi$ for some c, then w has its support in $c - b + V_\theta^\varphi$.
2. If $\rho * \delta_{-c} \in \mathscr{B}_\theta(\mathbf{R}^n)$ for some c, then $w * \delta_{b-c} \in \mathscr{B}_\theta(\mathbf{R}^n)$.
3. If the support of $\rho \in c + \mathscr{A}_\theta(\mathbf{R}^n)$ is contained in some half space $\{x \in \mathbf{R}^n;\ \theta \cdot x \geqslant s\}$, then w has its support in the half space $\{x \in \mathbf{R}^n;\ \theta \cdot x \geqslant s - \theta \cdot b\}$.

Proof. Let w_0 be the solution to $\nu * w_0 = \delta_0$ which vanishes when $\theta \cdot x$ is negative and sufficiently large. Then $w = \rho * w_0$ is well defined and is the solution with the desired properties in each of the three cases mentioned (Cases γ' and ε'). For instance, if in the second case $\operatorname{supp} \rho \in c + V_\theta^\psi$ with $\psi \in \Phi$, then $\operatorname{supp} w \in c - b + V_\theta^{\varphi \vee \psi}$, and, as noted, $\varphi \vee \psi \in \Phi$. □

Formula (4.1) lends itself to estimates of the solution. However, it seems to be difficult to get estimates as good as those that can be obtained with the help of the Fourier transformation.

5. Inverses of convolution operators

Let now $\mu \in \mathscr{B}_\theta^\varphi(\mathbf{R}^n)$ (see Definition 3.4) for some nonzero vector θ and some function $\varphi \in \Phi$. Let u be the solution to $(\delta_0 - \mu) * u = \delta_0$ which vanishes when $\theta \cdot x$

is negative and sufficiently large. We know then that u has its support contained in V_θ^φ.

With this vector θ we define $\mathscr{C}_\theta(\mathbf{R}^n)$ as the set of all functions $v \in l^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that $v(x)$ is nonzero only if $\theta \cdot x \geqslant -C_v$ for some constant C_v. Then, defining

$$P(v) = u * v,\ v \in \mathscr{C}_\theta(\mathbf{R}^n),\ \text{and}\ \ Q(w) = (\delta_0 - \mu) * w,\ w \in \mathscr{C}_\theta(\mathbf{R}^n),$$

(Case ε'), we obtain

$$Q(P(v)) = (\delta_0 - \mu) * (u * v) = ((\delta_0 - \mu) * u) * v = \delta_0 * v = v, \qquad v \in \mathscr{C}_\theta(\mathbf{R}^n),$$

and

$$P(Q(w)) = u * ((\delta_0 - \mu) * w) = (u * (\delta_0 - \mu)) * w = \delta_0 * w = w, \qquad w \in \mathscr{C}_\theta(\mathbf{R}^n).$$

The associative law holds under the conditions mentioned.

6. Infimal convolution

Tropicalization means, roughly speaking, that we replace an integral or a sum by a supremum.

A typical example is the tropicalization of the l^p-norm:

$$\|x\|_p = \Bigl(\sum |x_j|^p\Bigr)^{1/p} \quad \text{is replaced by} \quad (\sup |x_j|^p)^{1/p} = \sup |x_j| = \|x\|_\infty.$$

In this case we have convergence:

$$\|x\|_p = \Bigl(\sum |x_j|^p\Bigr)^{1/p} \to \sup_j |x_j| = \|x\|_\infty \quad \text{as} \quad p \to +\infty,\ x \in \mathbf{R}^n.$$

To understand how infimal convolution can be viewed as a tropicalization of ordinary convolution, let us study the convolution product of two functions of the form e^{-f}:

$$e^{-h_1(x)} = \sum_{y \in \mathbf{R}^n} e^{-f(x-y)} e^{-g(y)}, \qquad x \in \mathbf{R}^n,$$

assuming that f, g are equal to $+\infty$ outside some discrete set. If for instance f, g have their support in $\mathbf{Z}^n$ and $f(x), g(x) \geqslant \varepsilon\|x\| - C$, we have good convergence. The tropicalization of this convolution product is

$$e^{-h_\infty(x)} = \sup_{y \in \mathbf{R}^n} e^{-f(x-y)} e^{-g(y)}, \qquad x \in \mathbf{R}^n,$$

which can be written

$$h_\infty(x) = \inf_{y \in \mathbf{R}^n} (f(x-y) + g(y)), \qquad x \in \mathbf{R}^n.$$

If we define h_p by

$$e^{-p h_p(x)} = \sum_{y \in \mathbf{R}^n} e^{-p f(x-y)} e^{-p g(y)}, \qquad x \in \mathbf{R}^n,\ 0 < p < +\infty,$$

then h_p converges to h_∞ as $p \to +\infty$. So, also in this case we have nice convergence as p tends to infinity.

