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On some especially interesting distributions

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Resumo: *Pri iuj aparte interesaj distribucioj*
Por miaj kursanoj mi faros detalan pritrakton de distribucioj difinitaj per la precipa valoro kaj la finia parto de diverĝaj integraĵoj. Krome mi studos ecojn pri kontinuo de la distribucioj difinitaj per holomorfaj funkcioj.

Abstract: For my students I shall consider in detail distributions defined by the principal value and the finite part of divergent integrals. I shall also consider continuity properties of distributions defined by holomorphic functions.

1. Introduction

The purpose of this note is to discuss the notions of principal part of a divergent integral, the finite part of a divergent integral, and, finally, the continuity properties of the distributions that appear in Plemelj's formulas.

2. The principal value

Let f be a continuous function on the real line, or more generally a locally integrable function. The integral

$$\int_{\mathbf{R}} f(x)dx$$

is said to exist in the *generalized sense* if the limit

$$\lim_{a,b\to+\infty} \int_{-a}^{b} f(x)dx$$

exists. We say that the *principal value* of the integral exists if the limit exists when we impose the condition $b = a$, thus

$$\mathsf{vp} \int_{\mathbf{R}} f(x)dx = \lim_{a\to+\infty} \int_{-a}^{a} f(x)dx.$$

Example 2.1. If $f(x) = (\sin x)/x$, then its integral exists in the generalized sense, although the function is not Lebesgue integrable. If $f(x) = x/(1 + x^2)$, then its integral does not exist in the generalized sense, but the principal value exists (and is zero).

We can replace the singularity at infinity by a singularity at any other point. Let $f \in L^1_{\mathrm{loc}}(\mathbf{R} \smallsetminus \{0\})$. Suppose for simplicity that f has compact support so that there is no difficulty at infinity. Then we say that its integral exists in the *generalized sense* if the limit

$$\lim_{a,b\to 0+} \left(\int_{-\infty}^{-a} f(x)dx + \int_b^{\infty} f(x)dx \right)$$

exists. We say that the *principal value* exists if the limit exists when we restrict a and b to satisfy $a = b$, thus

$$\mathsf{vp} \int_{\mathbf{R}} f(x)dx = \lim_{\varepsilon\to 0+} \int_{|x|>\varepsilon} f(x)dx.$$

The idea is thus that we remove a *symmetric* neighborhood of a singular point and then pass to the limit. Large negative and positive values of the function can balance each other.

Exercise 2.2. Prove that a function $f \in L^1_{\mathrm{loc}}(\mathbf{R} \smallsetminus \{0\})$ such that its integral over $[-1, 1]$ exists in the generalized sense defines a distribution of order at most 1.

Example 2.3. The principal values

$$\mathsf{vp} \int_{-1}^{1} x^m dx$$

exist whenever m is an odd integer; the value is of course zero.

Example 2.4. The principal values

$$\mathsf{vp} \int_{\mathbf{R}} \frac{\varphi(x)}{x} dx$$

exists if $\varphi \in \mathcal{D}(\mathbf{R})$; more generally if $f \in C^1_0(\mathbf{R})$, for we can write

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx = \int_{\varepsilon}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx = \int_{\varepsilon}^{\infty} dx \int_{-1}^{1} \varphi'(tx)dt.$$

The existence of the limit is now obvious, and we can define a distribution $\mathsf{vp}(1/x)$ by the formula

$$\left(\mathsf{vp}\frac{1}{x}\right)(\varphi) = \int_0^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx = \int_0^{\infty} dx \int_{-1}^{1} \varphi'(tx)dt, \quad \varphi \in \mathcal{D}(\mathbf{R}). \tag{2.1}$$

