Division of mappings between complete lattices

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Abstract The importance for image processing of a good theory for morphological operators in complete lattices is now well understood. In this paper we introduce inverses and quotients of mappings between complete lattices which are analogous to inverses and quotients of positive numbers. These concepts are then used to create a convenient formalism for dilations and erosions as well as for cleistomorphisms (closure operators) and anoiktomorphisms (kernel operators).

Keywords: complete lattice, generalized inverse of a mapping, division of mappings, epigraph, hypograph, dilation, erosion, ethmomorphism, morphological filter, cleistomorphism, closure operator, anoiktomorphism, kernel operator, residuated mapping, Galois connection.

1. Introduction

Lattice theory is a mature mathematical theory thanks to the pioneering work by Garrett Birkhoff, Øystein Ore and others in the first half of the twentieth century. A standard reference is still Birkhoff's book (1995) [1], first published in 1940. The importance for image processing of a good theory for morphological operators in complete lattices is now well understood. See for instance the books by Matheron (1975) [9], Serra (1982, 1988) [13], [14] and Heijmans (1994) [6], and the articles by Heijmans and Ronse (1990) [7], Ronse (1990) [11], Ronse and Heijmans (1991) [12], and Serra (2006) [15].

In this paper we shall introduce inverses and quotients of mappings between complete lattices which are analogous to inverses $1/y$ and quotients x/y of positive numbers. These concepts are then used to create a convenient formalism for a unified treatment of dilations $\delta\colon L \to M$ and erosions $\varepsilon\colon L \to M$ as well as of cleistomorphisms (closure operators) $\kappa\colon L \to L$ and anoiktomorphisms (kernel operators) $\alpha\colon L \to L$. The theory for inverses in Section 3 generalizes the theory of Galois connections, which is equivalent to residuation theory.

To define an inverse of a general mapping seems to be a hopeless task. However, if the mapping is between preordered sets, there is some hope of constructing mappings that can serve in certain contexts just like inverses do.

2. Definitions

Definition 2.1. A *preorder* in a set X is a binary relation which is reflexive (for all $x \in X$, $x \leqslant x$) and transitive (for all $x, y, z \in X$, $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$). An *order* is a preorder which is antisymmetric (for all $x, y \in X$, $x \leqslant y$ and $y \leqslant x$ imply $x = y$).

To any preorder $\leqslant$ we introduce an equivalence relation $x \sim y$ defined as $x \leqslant y$ and $y \leqslant x$. If $\leqslant$ is an order, this equivalence relation is just equality. If we have two preorders, we say that $\leqslant_1$ is *stronger than* or *finer than* $\leqslant_2$ if for all x and y, $x \leqslant_1 y$ implies $x \leqslant_2 y$ We also say that $\leqslant_2$ is *weaker than* or *coarser than* $\leqslant_1$. The finest preorder is the *discrete preorder*, defined as equality; the coarsest preorder is the *chaotic preorder* given by $x \leqslant y$ for all x and y.

Definition 2.2. A *complete lattice* is an ordered set such that any family $(x_j)_{j \in J}$ of elements possesses a smallest majorant and a largest minorant. We denote them by $\bigvee_{j \in J} x_j$ and $\bigwedge_{j \in J} x_j$, respectively.

Definition 2.3. If $f \colon X \to Y$ is a mapping of a set into another, we define its *graph* as the set

$$\operatorname{graph} f = \{(x, y) \in X \times Y; y = f(x)\}.$$

If Y is preordered, we define also its *epigraph* and its *hypograph* as

$$\operatorname{epi} f = \{(x, y) \in X \times Y; f(x) \leqslant y\}, \qquad \operatorname{hypo} f = \{(x, y) \in X \times Y; y \leqslant f(x)\}.$$

If X and Y are given, any mapping $X \to Y$ is determined by its graph, and, if Y is a complete lattice, also by its epigraph and by its hypograph. It is often convenient to express properties of mappings in terms of their epigraphs or hypographs.

Definition 2.4. If two preordered sets X and Y and a mapping $f \colon X \to Y$ are given, we shall say that f is *increasing* if

$$\text{for all } x, x' \in X,\ x \leqslant_X x' \ \Rightarrow\ f(x) \leqslant_Y f(x'),$$

and that f is *coincreasing* if

$$\text{for all } x, x' \in X,\ f(x) \leqslant_Y f(x') \ \Rightarrow\ x \leqslant_X x'.$$

Blyth and Janowitz (1972:6) [3] and Blyth (2005:5) [2] call an increasing mapping *isotone*. The term *coincreasing* appears in my lecture notes (2002:12) [8].

