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FORMELSAMLING
Transformer för beräkningar
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1. Notation

We shall denote the set $\{0, 1, 2, ...\}$ by $\mathbf{N}$, and the set $\{1, 2, 3, ...\}$ by $\mathbf{N}^*$.[1] Other common notation is $\mathbf{Z}$ for the ring of integers, $\mathbf{Q}$ for the field of rationals, $\mathbf{R}$ for the field of real numbers, and $\mathbf{C}$ for the field of complex numbers.

2. Bases and dimensions

2.1. An *algebraic basis* (also known as a *Hamel basis*) of a vector space E is an indexed family $(e_i)_{i\in I}$ such that every element $x \in E$ can be written in a unique way as $x = \sum_{i\in I} x_i e_i$ with only finitely many coordinates x_i different from zero. If there exists a finite basis, then all bases have the same number of elements, and this number is called the *dimension* of the space.

2.2. If $f: V \to W$ is a linear mapping of one vector space V into another, called W, then $\dim \ker f + \dim \operatorname{im} f = \dim V$, with a natural interpretation even if some of the dimensions are infinite (two or three).

3. Normed spaces and inner product spaces

3.1. A *norm* on a vector space E is a function $E \ni x \mapsto \|x\| \in \mathbf{R}$ such that (i) $\|x\| \geqslant 0$ with equality only if $x = 0$; (ii) $\|tx\| = |t|\|x\|$ for all $x \in E$ and all scalars t; (iii) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in E$. *Examples:* $\|x\|_p = \left(\sum |x(j)|^p\right)^{1/p}$, $1 \leqslant p < +\infty$ and $\|x\|_\infty = \sup_j |x(j)|$ for vectors such that these expressions are finite. We denote by $l^p(\mathbf{Z}_N)$ the space $\mathbf{C}^N$ equipped with the norm $\|\cdot\|_p$, and by $l^p(\mathbf{Z})$ the space of all sequences $(z(n))_n$ such that $\|z\|_p$ is finite. The space $c_0(\mathbf{Z})$ of all sequences $z = (z(n))_{n\in\mathbf{Z}}$ tending to zero as $n \to \pm\infty$ is a closed subspace of $l^\infty(\mathbf{Z})$. We have

$$l^1(\mathbf{Z}) \subset l^2(\mathbf{Z}) \subset c_0(\mathbf{Z}) \subset l^\infty(\mathbf{Z}).$$

3.2. An *inner product* on a vector space E is a sesquilinear form $E \times E \ni (x, y) \mapsto \langle x, y\rangle$ such that $\langle x, x\rangle \geqslant 0$ with equality only for $x = 0$. (Another common notation is $(x \mid y)$.) Then $\sqrt{\langle x, x\rangle}$ is a norm. *Example:* $\langle x, y\rangle = \sum x(j)\overline{y(j)}$ for $x, y \in l^2(\mathbf{Z}_N)$. If $(e_i)_{i\in I}$ with I finite is an orthonormal basis, then $z = \sum_{i\in I}\langle z, e_i\rangle e_i$. In $l^2(\mathbf{Z}_N)$ the family $(\delta_k)_{k=0}^{N-1}$ of all Dirac functions forms an ON-basis ($\delta_k(j) = 1$ if $j = k$ and zero otherwise)

[1] Michael W. Frazier, in his book *An Introduction to Wavelets Through Linear Algebra.* Springer, 1999, on which this *Formelsamling* is based, writes $\mathbf{N}$ for $\{1, 2, 3, ...\}$. Here I use $\mathbf{N}$ and $\mathbf{N}^*$ according to the more usual convention. Otherwise I follow Frazier closely.