From the last expression we can estimate the values as follows:

$$\left| \left(\mathsf{vp}\frac{1}{x}\right)(\varphi) \right| \leqslant \int_0^A dx\, 2\|\varphi'\|_\infty = 2A\|\varphi'\|_\infty, \tag{2.2}$$

if A is so large that the support of φ is contained in $[-A, A]$. This shows that $\mathsf{vp}(1/x)$ is a distribution of order at most 1. (Show that it is of order at least 1, i.e., that it is not a measure!) However, the estimate (2.2) is not translation invariant. We therefore subdivide the interval $[0,+\infty[$ into two and use different estimates for each part:

$$\left|\left(\mathsf{vp}\frac{1}{x}\right)(\varphi)\right| \leqslant \int_0^c \left(\int_{-1}^1 |\varphi'(tx)|dt\right)dx + \int_c^\infty \frac{|\varphi(x)|+|\varphi(-x)|}{x}dx \\ \leqslant 2c\|\varphi'\|_\infty + c^{-1}\|\varphi\|_1, \tag{2.3}$$

which is a translation-invariant estimate. The translation invariance implies that we can estimate also convolutions:

$$\left\|\left(\mathsf{vp}\frac{1}{x}\right)*\varphi\right\|_\infty \leqslant 2c\|\varphi'\|_\infty + c^{-1}\|\varphi\|_1. \tag{2.4}$$

The best choice of c here, by the way, is

$$c = \sqrt{\frac{\|\varphi\|_1}{2\|\varphi'\|_\infty}},$$

which yields

$$\left|\left(\mathsf{vp}\frac{1}{x}\right)(\varphi)\right| \leqslant 2\sqrt{2\|\varphi'\|_\infty\|\varphi\|_1}, \quad \varphi \in \mathcal{D}(\mathbf{R}). \tag{2.5}$$

Exercise 2.5. How sharp is the estimate (2.5)? Prove that it can be improved to

$$\left|\left(\mathsf{vp}\frac{1}{x}\right)(\varphi)\right| \leqslant 2\sqrt{\|\varphi'\|_\infty\|\varphi\|_1}, \quad \varphi \in \mathcal{D}(\mathbf{R}).$$

Prove that, on the other hand, in any estimate

$$\left|\left(\mathsf{vp}\frac{1}{x}\right)(\varphi)\right| \leqslant C\sqrt{\|\varphi'\|_\infty\|\varphi\|_1}, \quad \varphi \in \mathcal{D}(\mathbf{R}),$$

we must have $C \geqslant 2\log 2 > 1.3862$.

Exercise 2.6. Prove the estimate

$$\left|\left(\mathsf{vp}\frac{1}{x}\right)(\varphi)\right| \leqslant C\|\varphi'\|_\infty^{1/3}\|\varphi\|_2^{2/3}.$$

Exercise 2.7. Prove that $x\mathsf{vp}(1/x) = 1$. The solutions to the equation $xu = 1$ are $u = \mathsf{vp}(1/x) + C\delta$, $C \in \mathbf{C}$.

Example 2.8. Is it possible to define a distribution $\mathsf{vp}(1/x^3)$? No, for the balance between negative and positive values in Example 2.3 for $m = -3$ is not sufficiently stable to allow multiplication by a smooth function. This follows from the formula

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^3}dx = \int_\varepsilon^\infty \frac{\varphi(x)-\varphi(-x)}{x^3}dx,$$

where the limit does not exist if we take φ such that $\varphi(x) = x$ near the origin. More generally, we see that we can define $\mathsf{vp}f$ if f is an odd function which is locally integrable in $\mathbf{R} \smallsetminus \{0\}$ and such that for some $\alpha > -2$, $|f(x)| \leqslant Cx^\alpha$, $0 < x < 1$. For such functions we can also change variables in the integral. In conclusion, odd symmetry can kill singularities, but only the mild ones.