The function h_∞ is the *infimal convolution* of f and g, denoted by $f \sqcap g$. Here of course we need not assume that f and g are equal to $+\infty$ outside a discrete set. More generally, we define it when f and g take values in $\mathbf{R}_!$ using upper addition:

$$(f \sqcap g)(x) = \inf_{y \in \mathbf{R}^n} (f(x-y) \dotplus g(y)), \qquad x \in \mathbf{R}^n,\ f, g \in \mathscr{F}(\mathbf{R}^n, \mathbf{R}_!).$$

The function $\mathbf{ind}_{\{0\}}$ is a neutral element for infimal convolution: $f \sqcap \mathbf{ind}_{\{0\}} = f$ for all f.

Of course we have also the *supremal convolution* defined as

$$(f \sqcup g)(x) = \sup_{y \in \mathbf{R}^n} (f(x-y) \underset{\cdot}{+} g(y)), \qquad x \in \mathbf{R}^n,\ f, g \in \mathscr{F}(\mathbf{R}^n, \mathbf{R}_!).$$

The superadditivity of a function $\varphi \in \Phi$ can for example be described by the inequality $\varphi \sqcup \varphi \leqslant \varphi$, understanding that $\varphi(t) = -\infty$ for $t < 0$.

Remark 6.1. Infimal convolution generalizes Minkowski addition: $\mathbf{ind}_A \sqcap \mathbf{ind}_B = \mathbf{ind}_{A+B}$ if A and B are arbitrary subsets of $\mathbf{R}^n$.

In the other direction, Minkowski addition in a higher dimension can be used to define infimal convolution: $\operatorname{epi}_{\mathrm{s}}^{\mathrm{F}}(f \sqcap g) = \operatorname{epi}_{\mathrm{s}}^{\mathrm{F}}(f) + \operatorname{epi}_{\mathrm{s}}^{\mathrm{F}}(g)$, where $\operatorname{epi}_{\mathrm{s}}^{\mathrm{F}}(f)$ is the *strict finite epigraph* of $f \colon \mathbf{R}^n \to \mathbf{R}_!$, defined as

$$\operatorname{epi}_{\mathrm{s}}^{\mathrm{F}}(f) = \{(x, t) \in \mathbf{R}^n \times \mathbf{R};\ f(x) < t\}. \qquad \square$$

7. The Fenchel transformation

The *Fenchel transform* $\tilde{f}$ of a function $f \colon \mathbf{R}^n \to \mathbf{R}_!$ is defined as

$$\tilde{f}(\xi) = \sup_{x \in \mathbf{R}^n} (\xi \cdot x - f(x)), \qquad \xi \in \mathbf{R}^n.$$

Clearly $\xi \cdot x - f(x) \leqslant \tilde{f}(\xi)$, which can be written as

$$\xi \cdot x \leqslant f(x) \dotplus \tilde{f}(\xi), \qquad (\xi, x) \in \mathbf{R}^n \times \mathbf{R}^n,$$

called *Fenchel's inequality.* It follows that the second transform $\tilde{\tilde{f}}$ satisfies $\tilde{\tilde{f}} \leqslant f$. We have equality here if and only if f is convex; lower semicontinuous; and takes the value $-\infty$ only if it is $-\infty$ everywhere.