To emphasize the symmetry between the two notions, we define, given any mapping $f \colon X \to Y$, a preorder $\leqslant_f$ in X by the requirement that $x \leqslant_f x'$ if and only if $f(x) \leqslant_Y f(x')$. Then f is increasing if and only if $\leqslant_X$ is finer than $\leqslant_f$, and f is coincreasing if and only if $\leqslant_f$ is finer than $\leqslant_X$.

A comparison with topology is in order here. If $f\colon X \to Y$ is a mapping of a topological space X into a topological space Y with topologies τ_X and τ_Y, we can define a new topology τ_f in X as the family of all sets $\{x \in X; f(x) \in V\}$, $V \in \tau_Y$. Then f is continuous if and only if τ_X is finer than τ_f.

Definition 2.5. A mapping $f\colon L \to M$ of a complete lattice L into a complete lattice M is said to be a *dilation* if $f\left(\bigvee_{j\in J} x_j\right) = \bigvee_{j\in J} f(x_j)$ for all families $(x_j)_{j\in J}$ of elements in L. The mapping is said to be an *erosion* if $f\left(\bigwedge_{j\in J} x_j\right) = \bigwedge_{j\in J} f(x_j)$ for all families $(x_j)_{j\in J}$.

Singer (1997:172) [16] calls a mapping $f\colon L \to M$ a *duality* if $f\left(\bigwedge_{j\in J} x_j\right) = \bigvee_{j\in J} f(x_j)$ for all families $(x_j)_{j\in J}$ of elements in L. Thus a duality induces a dilation $L^{\mathrm{op}} \to M$ and an erosion $L \to M^{\mathrm{op}}$ if we change the order in L or M to the opposite order; the study of dualities in the sense of Singer is equivalent to that of dilations or erosions.

Definition 2.6. A mapping $f\colon X \to X$ of a preordered set X into itself is said to be an *ethmomorphism* if it is increasing and idempotent. If in addition it is extensive, i.e., $f(x) \geqslant x$ for all $x \in X$, then it is said to be a *cleistomorphism*; if it is antiextensive, i.e., $f(x) \leqslant x$ for all $x \in X$, then it is called an *anoiktomorphism*.[1]

For the notions just defined many terms have been used. Other terms for ethmomorphism are *morphological filter* (Serra 1988:104 [14]), *projection operator* and *projection* (Gierz et al. 2003:26 [5]). For cleistomorphism other terms include *closure mapping* (Blyth and Janowitz 1972:9 [3]), *closing* (Matheron 1975:187 [9]; Serra 1982:56 [13]), *hull operator* (Singer 1997:8 [16]), *closure operator* (Gierz et al. 2003:26 [5]). For anoiktomorphism there are several other terms: *dual closure mapping* (Blyth and Janowitz 1972:9 [3]), *opening* (Matheron 1975:187 [9]; Serra 1982:56 [13]), *kernel operator* (Gierz et al. 2003:26 [5]).

3. Inverses of mappings

In general a mapping $g\colon X \to Y$ between sets does not have an inverse. If g is injective, we may define a left inverse $u\colon Y \to X$, thus with $u \circ g = \mathsf{id}_X$. If g is surjective, we may define a right inverse $v\colon Y \to X$, thus with $g \circ v = \mathsf{id}_Y$. We then need to define $v(y)$ as an element of $\{x; g(x) = y\}$. In the general situation this has to be done using the axiom of choice. In a complete lattice,

[1] Cf. the noun *ēthmós* ‘strainer’ and the adjectives *kleistós* ‘closed’ and *anoiktós* ‘open’ in Classical Greek. I am grateful to Ebbe Vilborg for help with these words.

however, it could be interesting to define $v(y)$ as the supremum or infimum of all x such that $g(x) = y$, even though this supremum or infimum need not belong to the set. However, for various purposes it is convenient to take instead the infimum of all x such that $g(x) \geqslant y$ or the supremum of all x such that $g(x) \leqslant y$. This yields better monotonicity properties. (The case $g(x) = y$ is covered if we let the preorder in Y be the discrete preorder.)