3. Pseudofunctions

The pseudofunction defined by $1/x^2$ is a distribution defined as follows:

$$\left(\mathsf{pf}\frac{1}{x^2}\right)(\varphi) = \lim_{\varepsilon\to 0}\left(\int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx - \frac{C}{\varepsilon}\right),$$

where C is the constant, if any, such that the limit exists. Of course the limit cannot exist for more that one choice of C. The notation pf is chosen to make us think not only of a pseudofunction but also of the finite part (*la partie finie*) of Hadamard. We write

$$\begin{aligned}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx &= \int_\varepsilon^A \frac{\varphi(x)+\varphi(-x)}{x^2}dx\\ &= \int_\varepsilon^A \frac{\varphi(x)-2\varphi(0)+\varphi(-x)}{x^2}dx + 2\varphi(0)\left(\frac{1}{\varepsilon}-\frac{1}{A}\right),\end{aligned}$$

where A is so large that $[-A, A]$ contains the support of the test function. Hence $C = 2\varphi(0)$ is the only choice, and we define

$$\left(\mathsf{pf}\frac{1}{x^2}\right)(\varphi) = \int_0^A \frac{\varphi(x)-2\varphi(0)+\varphi(-x)}{x^2}dx - \frac{2\varphi(0)}{A}, \quad \varphi\in\mathcal{D}(\mathbf{R}) \tag{3.1}$$

Note that the right-hand side is independent of A as long as A is large (calculate the derivative of the right-hand side with respect to A). Since the integral of $1/x^2$ is convergent at infinity, we can let A tend to infinity here, thus the definition can also be written:

$$\left(\mathsf{pf}\frac{1}{x^2}\right)(\varphi) = \int_0^\infty \frac{\varphi(x)-2\varphi(0)+\varphi(-x)}{x^2}dx, \quad \varphi\in\mathcal{D}(\mathbf{R}). \tag{3.2}$$

To estimate the integral we now write

$$\frac{\varphi(x)-2\varphi(0)+\varphi(-x)}{x^2} = \int_0^1 s\,ds\int_{-1}^1 \varphi''(stx)dt, \tag{3.3}$$

which shows that the pseudofunction can be estimated as

$$\left|\left(\mathsf{pf}\frac{1}{x^2}\right)(\varphi)\right| \leqslant \int_0^A\int_0^1 sds\int_{-1}^1 \|\varphi''\|_\infty dt = A\|\varphi''\|_\infty,$$

where A is so large that the support of φ is contained in $[-A, A]$. This shows that $\mathsf{pf}(1/x^2)$ is a distribution of order at most two. (Show that the order is two!) Since the estimate is not translation invariant, we shall modify it as in the case of the principal value. We use the representation (3.2) only for $x \in [0, c]$:

$$\left(\mathsf{pf}\frac{1}{x^2}\right)(\varphi) = \int_0^c \int_0^1 s\,ds \int_{-1}^1 \varphi''(stx)dt + \int_c^\infty \frac{\varphi(x) - 2\varphi(0) + \varphi(-x)}{x^2}dx,$$

which can be estimated as

$$\left|\left(\mathsf{pf}\frac{1}{x^2}\right)(\varphi)\right| \leqslant c\|\varphi''\|_\infty + 2c^{-1}\|\varphi\|_\infty + c^{-2}\|\varphi\|_1,$$

or perhaps simpler as

$$\left|\left(\mathsf{pf}\frac{1}{x^2}\right)(\varphi)\right| \leqslant c\|\varphi''\|_\infty + 4c^{-1}\|\varphi\|_\infty;$$

both are translation-invariant estimates. What is the best choice of c?

Exercise 3.1. Prove that $x^2\mathsf{pf}(1/x^2) = 1$. The solutions to $x^2u = 1$ are $u = \mathsf{pf}(1/x^2) + C_0\delta + C_1\delta'$.