The Fenchel transformation $f \mapsto \tilde{f}$, named for Werner Fenchel (1905–1988), is a tropical counterpart of the Fourier transformation. This is perhaps even more obvious if we look at the Laplace transform of a function g: $(\mathscr{L}g)(\xi) = \int_0^\infty g(x) e^{-\xi x} dx$, $\xi \in \mathbf{R}$. If we replace the integral by a supremum and take the logarithm, we get

$$\log(\mathscr{L}_{\mathrm{trop}}\, g)(\xi) = \sup_x (\log g(x) - \xi x) = \tilde{f}(-\xi), \qquad f(x) = -\log g(x).$$

We have

$$(f \sqcap g)\tilde{} = \tilde{f} \underset{\cdot}{+} \tilde{g} \leqslant \tilde{f} \dotplus \tilde{g}.$$

If φ and ψ are convex, then $\varphi \dotplus \psi$ is convex, but not always $\varphi \underset{\cdot}{+} \psi$. However, when $\varphi = \tilde{f}$ and $\psi = \tilde{g}$, this is true: $\tilde{f} \underset{\cdot}{+} \tilde{g}$ is always convex, and is often equal to $\tilde{f} \dotplus \tilde{g}$. In fact equality holds except for a few special cases.

7.1. Transforms of indicator functions: supporting functions

If f is an indicator function, then $\tilde{f}$ is positively homogeneous of degree 1 as the supremum of a family of linear functions:

$$\tilde{f}(\xi) = \sup_{\substack{x \in \mathbf{R}^n \\ f(x)=0}} \xi \cdot x, \qquad \xi \in \mathbf{R}^n.$$

Thus the Fenchel transform of an indicator function $\mathbf{ind}_A$ is positively homogeneous of degree 1, and actually equal to the *supporting function* H_A of A, which is defined as

$$H_A(\xi) = (\mathbf{ind}_A)\tilde{}(\xi) = \sup_{x \in A} \xi \cdot x, \qquad \xi \in \mathbf{R}^n.$$

7.2. Transforms of positively homogeneous functions

Conversely, if φ is positively homogeneous of degree one, then $\tilde{\varphi}$ can take only the values 0, $+\infty$, $-\infty$. Indeed, if $\varphi(tx) = t\varphi(x)$ for all $t > 0$, then $t\tilde{\varphi} = \tilde{\varphi}$ for all $t > 0$, and this is only true for the three values 0, $+\infty$, $-\infty$. If φ is not identically $+\infty$, $\tilde{\varphi}$ is an indicator function: $\tilde{\varphi} = \mathbf{ind}_M$ for some set M.

If A is a set such that $H_A = (\mathbf{ind}_A)\tilde{} = \varphi$, then $\tilde{\varphi}$ is the second Fenchel transform of $\mathbf{ind}_A$, equal to the indicator function of the closure $\overline{\mathbf{cvxh}(A)}$ of the convex hull $\mathbf{cvxh}(A)$ of A.

Now if φ is positively homogeneous of degree one and not identically $+\infty$ (so that $\tilde{\varphi}$ does not take the value $-\infty$), we define M as the set such that $\tilde{\varphi} = \mathbf{ind}_M$. Fenchel's inequality says that

$$-\varphi(x) \leqslant -\eta \cdot x + \tilde{\varphi}(\eta) = -\eta \cdot x, \qquad x \in \mathbf{R}^n,\ \eta \in M. \tag{7.1}$$

This inequality can be used to determine the domain of holomorphy of the Fourier transform of a function f with support in $\mathbf{N}^n$ and satisfying $|f(x)| \leqslant Ce^{-\varphi(x)}$, $x \in \mathbf{N}^n$.

7.3. Transforms of indicator functions that are positively homogeneous

Let $f = \mathbf{ind}_C$ be an indicator function which is also positively homogeneous. Then C is a cone, and the Fenchel transform of f is also both positively homogeneous and an indicator function, say $\tilde{f} = \mathbf{ind}_\Gamma$, where Γ is a cone, necessarily closed and convex since $\tilde{f}$ is lower semicontinuous and convex.

The dual of a cone C is a closed convex cone: $C^{\mathrm{dual}} = -\Gamma$, where $\mathbf{ind}_\Gamma = (\mathbf{ind}_C)\tilde{}$.

8. Measuring the growth: The radial indicators

Definition 8.1. Given any subset A of $\mathbf{R}^n$ we define A_∞ as the union of $\{0\}$ and the set of all $x \in \mathbf{R}^n \smallsetminus \{0\}$ such that there exists a sequence $(a^{(j)})_{j \in \mathbf{N}}$ of points in A with $\|a^{(j)}\|$ tending to $+\infty$ and $a^{(j)}/\|a^{(j)}\| \to x/\|x\|$.