Definition 3.1. Let L be a complete lattice, Y a preordered set, and $g\colon L \to Y$ any mapping. We then define the *upper inverse* $g^{[-1]}\colon Y \to L$ and the *lower inverse* $g_{[-1]}\colon Y \to L$ as the mappings

$$g^{[-1]}(y) = \bigwedge_{x\in L}(x; g(x) \geqslant_Y y) = \bigwedge_{x\in L}(x; (x, y) \in \operatorname{hypo} g), \qquad y \in Y; \tag{1}$$

$$g_{[-1]}(y) = \bigvee_{x\in L}(x; g(x) \leqslant_Y y) = \bigvee_{x\in L}(x; (x, y) \in \operatorname{epi} g), \qquad y \in Y. \tag{2}$$

As a first observation, let us note that these inverses are always increasing. If Y possesses a smallest element $\mathbf{0}_Y$, then $g^{[-1]}(\mathbf{0}_Y) = \mathbf{0}_L$. Similarly, if there is a largest element $\mathbf{1}_Y$, then $g_{[-1]}(\mathbf{1}_Y) = \mathbf{1}_L$. If Y has the chaotic preorder, then both inverses are constant, $g^{[-1]} = \mathbf{0}_L$ and $g_{[-1]} = \mathbf{1}_L$ identically.

We note that we always have

$$\left(\operatorname{epi} g^{[-1]}\right)^{-1} \supset \operatorname{hypo} g \text{ and } \left(\operatorname{hypo} g_{[-1]}\right)^{-1} \supset \operatorname{epi} g. \tag{3}$$

Here $S^{-1} = \{(y, x) \in Y \times L; (x, y) \in S\}$ for any subset S of $L \times Y$. In general these inclusions are strict as we shall see below.

Note that we do not require in (2) that the set of all x such that $g(x) \leqslant_Y y$ shall have a largest element. In other words, the supremum in (2) is not necessarily a maximum.

The special situation when the supremum in (2) is a maximum, in other words when $g(g_{[-1]}(y)) \leqslant_Y y$ for all y, has been studied for a long time, and from various aspects. Let us mention a few examples.

1. When the supremum in (2) is a maximum, the pair $(g, g_{[-1]})$ is said to be a *Galois connection* (Gierz et al. 2003:22) [5], a concept which goes back to Évariste Galois' work on the automorphism groups of a field. Ore (1944:495) [10] called a variant of the pair of mappings $(g, g_{[-1]})$ a *Galois connexion*.

2. One also says in this special case that g is *residuated* and that $g_{[-1]}$ is its *residual* (Blyth and Janowitz 1972:11 [3]; Blyth 2005:7 [2]). If the infimum in (1) is a minimum, g is said to be *dually residuated* and $g^{[-1]}$ is called its *dual residual*; the pair $(g^{[-1]}, g)$ is a Galois connection between Y and L. Residuation theory goes back at least to a paper by Ward and Dilworth (1939) [17]. In an ordered groupoid one fixes an element c and assumes that the set of all x such that $cx \leqslant y$ has a largest element, which is denoted by $y : c$ (we consider for simplicity only the commutative case).

We see that this is $g_{[-1]}$ if $g\colon L \to L$ is the mapping $g(x) = cx$. Thus $cx \leqslant y$ if and only if $x \leqslant y : c$.

3. The pair $(g, g_{[-1]})$ is also said to be an *adjunction* (Gierz et al. 2003:22 [5]) in this special case. This aspect probably originates in logic, and is important in image processing.

4. Finally, there is duality in convexity theory. The Fenchel transformation (Fenchel 1949 [4]) of a function $\varphi\colon \mathbf{R}^n \to [-\infty, +\infty]$ is defined as

$$\tilde{\varphi}(\xi) = \sup_{x \in \mathbf{R}^n} \big(\xi \cdot x - \varphi(x)\big), \qquad \xi \in \mathbf{R}^n,$$

and satisfies

$$\tilde{\varphi} \leqslant f \iff \tilde{f} \leqslant \varphi.$$

After a change of order on one of the sides it satisfies (3) with equality, which means that we have a Galois connection (see condition (C) in Theorem 3.2). It is also the case that

$$\big(\inf_{j \in J} \varphi\big)\tilde{} = \sup_{j \in J} \tilde{\varphi},$$

so that we have a duality in the sense of Singer; i.e., after a change of the order relation we have a dilation or erosion (see condition (A) in Theorem 3.2). Singer (1997) [16] studies several other dualities in convexity theory.

The results of the present section generalize residuation theory, equivalently the theory of Galois connections, to a more general situation, a situation which appears even in very simple examples as we shall see now.