4. Bases in a normed space

4.1. A *Schauder basis* in a normed space E is an indexed family $(e_i)_{i\in I}$ such that every element can be written in a unique way as a convergent sum $z = \sum_{i\in I} z_i e_i$ for some scalars z_i. The convergence means here that for every positive ε there exists a finite set $J_\varepsilon \subset I$ such that $\|z - \sum_{i\in J} z_i e_i\| < \varepsilon$ for all finite subsets $J \supset J_\varepsilon$. If $(e_i)_{i\in I}$ is an orthonormal Schauder basis in an inner-product space, then $z = \sum_{i\in I}\langle z, e_i\rangle e_i$ with convergence in norm, i.e., the coordinates are just $z_i = \langle z, e_i\rangle$. The family of all Dirac functions $(\delta_k)_{k\in\mathbf{Z}}$ form a Schauder basis (not an algebraic basis) in the spaces $l^p(\mathbf{Z})$, $1 \leqslant p < \infty$, as well as in $c_0(\mathbf{Z})$. For $p = 2$ they form an ON-basis. (In an infinite-dimensional space, an algebraic basis must have many more elements than a Schauder basis.) An orthogonal indexed family $(e_i)_{i\in I}$ is said to be *complete* if it is a basis, which is equivalent to saying that the closure of the vector space spanned by all finite linear combinations of the e_i is equal to the whole space.

5. Convolution on a finite group

5.1. On any finite group G we may define

$$(f * g)(z) = \sum f(x)g(y),$$

where the sum is extended over all elements $x, y \in G$ such that $xy = z$. If the group is commutative it is customary to write the group operation as addition, thus

$$(f * g)(z) = \sum_{y\in G} f(z-y)g(y), \qquad z \in G.$$

Convolution is commutative and associative if the group is finite and commutative.

5.2. We consider the finite cyclic group $\mathbf{Z}_N = \mathbf{Z} \bmod N = \mathbf{Z}/N\mathbf{Z}$. A function on that space can be represented by its values $(z(0), z(1), ..., z(N-1))$, but equally well by its values $(z(1), z(2), ..., z(N))$ or any other period. Convolution of two functions or vectors is defined as

$$(z * w)(n) = \sum_j z(j)w(n-j), \qquad n \in \mathbf{Z}_N,$$

where it is understood that summation is extended over a period, the vectors being periodic functions. The *translation operator* R_k is defined by $(R_k z)(n) = z(n-k)$. We have $z * \delta_k = R_k z$, where δ_k is the Dirac delta placed at position k.

5.3. Any linear translation-invariant operator $T\colon l^2(\mathbf{Z}_N) \to l^2(\mathbf{Z}_N)$ is given by convolution by some vector b:

$$T(z) = b * z, \qquad z \in l^2(\mathbf{Z}_N).$$

6. The discrete Fourier transform

6.1. For $z \in l^2(\mathbf{Z}_N)$ we define its Fourier[2] transform

$$\hat{z}(n) = \sum_j z(j)\omega^{jn} = \sum_j z(j)e^{-2\pi ijn/N}, \quad n \in \mathbf{Z}_N.$$

[2]Jean-Baptiste-Joseph Fourier, 1768–1830.

Here $\omega = e^{-2\pi i/N}$ is a root of unity: $\omega^N = 1$. It is easy to express the Fourier transform of a convolution product in terms of that of the factors:

$$\widehat{z * w}(n) = \hat{z}(n)\widehat{w}(n), \qquad n \in \mathbf{Z}_N.$$

6.2. For any $z \in l^2(\mathbf{Z}_N)$, we define $\tilde{z} \in l^2(\mathbf{Z}_N)$ by $\tilde{z}(n) = \overline{z(-n)}$. We note that $\widehat{\tilde{z}} = \overline{\hat{z}}$, that $\widehat{\overline{z}} = \widetilde{\hat{z}}$, and that $(z * \widetilde{w})(k) = \langle z, R_k w\rangle$.

6.3. Any linear operator $T\colon l^2(\mathbf{Z}_N) \to l^2(\mathbf{Z}_N)$ which commutes with translation is diagonalized by the Fourier transformation in that we have

$$\widehat{T(z)} = \hat{b} \cdot \hat{z}, \qquad z \in l^2(\mathbf{Z}_N),$$

for some $b \in l^2(\mathbf{Z}_N)$.