In contrast to the principal value, we can consider pseudofunctions defined by one-sided integrals. Thus the distribution $\mathsf{pf}(1/x_+)$ is defined by

$$\left(\mathsf{pf}\frac{1}{x_+}\right)(\varphi) = \lim_{\varepsilon\to 0}\left(\int_\varepsilon^\infty \frac{\varphi(x)}{x}dx + C\log\varepsilon\right),$$

where again C is the only constant such that the limit exists. We can write

$$\int_\varepsilon^A \frac{\varphi(x)}{x}dx + C\log\varepsilon = \int_\varepsilon^A \frac{\varphi(x) - C}{x}dx + C\log A,$$

which makes it obvious that the only choice is $C = \varphi(0)$. Thus

$$\left(\mathsf{pf}\frac{1}{x_+}\right)(\varphi) = \int_0^A \frac{\varphi(x) - \varphi(0)}{x}dx + \varphi(0)\log A, \quad \varphi \in \mathcal{D}(\mathbf{R}).$$

(Here we cannot let A tend to infinity.)

Similarly for $\mathsf{pf}(1/x_+^2)$:

$$\left(\mathsf{pf}\frac{1}{x_+^2}\right)(\varphi) = \lim_{\varepsilon\to 0}\left(\int_\varepsilon^A \frac{\varphi(x)}{x^2}dx - \frac{C_0}{\varepsilon} + C_1\log\varepsilon\right).$$

Again only one choice of constants is possible, and we can write

$$\int_\varepsilon^A \frac{\varphi(x)}{x^2}dx - \frac{C_0}{\varepsilon} + C_1\log\varepsilon = \int_\varepsilon^A \frac{\varphi(x) - C_0 - C_1x}{x^2}dx - \frac{C_0}{A} + C_1\log A,$$

which shows that the constants must be $C_0 = \varphi(0)$, $C_1 = \varphi'(0)$, so that

$$\left(\mathsf{pf}\frac{1}{x_+^2}\right)(\varphi) = \int_0^A \frac{\varphi(x) - \varphi(0) - \varphi'(0)x}{x^2}dx - \frac{\varphi(0)}{A} + \varphi'(0)\log A, \quad \varphi \in \mathcal{D}(\mathbf{R}).$$

We can perhaps formulate a rule for these pseudofunctions as follows: we subtract a part of the Taylor expansion of the function to make the integral convergent at the singular point (the origin in our case), but then we must compensate by a function of A, the upper limit of the integral, to make the whole expression independent of A. This sounds sloppy, but if we add the information that the right-hand side shall be zero for $\varphi = 0$, then it is actually uniquely determined.

4. On Plemelj's formulas

Let us consider the distributions u_s defined for real nonzero s by

$$u_s(\varphi) = \int_{\mathbf{R}} \frac{\varphi(x)}{x + is}dx, \quad \varphi \in \mathcal{D}(\mathbf{R}). \tag{4.1}$$

Thus they are defined by smooth functions, but the interesting question is what happens when the parameter s tends to zero. We rewrite the definition as follows:

$$u_s(\varphi) = \int_0^\infty \frac{\varphi(x) - \varphi(-x)}{x^2 + s^2}xdx - is\int_0^\infty \frac{\varphi(x) + \varphi(-x)}{x^2 + s^2}dx. \tag{4.2}$$

Now it is easy to see that u_s converges to a distribution u_{0+} as s tends to zero while being positive:

$$u_{0+}(\varphi) = \int_0^\infty \frac{\varphi(x) - \varphi(-x)}{x}dx - i\pi\varphi(0), \quad \varphi \in \mathcal{D}(\mathbf{R}), \tag{4.3}$$

thus

$$\lim_{s\to 0+} u_s = u_{0+} = \mathsf{vp}\left(\frac{1}{x}\right) - i\pi\delta, \tag{4.4}$$

a relation known as Plemelj's formula, or Sohockij–Plemelj's formula. (Do the necessary deliberations.) Note that the passage to the limit works for any function which is of class C^1 in a neighborhood of the origin and such that $x^\varepsilon\varphi(x)$ is bounded for some positive ε.