It seems that this generalization of residuation theory has not been considered in the contexts of the branches of mathematics mentioned under 1, 2, and 3 above. However, Singer (1997:176) [16] defines the *dual* $M \to L$ of a duality $L \to M$, which, after a change of order in L, is the lower inverse defined here. He notes the inclusion corresponding to the second inclusion in (3) and proves that it is an equality when g is a dilation.

Example 3.1. Take $Y = L$ in Definition 3.1, fix an element c of L, and define a mapping $g\colon L \to L$ by $g(x) = x \vee c$, $x \in L$. In this case, the supremum in (2) is a maximum if $y \geqslant c$ but only then. Thus g is not residuated unless $c = \mathbf{0}$. But it is easy to determine its lower inverse: $g_{[-1]}(y) = y$ if $y \geqslant c$ and $g_{[-1]}(y) = \mathbf{0}$ otherwise. We have

$$\operatorname{epi} g = \{(x, y) \in L^2; y \geqslant x \vee c\},$$

while

$$\big(\operatorname{hypo} g_{[-1]}\big)^{-1} = \operatorname{epi} g \cup \{(\mathbf{0}, y) \in L^2; y \ngeqslant c\},$$

so that

$$\big(\operatorname{hypo} g_{[-1]}\big)^{-1} \smallsetminus \operatorname{epi} g = \{(\mathbf{0}, y) \in L^2; y \ngeqslant c\} \neq \varnothing \text{ if } c \neq \mathbf{0}.$$

For the upper inverse, we can only say that $g^{[-1]} = \mathbf{0}$ if $y \leqslant c$ and that $g^{[-1]}(y) \leqslant y$ for $y \nleqslant c$. Both equality and strict inequality can occur here as we shall see. $\square$

Example 3.2. We let g be as in Example 3.1 and assume in addition that L is totally ordered. We have already determined $g_{[-1]}$ in Example 3.1, and we know that $g^{[-1]}(y) = \mathbf{0}$ for $y \leqslant c$. In the case of total order, we have $g^{[-1]}(y) = y$ for all $y > c$. In the notation which Singer (1997:335) [16] uses for $L = [-\infty, +\infty]$, we can write $g^{[-1]}(y) = y \top c$, $y \in L$. Thus $g_{[-1]}$ and $g^{[-1]}$ are equal except for $y = c$, where we get $g^{[-1]}(c) = \mathbf{0} \leqslant c = g_{[-1]}(c)$. Moreover we have

$$\left(\operatorname{epi} g^{[-1]}\right)^{-1} = \operatorname{hypo} g = \{(x, y) \in L^2; y \leqslant x \vee c\},$$

which, in view of Corollary 3.1 means that $g^{[-1]}$ is dually residuated with dual residual g, or that $(g^{[-1]}, g)$ is a Galois connection. □

Example 3.3. Let now L be $[0, 1]^2$, the Cartesian product of two intervals. The lower inverse is already known from Example 3.1. The upper inverse is

$$g^{[-1]}(y) = \begin{cases} \mathbf{0}, & y \leqslant c; \\ (0, y_2), & y_1 \leqslant c_1,\ y_2 > c_2; \\ (y_1, 0), & y_1 > c_1,\ y_2 \leqslant c_2; \\ y, & y_1 > c_1, y_2 > c_2. \end{cases}$$

Thus strict inequality in $g^{[-1]}(y) \leqslant y$ can occur. We have $\left(\operatorname{epi}(g^{[-1]})\right)^{-1} = \operatorname{hypo} g$. □

Example 3.4. Let now L be $\{0, 1\}^2$ with the coordinatewise order, and let g be as in Example 3.1. We choose $c = (1, 0)$ and denote $(0, 1)$ by a so that L consists of the four element $\mathbf{0} = (0, 0)$, $a = (0, 1)$, $c = (1, 0)$, and $\mathbf{1} = (1, 1)$. From Example 3.1 we know that $g_{[-1]}(y) = y$ if $y \geqslant c$ and $g(y) = \mathbf{0}$ otherwise. Thus

$$g_{[-1]}(y) = \begin{cases} \mathbf{0}, & y = \mathbf{0}; \\ \mathbf{0}, & y = a; \\ c, & y = c; \\ \mathbf{1}, & y = \mathbf{1}. \end{cases}$$

We find that

$$\left(\operatorname{hypo} g_{[-1]}\right)^{-1} \smallsetminus \operatorname{epi} g = \{(\mathbf{0}, \mathbf{0}), (\mathbf{0}, a)\} \neq \varnothing.$$

Thus g is not residuated.