7. Wavelets on a finite cyclic group

7.1. *Downsampling and upsampling.* We define $D\colon l^2(\mathbf{Z}_N) \to l^2(\mathbf{Z}_M)$, where $N = 2M$, by $D(z)(n) = z(2n)$, $n \in \mathbf{Z}_M$. We define $U\colon l^2(\mathbf{Z}_M) \to l^2(\mathbf{Z}_N)$ by $U(z)(n) = z(n/2)$ when n is even, $U(z) = 0$ when n is odd. These are the *downsampling* and *upsampling operators*; $D \circ U$ is the identity in $l^2(\mathbf{Z}_M)$. We note that U is a homomorphism between two convolution algebras: it is linear and $U(z * w) = U(z) * U(w)$, while in general $D(z * w) \neq D(z) * D(w)$; however there is equality here when $w = U(x)$ for some x.

7.2. A *first stage wavelet basis* is an orthonormal sequence of vectors $R_{2k}u$, $R_{2k}v$, $k = 0, 1..., M-1$, where $u, v \in l^2(\mathbf{Z}_N)$, $N = 2M$.

7.3. If N is even, $N = 2M$, and $u, v \in l^2(\mathbf{Z}_N)$, we define the *system matrix* $A(n)$ of u and v by

$$A(n) = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{u}(n) & \hat{v}(n) \\ \hat{u}(n+M) & \hat{v}(n+M) \end{pmatrix}.$$

Then the N vectors $R_k v$, $R_k u$, $k = 0, ..., M-1$, form an orthonormal system if and only if $A(n)$ is unitary for every n.

7.4. An arbitrary signal z can be reconstructed from $U(D(z * \tilde{v}))$ and $U(D(z * \tilde{u}))$ as

$$\tilde{t} * U(D(z * \tilde{v})) + \tilde{s} * U(D(z * \tilde{u})) = z$$

if and only if

$$A(n) \begin{pmatrix} \hat{s}(n) \\ \hat{t}(n) \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \qquad \text{for all } n.$$

7.5. *The first stage Shannon*[3] *basis.* Here the basis vectors are defined by

$$\hat{u}(n) = \begin{cases} \sqrt{2}, & n = 0, 1, ..., \frac{1}{4}N - 1 \text{ or } n = \frac{3}{4}N, \frac{3}{4}N + 1, ..., N-1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\hat{v}(n) = \begin{cases} \sqrt{2}, & n = \frac{1}{4}N, \frac{1}{4}N + 1, ..., \frac{3}{4}N - 1, \\ 0 & \text{otherwise,} \end{cases}$$

[3] Claude Elwood Shannon, b. 1916 04 30 – 2001 02 24.

assuming that N is divisible by 4.

7.6. *The first stage Haar*[4] *basis.* Here the basis vectors are

$$u(n) = \begin{cases} \frac{1}{\sqrt{2}}, & n = 0, 1, \\ 0 & \text{otherwise;} \end{cases}$$
$$v(n) = \begin{cases} \frac{1}{\sqrt{2}}, & n = 0, \\ -\frac{1}{\sqrt{2}}, & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

7.7. A p^{th} *stage wavelet filter sequence* is a sequence of vectors $u_1, v_1, u_2, v_2, ..., u_p, v_p$, such that, for each $j = 1, 2, ..., p$, we have $u_j, v_j \in l^2(\mathbf{Z}_{N/2^{j-1}})$ and the system matrix

$$A_j(n) = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{u}_j(n) & \hat{v}_j(n) \\ \hat{u}_j(n + \frac{1}{2}N) & \hat{v}_j(n + \frac{1}{2}N) \end{pmatrix}$$

is unitary for all $n = 0, 1, ..., N/2^j - 1$. For any input z we define $x_1 = D(z * \tilde{v}_1)$, $y_1 = D(z * \tilde{u}_1)$, $x_j = D(y_{j-1} * \tilde{v}_j) \in l^2(\mathbf{Z}_{N/2^j})$, $y_j = D(y_{j-1} * \tilde{u}_j) \in l^2(\mathbf{Z}_{N/2^j})$. The *output of the analysis phase* is the sequence of vectors $x_1, x_2, ..., x_p, y_p$.