We can estimate the integrals defining u_s as follows, taking a number A so large that $[-A, A]$ contains the support of φ:

$$|u_s(\varphi)| \leqslant 2\|\varphi'\|_\infty \int_0^A \frac{x^2}{x^2 + s^2}dx + 2\|\varphi\|_\infty \int_0^A \frac{s}{x^2 + s^2}dx.$$

Here the first integral is not larger than A, the second not larger than $\pi/2$; thus

$$|u_s(\varphi)| \leqslant 2A\|\varphi'\|_\infty + \pi\|\varphi\|_\infty, \quad \varphi \in \mathcal{D}(\mathbf{R});$$

cf. (2.2). As we noted several times already, such estimates containing a number A depending on the support of the test function are not translation invariant. Instead we write

(4.5)
$$\begin{aligned}|u_s(\varphi)| &\leqslant \int_0^c \frac{|\varphi(x)-\varphi(-x)|}{x}\frac{x^2}{x^2+s^2}dx + \int_c^\infty \frac{|\varphi(x)-\varphi(-x)|}{x}\frac{x^2}{x^2+s^2}dx + \pi\|\varphi\|_\infty \\ &\leqslant 2c\|\varphi'\|_\infty + c^{-1}\|\varphi\|_1 + \pi\|\varphi\|_\infty.\end{aligned}$$

This holds for all $c > 0$ and all $s \in \mathbf{R}$, the case $s = 0$ including both u_{0+} and u_{0-}. We can also estimate in terms of the L^p-norm for any $p > 1$; in particular we get for the L^2-norm,

$$(4.6) \qquad |u_s(\varphi)| \leqslant 2c\|\varphi'\|_\infty + c^{-1/2}\|\varphi\|_2 + \pi\|\varphi\|_\infty, \qquad \varphi \in \mathcal{D}(\mathbf{R}),\ s \in \mathbf{R}.$$

These estimates make it possible to let u_s act on more general functions than the usual test functions in $\mathcal{D}(\mathbf{R})$. We formulate this result as a lemma.

Lemma 4.1. *Let $C^1_\diamond(\mathbf{R})$ denote the space of all $f \in C^1(\mathbf{R})$ such that $f \in L^2 \cap L^\infty$ and $f' \in L^\infty$. We equip $C^1_\diamond(\mathbf{R})$ with the norm*

$$\|\varphi\|_\diamond = \|\varphi\|_2 + \pi\|\varphi\|_\infty + 2\|\varphi'\|_\infty.$$

Then the distributions u_s, $s \in \mathbf{R} \smallsetminus \{0\}$, and u_{0+}, u_{0-} can be extended from $\mathcal{D}(\mathbf{R})$ to $C^1_\diamond(\mathbf{R})$ as defined by (4.2) and (4.3) and they satisfy the estimate $|u_s(\varphi)| \leqslant \|\varphi\|_\diamond$, $\varphi \in C^1_\diamond(\mathbf{R})$; equivalently

$$(4.7) \qquad \|u_s * \varphi\|_\infty \leqslant \|\varphi\|_\diamond, \qquad \varphi \in C^1_\diamond(\mathbf{R}).$$

It can be seen easily that $u_s * \varphi$, $u_{0+} * \varphi$, and $u_{0-} * \varphi$ all belong to $C^1_\diamond(\mathbf{R})$, and we shall soon apply (4.7) to such more general functions.

Let us now look at the difference $u_s - u_{0+}$:

$$(u_s - u_{0+})(\varphi) = -s^2\int_0^\infty \frac{\varphi(x)-\varphi(-x)}{x(x^2+s^2)}dx - is\int_0^\infty \frac{\varphi(x)-2\varphi(0)+\varphi(-x)}{x^2+s^2}dx.$$

We can estimate these integrals by the methods already used in the proof of (4.5) and get

$$(4.8) \qquad |(u_s - u_{0+})(\varphi)| \leqslant s\left(4c^{-1}\|\varphi\|_\infty + \pi\|\varphi'\|_\infty + c\|\varphi''\|_\infty\right).$$