We also find that

$$g^{[-1]}(y) = \begin{cases} \mathbf{0}, & y = \mathbf{0}; \\ a, & y = a; \\ \mathbf{0}, & y = c; \\ a, & y = \mathbf{1}. \end{cases}$$

The infimum is in all cases a minimum, meaning that $g^{[-1]}$ is dually residuated, in other words, $(g^{[-1]}, g)$ is a Galois connection. We have

$$\left(\operatorname{epi} g^{[-1]}\right)^{-1} = \operatorname{hypo} g = L^2 \smallsetminus \{(\mathbf{0}, a), (\mathbf{0}, \mathbf{1}), (c, a), (c, \mathbf{1})\}. \quad \square$$

If, given a mapping $g\colon L \to Y$, we can find a mapping $u\colon Y \to L$ such that $\operatorname{epi} u = (\operatorname{hypo} g)^{-1}$ we would be content to have a kind of inverse to g. However, usually the best we can do is to study mappings with $\operatorname{epi} u \supset (\operatorname{hypo} g)^{-1}$ or $\operatorname{epi} v \subset (\operatorname{hypo} g)^{-1}$. This we shall do in the following proposition, which shows that the upper and lower inverses are solutions to certain extremal problems.

Proposition 3.1. *Let L be a complete lattice, Y a preordered set, and let $g\colon L \to Y$, $u, v\colon Y \to L$ be mappings. If $\operatorname{epi} u \supset (\operatorname{hypo} g)^{-1} \supset \operatorname{epi} v$, then $u \leqslant g^{[-1]} \leqslant v$ and*

$$\operatorname{epi} u \supset \operatorname{epi} g^{[-1]} \supset (\operatorname{hypo} g)^{-1} \supset \operatorname{epi} v.$$

Hence $g^{[-1]}$ is the largest mapping u such that $\operatorname{epi} u$ contains $(\operatorname{hypo} g)^{-1}$. Similarly, if $\operatorname{hypo} u \subset (\operatorname{epi} g)^{-1} \subset \operatorname{hypo} v$, then $u \leqslant g_{[-1]} \leqslant v$ and

$$\operatorname{hypo} u \subset (\operatorname{epi} g)^{-1} \subset \operatorname{hypo} g_{[-1]} \subset \operatorname{hypo} v.$$

Hence $g_{[-1]}$ is the smallest mapping v which satisfies $\operatorname{hypo} v \supset (\operatorname{epi} g)^{-1}$.

Corollary 3.1. *With g, u and v given as in the proposition, assume that $(\operatorname{epi} u)^{-1} = \operatorname{hypo} g$. Then $u = g^{[-1]}$. Similarly, if $(\operatorname{hypo} v)^{-1} = \operatorname{epi} g$, then $v = g_{[-1]}$. If also Y is a complete lattice, then $\operatorname{epi} u = (\operatorname{hypo} g)^{-1}$ implies that $u_{[-1]} = g$ in addition to $u = g^{[-1]}$. Similarly, $\operatorname{hypo} v = (\operatorname{epi} g)^{-1}$ implies $v^{[-1]} = g$ in addition to $v = g_{[-1]}$.*

The corollary singles out the special case of adjunctions between L and Y among all pairs $\left(g, g_{[-1]}\right)$ and adjunctions between Y and L among all pairs $\left(g^{[-1]}, g\right)$.

An ideal inverse u would satisfy $u \circ g = \mathsf{id}_L$, $g \circ u = \mathsf{id}_Y$, and the inverse of u would be g. It is therefore natural to compare $g^{[-1]} \circ g$ and $g_{[-1]} \circ g$ with id_L; $g \circ g^{[-1]}$ and $g \circ g_{[-1]}$ with id_Y; and $\left(g_{[-1]}\right)^{[-1]}$ and $\left(g^{[-1]}\right)_{[-1]}$ with g. This is what we shall do now.

Left inverses

We shall now investigate to what extent $g^{[-1]}$ and $g_{[-1]}$ can serve as left inverses to g.

Proposition 3.2. *Suppose that L is a complete lattice and Y a preordered set. Then for all mappings $g\colon L \to Y$ one has $g^{[-1]} \circ g \leqslant \mathsf{id}_L \leqslant g_{[-1]} \circ g$. The following three conditions are equivalent:*
(α) g is coincreasing;
(β) $g^{[-1]} \circ g = \mathsf{id}_L$;

(γ) $g_{[-1]} \circ g = \mathsf{id}_L$.