7.8. Suppose N is divisible by 2^p and let $u_1, v_1, u_2, v_2, ..., u_p, v_p$ be a p^{th}-stage wavelet filter sequence. Define $f_1, f_2, ..., f_p, g_1, g_2, ..., g_p$ by

$$f_j = g_{j-1} * U^{j-1}(v_j), \qquad g_j = g_{j-1} * U^{j-1}(u_j).$$

Then $f_1, f_2, ..., f_p, g_p$ generate a p^{th}*-stage wavelet basis* for $l^2(\mathbf{Z}_N)$, i.e.,

$$R_{2kf_1},\ k = 0, ..., \tfrac{1}{2}N - 1;\ R_{4k}f_2,\ k = 0, ..., \tfrac{1}{4}N - 1; \ldots;\ R_{2^pk}f_j,\ k = 0, ..., N/2^p - 1,$$
$$R_{2^pk}g_p,\ k = 0, ..., N/2^p - 1,$$

form an orthonormal basis for $l^2(\mathbf{Z}_N)$. With the notation

$$\psi_{-j,k} = R_{2^jk}f_j, \quad \varphi_{-j,k} = R_{2^jk}g_j, \qquad j = 1, 2, ..., p,\ k = 0, 1, ..., N/2^j - 1,$$

the p^{th}-stage wavelet basis generated by $f_1, f_2, ..., f_p, g_p$ is

$$\psi_{-1,k},\ k = 0, ..., \tfrac{1}{2}N - 1; \quad \psi_{-2,k},\ k = 0, ..., \tfrac{1}{4}N - 1; ...;\ \psi_{-p,k},\ k = 0, ..., N/2^p - 1;$$
$$\varphi_{-p,k},\ k = 0, ..., N/2^p - 1.$$

We see that the space V_{-j} spanned by $\varphi_{-j,0}, ..., \varphi_{-j,N/2^j-1}$ and the space W_{-j} spanned by $\psi_{-j,0}, ..., \psi_{-j,N/2^j-1}$ satisfy $V_{-j} = V_{-j-1} + W_{-j-1}$, while $V_0 = l^2(\mathbf{Z}_N)$.

8. Convolution on the integers

8.1. The convolution product of two vectors z, w is defined by

$$(z * w)(n) = \sum_{j \in \mathbf{Z}} z(j)w(n - j), \qquad n \in \mathbf{Z},$$

[4] Alfred Haar, 1885 10 11 – 1933 03 16.

whenever the sum has a sense. The following cases are noteworthy. If one of the factors has only finitely many values different from zero, then the convolution product always exists. If both factors are zero for large negative values of the index, i.e., if $z(j) = w(j) = 0$ when $j \ll 0$, then the convolution product exists. If one of the factors is in $l^1(\mathbf{Z})$ and the other is in $l^p(\mathbf{Z})$, then then the convolution product exists and belongs to $l^p(\mathbf{Z})$. In particular, $l^1(\mathbf{Z})$ is a convolution algebra. If both factors belong to $l^2(\mathbf{Z})$, then the convolution exists and belongs to $c_0(\mathbf{Z})$. Connected with these statements are the inequalities

$$\|z * w\|_p \leqslant \|z\|_1 \cdot \|w\|_p \text{ for } 1 \leqslant p \leqslant +\infty; \quad \|z * w\|_\infty \leqslant \|z\|_2 \cdot \|w\|_2.$$

Convolution is commutative and associative under reasonable hypotheses. Note, however, examples like this:

$$1 = 1 * \delta = 1 * ((\delta_0 - \delta_1) * h) \neq (1 * (\delta_0 - \delta_1)) * h = 0 * h = 0,$$

where $h(n) = 0$ for $n < 0$, $h(n) = 1$ for $n \geqslant 0$. Here all convolutions listed in the formula do exist, but the convolution $1 * h$ does not.

8.2. *Support.* The support of a sequence $(z(n))_{n\in\mathbf{Z}}$ is the set of all indices n such that $z(n) \neq 0$. We denote this set by $\operatorname{supp} z$. It is clear that $\operatorname{supp}(z*w) \subset \operatorname{supp} z + \operatorname{supp} w$, but the inclusion may be strict. If we denote by $K(z)$ the smallest interval which contains $\operatorname{supp} z$, then we have an equality $K(z * w) = K(z) + K(w)$ valid for all vectors z, w with finite support. Actually $K(z) = \operatorname{cvx} \operatorname{supp} z$; see Chapter 11.