If $A = \mathbf{Z}^n$, then $A_\infty = \mathbf{R}^n$. The same holds for all asymptotically dense sets A as defined in Section 2.

Definition 8.2. Given a function $f\colon A \to \mathbf{C}$ we define its *upper radial indicator* as

$$p_f(x) = \limsup_{a \in A} \frac{\|x\|}{\|a\|} \log |f(a)|, \qquad x \in A_\infty \smallsetminus \{0\},$$

where the limit superior is taken over all $a \in A$ such that $\|a\| \to +\infty$ and $a/\|a\| \to x/\|x\|$. Similarly, we define its *lower radial indicator* as

$$q_f(x) = \liminf_{a \in A} \frac{\|x\|}{\|a\|} \log |f(a)|, \qquad x \in A_\infty \smallsetminus \{0\}.$$

Finally, we define $p_f(0) = q_f(0) = 0$.

Theorem 8.3. *Let A be any subset of $\mathbf{R}^n$ and $f\colon A \to \mathbf{C}$ any function defined on A. Fix $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbf{R}^n$. Then the following four properties are equivalent.*
(A). *For every positive ε there exists a constant C_ε such that*

$$|f(a)| \leqslant C_\varepsilon e^{\sigma \cdot a + \varepsilon \|a\|}, \qquad a \in A.$$

(A′). *The upper radial indicator of f satisfies*

$$p_f(x) \leqslant \sigma \cdot x \qquad x \in A_\infty.$$

(A″). *The Fenchel transform of $-p_f$ satisfies $(-p_f)^{\tilde{}}(-\sigma) \leqslant 0$. (Here we take $p_f = -\infty$ in $\mathbf{R}^n \smallsetminus A_\infty$.)*
(A‴). *$-\sigma \in M_f$, where M_f is the set such that $(-p_f)^{\tilde{}} = \mathbf{ind}_{M_f}$.*

Proof. That (A′), (A″) and (A‴) are all equivalent is obvious from the definitions.

Now suppose that (A) holds. If $x \in A_\infty$ and a converges as described in the definition of A_∞, we get

$$\frac{\|x\|}{\|a\|} \log |f(a)| \leqslant \frac{\|x\|}{\|a\|} \log C_\varepsilon + \frac{\|x\|}{\|a\|} \sigma \cdot a + \varepsilon \|x\| \to \sigma \cdot x + \varepsilon \|x\|.$$

Hence $p_f(x) \leqslant \sigma \cdot x + \varepsilon \|x\|$ for all $x \in A_\infty$. Letting $\varepsilon \to 0$ we get (A′).

Conversely, let us assume that (A) does not hold. Then there exists a positive number ε such that for every $k \in \mathbf{N}$ there is a point $a^{(k)} \in A$ such that

$$|f(a^{(k)})| > k e^{\sigma \cdot a^{(k)} + \varepsilon \|a^{(k)}\|}.$$

It follows that $\|a^{(k)}\|$ must tend to $+\infty$. Define $b^{(k)} = a^{(k)}/\|a^{(k)}\|$. The points $b^{(k)}$ belong to the unit sphere, a compact set, so there exists a subsequence which converges to some point b on the sphere. After a change of notation we may assume that the sequence $(b^{(k)})_k$ itself converges to b. We obtain

$$\begin{aligned} p_f(b) &\geqslant \limsup_{k \to \infty} \frac{1}{\|a^{(k)}\|} \log |f(a^{(k)})| \geqslant \limsup_{k \to \infty} \frac{1}{\|a^{(k)}\|} (\sigma \cdot a^{(k)} + \varepsilon \|a^{(k)}\|) \\ &= \limsup_{k \to \infty} (\sigma \cdot b^{(k)} + \varepsilon) = \sigma \cdot b + \varepsilon > \sigma \cdot b. \end{aligned}$$

Hence (A′) does not hold. We are done. ☐

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Affiliation: Uppsala University, Department of Information Technology,
Division of Visual Information and Interaction,
Computerized Image Analysis and Human-Computer Interaction.

Paper mail address: P. O. Box 337, SE-751 05 Uppsala, Sweden

Amber addresses: `kiselman@it.uu.se`, `christer@kiselman.eu`