Corollary 3.2. *Let L be a complete lattice and Y a preordered set. Then $g^{[-1]}(y) \leqslant g_{[-1]}(y)$ for all $y \in \operatorname{im} g$, and also for all y majorizing or minorizing $\operatorname{im} g$. In particular, $g^{[-1]} \leqslant g_{[-1]}$ if g is surjective.*

Proposition 3.3. *If u, v are increasing mappings such that $u \circ g \leqslant \mathsf{id}_L \leqslant v \circ g$, then $u \leqslant g^{[-1]}$ and $v \geqslant g_{[-1]}$. Hence, in view of Proposition 3.2, $g^{[-1]}$ is the largest increasing mapping u such that $u \circ g \leqslant \mathsf{id}_L$, and $g_{[-1]}$ is the smallest increasing mapping v such that $v \circ g \geqslant \mathsf{id}_L$.*

Theorem 3.1. *Let L be a complete lattice and Y a preordered set. Then the following six conditions are equivalent.*
(a) *g is coincreasing;*
(b) *$g^{[-1]} \circ g \geqslant \mathsf{id}_L$;*
(c) *$g^{[-1]} \circ g = \mathsf{id}_L$;*
(d) *$g_{[-1]} \circ g \leqslant \mathsf{id}_L$;*
(e) *$g_{[-1]} \circ g = \mathsf{id}_L$;*
(f) *$g_{[-1]} \leqslant g^{[-1]}$.*

Right inverses

Next we compose $g_{[-1]}$ with g in the other order: we shall see to what extent the inverses we have constructed can serve as right inverses. This will lead to a characterization of dilations—and, by duality, of erosions.

Theorem 3.2. *If L and M are complete lattices and $g\colon L \to M$ is any mapping, then the following five properties are equivalent.*
(A) *g is a dilation;*
(B) *$\big(\operatorname{hypo}(g_{[-1]})\big)^{-1} \subset \operatorname{epi} g$;*
(C) *$\big(\operatorname{hypo}(g_{[-1]})\big)^{-1} = \operatorname{epi} g$;*
(D) *g is increasing and $\big(\operatorname{graph}(g_{[-1]})\big)^{-1} \subset \operatorname{epi} g$;*
(E) *g is increasing and $g \circ g_{[-1]} \leqslant \mathsf{id}_M$.*

This theorem characterizes the special case when the supremum in (2) is a maximum (property (E)); equivalently, it characterizes the special case of residuated mappings or Galois connections (property (C)).

By duality we get a similar characterization of erosions; equivalently of the case when the infimum defining the upper inverse is a minimum.

Singer (1997:178, Proposition 5.3) [16] proves that (A) and (E) are equivalent (expressed for dualities).

Corollary 3.3. *If L and M are complete lattices and $g\colon L \to M$ and $u\colon M \to L$ are two mappings such that $\operatorname{epi} g = (\operatorname{hypo} u)^{-1}$, then u is a dilation and g is an erosion, and $g_{[-1]} = u$, $u^{[-1]} = g$.*

Inverses of inverses

Theorem 3.3. *If L and M are complete lattices and $g\colon L \to M$ is any mapping, then quite generally $\left(g_{[-1]}\right)^{[-1]} \leqslant g \leqslant \left(g^{[-1]}\right)_{[-1]}$. Equality holds at the first place if and only if g is a dilation; at the second place if and only if g is an erosion.*

Theorem 3.4. *If L and M are complete lattices and $\delta\colon L \to M$ is a dilation, then $\delta_{[-1]}\colon M \to L$ is an erosion. Similarly, if $\varepsilon\colon L \to M$ is an erosion, then $\varepsilon^{[-1]}$ is a dilation.*

Corollary 3.4. *For any dilation $\delta\colon L \to M$ we have $\delta \circ \delta_{[-1]} \circ \delta = \delta$ and $\delta_{[-1]} \circ \delta \circ \delta_{[-1]} = \delta_{[-1]}$. In particular, $\delta_{[-1]} \circ \delta$ and $\delta \circ \delta_{[-1]}$ are idempotent and therefore ethmomorphisms. The first is a cleistomorphism in L, the second an anoiktomorphism in M. Dually $\varepsilon \circ \varepsilon^{[-1]} \circ \varepsilon = \varepsilon$ and $\varepsilon^{[-1]} \circ \varepsilon \circ \varepsilon^{[-1]} = \varepsilon^{[-1]}$ for any erosion $\varepsilon\colon L \to M$. Also $\varepsilon^{[-1]} \circ \varepsilon$ and $\varepsilon \circ \varepsilon^{[-1]}$ are idempotent; the first an anoiktomorphism, the second a cleistomorphism.*

4. Division of mappings

We shall now generalize the definitions of upper and lower inverses.