This gives a quantitative idea of how fast u_s converges to u_{0+}. We may extend the validity of (4.8) to functions f such that $\varphi, \varphi' \in C^1_\diamond(\mathbf{R})$, or even a little farther:

Lemma 4.2. *Let $C^2_\infty(\mathbf{R}^2)$ denote the space of all functions $\varphi \in C^2(\mathbf{R})$ such that $\varphi, \varphi', \varphi'' \in L^\infty$ with norm*

$$\|\varphi\|_{2,\infty} = 4\|\varphi\|_\infty + \pi\|\varphi'\|_\infty + \|\varphi''\|_\infty.$$

Then $u_s - u_{0+}$ *can be extended from* $\mathcal{D}(\mathbf{R})$ *to* $C^2_\infty(\mathbf{R})$ *and satisfies*

$$\|(u_s - u_{0+}) * \varphi\|_\infty \leqslant s\|\varphi\|_{2,\infty}, \qquad \varphi \in C^2_\infty(\mathbf{R}). \tag{4.9}$$

We can also see that the derivative with respect to s exists:

$$\frac{u_s - u_{0+}}{s} \to \pi\delta' - i\,\mathsf{pf}\left(\frac{1}{x^2}\right), \quad s \to 0+.$$

One can prove, using the residue formula, that $u_s * u_t = 0$ if $st < 0$. (Do so! We have a ring with zero divisors!) If we could pass to the limit in this equation we would obtain $u_{0+} * u_{0-} = 0$; hence, in view of Plemelj's formula (4.4),

$$\mathsf{vp}\left(\frac{1}{x}\right) * \mathsf{vp}\left(\frac{1}{x}\right) = -\pi^2\delta. \tag{4.10}$$

What does the Titchmarsh support theorem say?

We shall now see that passage to the limit is legitimate. We state the result as a proposition.

Proposition 4.3. *Let* u_{0+} *be defined by (4.3) and let* u_{0-} *be its complex conjugate. Then* $u_{0+} * u_{0-} = 0$.

Proof. We write

$$-u_{0+} * u_{0+} = u_s * u_t - u_{0+} * u_{0-} = u_s * (u_t - u_{0-}) + (u_s - u_{0+}) * u_{0-}. \tag{4.11}$$

To estimate the first term we use (4.9) with s replaced by t, and $\varphi * u_s$ as a test function:

$$\|(\varphi * u_s) * (u_t - u_{0-})\|_\infty \leqslant |t|\|\varphi * u_s\|_{2,\infty} = |t|(4\|u_s * \varphi\|_\infty + \pi\|u_s * \varphi'\|_\infty + \|u_s * \varphi''\|_\infty).$$

Here the right-hand side can be estimated using (4.7); it does not exceed

$$|t|(4\|\varphi\|_\diamond + \pi\|\varphi'\|_\diamond + \|\varphi''\|_\diamond),$$

which is independent of s.

To take care of the second term in (4.11) we shall use (4.9) with $u_{0+} * \varphi$ as a test function:

$$\|(u_s - u_{0+}) * (u_s * \varphi)\|_\infty \leqslant s\|u_{0+} * \varphi\|_{2,\infty} \leqslant s(4\|\varphi\|_\diamond + \pi\|\varphi'\|_\diamond + \|\varphi''\|_\diamond).$$

These estimates for the two terms in (4.10) show that for $s > 0 > t$, we have

$$\|(u_{0+} * u_{0-}) * \varphi\|_\infty = \|(u_s * u_t - u_{0+} * u_{0-}) * \varphi\|_\infty \leqslant C_\varphi(s + |t|),$$

where $C_\varphi = 4\|\varphi\|_\diamond + \pi\|\varphi'\|_\diamond + \|\varphi''\|_\diamond$ is a constant depending on $\varphi \in \mathcal{D}(\mathbf{R})$. Thus $u_{0+} * u_{0-}$ is zero.

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