9. The Fourier transformation on the integers

9.1. For $z \in l^p(\mathbf{Z})$ we define its *Fourier transform* as the periodic function

$$\hat{z}(t) = \sum z(n)e^{int}, \qquad t \in \mathbf{R}.$$

The sum has to be interpreted as a limit in $L^2(I)$, $I = [-\pi, \pi[$, of finite sums. However, if $z \in l^1(\mathbf{Z})$, then the sum is absolutely convergent and the transform $\hat{z}$ is a continuous function.

9.2. The inverse Fourier transform $\check{f}$ of a function f on I is the sequence

$$\check{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int} dt.$$

If $f \in L^2(I)$, then $\check{f} \in l^2(\mathbf{Z})$; if $f \in L^1(I)$, then $\check{f} \in c_0(\mathbf{Z})$.

9.3. The Fourier transform of a convolution product is given by the usual formula $\widehat{z * w} = \hat{z}\widehat{w}$, at least if $z \in l^1(\mathbf{Z})$ and $w \in l^2(\mathbf{Z})$. (Distribution theory enables us to weaken these assumptions.)

10. Wavelets on the group of integers

10.1. *Downsampling and upsampling on* $\mathbf{Z}$. The downsampling operator $D\colon l^2(\mathbf{Z}) \to l^2(\mathbf{Z})$ and the upsampling operator $U\colon l^2(\mathbf{Z}) \to l^2(\mathbf{Z})$ are defined on $l^2(\mathbf{Z})$ by the same formulas as in the finite-dimensional case.

10.2. *A first-stage wavelet system* for $l^2(\mathbf{Z})$ is a complete orthonormal family $R_{2k}v$, $R_{2k}u$, $k \in \mathbf{Z}$, where $u, v \in l^1(\mathbf{Z})$.

10.3. *The system matrix* of two vectors $u, v \in l^2(\mathbf{Z})$ is

$$A(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{u}(t) & \hat{v}(t) \\ \hat{u}(t+\pi) & \hat{v}(t+\pi) \end{pmatrix}.$$

If $u, v \in l^1(\mathbf{Z})$, then $R_{2k}v$, $R_{2k}u$, $k \in \mathbf{Z}$, is a first-stage wavelet system if and only if $A(t)$ is unitary for all $t \in \mathbf{R}$.

10.4. Assume that $u, v \in l^1(\mathbf{Z})$ and that their system matrix $A(t)$ is unitary for all t. Let an element $g_{j-1} \in l^2(\mathbf{Z})$ be given such that $(R_{2k})_{k\in\mathbf{Z}}$ is orthonormal. Define

$$f_j = g_{j-1} * U^{j-1}(v), \qquad g_j = g_{j-1} * U^{j-1}(u).$$

Then the system $(R_{2^jk}f_j)_{k\in\mathbf{Z}}, (R_{2^jk}g_j)_{k\in\mathbf{Z}}$ is orthonormal.

10.5. Now define

$$V_{-j} = \left\{\textstyle\sum_{k\in\mathbf{Z}} z(k)R_{2^jk}g_j; z \in l^2(\mathbf{Z})\right\},$$
$$W_{-j} = \left\{\textstyle\sum_{k\in\mathbf{Z}} z(k)R_{2^jk}f_j; z \in l^2(\mathbf{Z})\right\}.$$

Then $V_{-j} + W_{-j} = V_{-j+1}$, and the sum is orthogonal.

10.6. A p^{th}*-stage wavelet system for* $l^2(\mathbf{Z})$ is an indexed family

$$(R_{2^jk}f_j, R_{2^jk}g_p)_{k\in\mathbf{Z}, j=1,\dots,p}$$

which is orthonormal and complete.