Definition 4.1. Let a set X, a complete lattice M, and a preordered set Y, as well as two mappings $f\colon X \to M$ and $g\colon X \to Y$ be given. We define two mappings $f/^{\star}g, f/_{\star}\, g\colon Y \to M$ by

$$(f/^{\star}g)(y) = \bigwedge_{x \in X} (f(x); g(x) \geqslant_Y y), \qquad y \in Y,$$

$$(f/_{\star}\, g)(y) = \bigvee_{x \in X} (f(x); g(x) \leqslant_Y y), \qquad y \in Y.$$

We shall call them the *upper quotient* and the *lower quotient* of f and g.

We shall often assume that X, M and Y are all complete lattices, but this is not necessary for the definitions to make sense.

The quotients $f/^{\star}g$ and $f/_{\star}\, g$ increase when f increases and they decrease when g increases—just as with division of positive numbers:

$$\text{If } f_1 \leqslant_M f_2 \text{ and } g_1 \geqslant_Y g_2, \text{ then } f_1/^{\star}g_1 \leqslant_M f_2/^{\star}g_2 \text{ and } f_1/_{\star}\, g_1 \leqslant_M f_2/_{\star}\, g_2.$$

The mappings $f/^{\star}g$ and $f/_{\star}\, g$ are always increasing. If $g(x) \geqslant_Y y$, then $f(x) \geqslant_M (f/^{\star}g)(y)$; if $g(x) \leqslant_Y y$, then $f(x) \leqslant_M (f/_{\star}\, g)(y)$. In particular, if $g(x) = y$, then $(f/^{\star}g)(y) \leqslant_M f(x) \leqslant_M (f/_{\star}\, g)(y)$.

If we specialize the definitions to the situation when $X = M$ and $f = \mathsf{id}_X$, then $f/^{\star}g = \mathsf{id}_X/^{\star}g = g^{[-1]}$ and $f/_{\star}\, g = \mathsf{id}_X/_{\star}\, g = g_{[-1]}$; cf. Definition 3.1.

We note another special case:

Proposition 4.1. *For all mappings $f\colon X \to M$ we have*

$$f /_{\star} f \leqslant \mathsf{id}_M \leqslant f /^{\star} f$$

and

$$(f /_{\star} f) \circ f = f = (f /^{\star} f) \circ f. \tag{4}$$

Proposition 4.2. *Let X be an arbitrary subset of a complete lattice M, let $Y = M$, and g the inclusion mapping $X \to M$. Then $f /^{\star} g = f^{\diamond}$ and $f /_{\star} g = f_{\diamond}$, where $f^{\diamond}$ is the largest increasing mapping $h\colon M \to M$ such that $h\big|_X$ minorizes f, i.e.,*

$$f^{\diamond}(y) = \sup_h \big(h(y); h \text{ is increasing and } h(x) \leqslant f(x) \text{ for all } x \in X\big);$$

and $f_{\diamond}$ is the smallest increasing mapping k such that $k\big|_X$ majorizes f, i.e.,

$$f_{\diamond}(y) = \inf_k \big(k(y); k \text{ is increasing and } k(x) \geqslant f(x) \text{ for all } x \in X\big).$$

If f itself is increasing, they are in fact extensions of f.

The definitions of $f^{\diamond}$ and $f_{\diamond}$ are taken from Matheron (1975:187) [9] and are generalized here to any complete lattice.

If we specialize further, letting also f be the inclusion mapping $X \to M$, we obtain

$$(f /_{\star} g)(y) = (f /_{\star} f)(y) = f_{\diamond}(y) = \bigvee_{x \in X} (x; x \leqslant y) = y^{\circ} \in M,$$

where the last equality defines y°. It is easy to verify that $y \mapsto y^{\circ}$ is an anoiktomorphism. A well-known situation is described in the following example.

Example 4.1. Let M be the complete lattice $[-\infty, +\infty]^E$ of functions on a vector space E with values in the extended reals, let F be a vector subspace of its algebraic dual $E^{\star}$ (the space of all linear forms on E), and let X be the set of all affine functions with linear part in F, i.e., functions of the form $\alpha(x) = \xi(x) + c$ for some linear form $\xi \in F$ and some real constant c. A function f such that $f^{\circ} = f$ is called X-*convex* by Singer (1997:10) [16].

We see that a function on E is X-convex in the sense of Singer if and only if it is equal to the supremum of all its affine minorants belonging to X.

We may ask for a characterization of the X-convex functions. A generalization of Fenchel's theorem to this setting gives the answer: this happens if and only if the function possesses three properties:
(a) it is convex in the usual sense;
(b) it is lower semicontinuous for the topology $\sigma(E, F)$ on E generated by the linear forms in F; and
(c) it does not take the value $-\infty$ except when it is equal to the constant $-\infty$. $\square$

Proposition 4.3. *If $f\colon X \to M$ is increasing and $g\colon X \to Y$ is coincreasing, then $f/_{\star}\, g \leqslant f/^{\star}g$.*

The upper quotient $f/^{\star}g$ is the optimal solution to an inequality:

Proposition 4.4. *For all mappings $f\colon X \to M$ and $g\colon X \to Y$ we have*

$$(f/^{\star}g) \circ g \leqslant f \leqslant (f/_{\star}\, g) \circ g,$$

with equality if f is increasing and g is coincreasing. From this we deduce that $(f/^{\star}g)(y) \leqslant (f/_{\star}\, g)(y)$ for all $y \in \operatorname{im} g$ as well as for all majorants and minorants of $\operatorname{im} g$. In particular, $f/^{\star}g \leqslant f/_{\star}\, g$ if g is surjective.

Conversely, if $u, v\colon Y \to M$ are two increasing functions such that $u \circ g \leqslant f \leqslant v \circ g$, then $u \leqslant f/^{\star}g$ and $v \geqslant f/_{\star}\, g$. Thus $f/^{\star}g$ is the largest increasing function u such that $u \circ g \leqslant f$, and $f/_{\star}\, g$ is the smallest increasing function v such that $f \leqslant v \circ g$.

In the special case $X = Y$ and $g = \mathsf{id}_X$ we obtain

$$f/^{\star}\mathsf{id}_X \leqslant f \leqslant f/_{\star}\, \mathsf{id}_X,$$

where $f/^{\star}\mathsf{id}_X$ is the largest increasing minorant of f and $f_{\star}/\mathsf{id}_X$ is the smallest increasing majorant of f; when f itself is increasing we therefore get equality.

We next compare the quotient $f/^{\star}g$ and the composition $f \circ g^{[-1]}$ (think of $x/y = x \cdot y^{-1}$ for positive numbers):

Proposition 4.5. *For every increasing mapping $f\colon X \to M$ and every mapping $g\colon X \to Y$ we have $f/^{\star}g \geqslant f \circ g^{[-1]}$ with equality if f is an erosion, and $f/_{\star}\, g \leqslant f \circ g_{[-1]}$ with equality if f is a dilation. If g is coincreasing, then $f/_{\star}\, g \leqslant f \circ g_{[-1]} \leqslant f \circ g^{[-1]} \leqslant f/^{\star}g$.*

Proposition 4.6. *If P is a preordered set and $h\colon M \to P$ is increasing, we have $h \circ (f/^{\star}g) \leqslant (h \circ f)/^{\star}g$ with equality if h is an erosion. Similarly $h \circ (f/_{\star}\, g) \geqslant (h \circ f)/_{\star}\, g$ with equality if h is a dilation. A special case is $h \circ (f/^{\star}\mathsf{id}_X) \leqslant (h \circ f)/^{\star}\mathsf{id}_X$ (take $X = Y$ and $g = \mathsf{id}_X$). Another special case is Proposition 4.5 (take $X = M$ and $f = \mathsf{id}_X$).*

Cleistomorphisms and anoiktomorphisms

Theorem 4.1. *Let $f\colon X \to M$ be any mapping from a set X into a complete lattice M. Then $\alpha = f/_{\star}\, f\colon M \to M$ is an anoiktomorphism. Conversely, any anoiktomorphism in M is of this form for some mapping $f\colon X \to M$ with $X = M$. By duality we get analogous statements for the upper quotient and cleistomorphisms.*

5. Conclusion

We have introduced the notions of upper and lower inverses and upper and lower quotients for mappings between complete lattices. Their most basic properties have been investigated, in particular how the inverses can serve as left and right inverses to a given mapping. Important morphological operators can be systematically treated in the calculus created. In particular, anoiktomorphisms are always lower quotients of the form $f/_{\star} f$, and cleistomorphisms are always upper quotients of the form $f/^{\star} f$.

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