10.7. Let $u_j, v_j \in l^1(\mathbf{Z})$, $j = 1, \dots, p$, and assume that the system matrix $A_j(t)$ of u_j and v_j is unitary for all $j = 1, \dots, p$ and all $t \in \mathbf{R}$. Define $f_j = v_1$, $g_j = u_1$ and inductively $f_j = g_{j-1} * U^{j-1}(v_j)$, $g_j = g_{j-1} * U^{j-1}(u_j)$. Then the system

$$(R_{2^jk}f_j, R_{2^jk}g_p)_{k\in\mathbf{Z}, j=1,\dots,p}$$

is a p^{th}-stage wavelet system for $l^2(\mathbf{Z})$.

10.8. A *homogeneous wavelet system* for $l^2(\mathbf{Z})$ is an indexed family $(R_{2^jk}f_j)_{k\in\mathbf{Z}, j\in\mathbf{N}^*}$ which is orthonormal and complete.

10.9. Suppose that $u_j, v_j \in l^1(\mathbf{Z})$, $j \in \mathbf{N}$, and that the system matrix $A_j(t)$ is unitary for all $t \in \mathbf{R}$. Define f_j and g_j as in 10.7 and V_{-j} as in 10.5. Then if the intersection of all the V_{-j} consists of the origin only, the system $(R_{2^jk}f_j)_{k\in\mathbf{Z}, j\in\mathbf{N}^*}$ is a homogeneous wavelet system.

10.10. *The Haar wavelets on* $\mathbf{Z}$. Define $u, v \in l^1(\mathbf{Z})$ by

$$u(n) = \begin{cases} \frac{1}{\sqrt{2}}, & n = 0, 1, \\ 0 & \text{otherwise}; \end{cases}$$
$$v(n) = \begin{cases} \frac{1}{\sqrt{2}}, & n = 0, \\ -\frac{1}{\sqrt{2}}, & n = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Then the pair u, v generates a first-stage wavelet system for $l^2(\mathbf{Z})$, and the intersection of all the spaces V_{-j}, defined as in 10.7, consists of the zero vector only. Hence $(R_{2^j k} f_j)_{k\in\mathbf{Z}, j\in\mathbf{N}^*}$ is a homogeneous wavelet system, called the *Haar system for* $l^2(\mathbf{Z})$.

10.11. *Daubechies*[5] *D6 wavelets on* $\mathbf{Z}$ are defined by taking $u(n) = 0$ for $n \neq 0, 1, 2, 3, 4, 5$, and then

$$\begin{aligned}&(u(0), u(1), u(2), u(3), u(4), u(5))\\ &\qquad = \frac{\sqrt{2}}{32}(b + c, 2a + 3b + 3c, 6a + 4b + 2c, 6a + 4b - 2c, 2a + 3b - 3c, b - c),\end{aligned}$$

where

$$a = 1 - \sqrt{10},\ b = 1 + \sqrt{10},\ c = \sqrt{5 + 2\sqrt{10}},$$

and then v defined by $v(-4) = -u(5)$, $v(-3) = u(4)$, $v(-2) = -u(3)$, $v(-1) = u(2)$, $v(-1) = u(2)$, $v(0) = -u(1)$, $v(1) = u(0)$, all other values being zero.

11. Convolution on the real axis

11.1. *Support.* The *support* of a function f defined on the real axis is the closure of the set of all points where the function is nonzero; it is denoted by $\operatorname{supp} f$. It is easy to prove that $\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g$. It is a deep theorem, called the *Titchmarsh*[6] *support theorem,* that $\operatorname{cvx}\operatorname{supp}(f * g) = \operatorname{cvx}\operatorname{supp} f + \operatorname{cvx}\operatorname{supp} g$ if both $\operatorname{supp} f$ and $\operatorname{supp} g$ are bounded. Here $\operatorname{cvx} A$ denotes the convex hull of a set A; the theorem is valid also in $\mathbf{R}^n$. On the real axis, $\operatorname{cvx}\operatorname{supp} f$ is the smallest closed interval outside of which the function vanishes.

12. The Fourier transformation on the real axis

12.1. For a reasonable function $f\colon \mathbf{R} \to \mathbf{C}$ on the real axis we define its *Fourier transform* $\hat{f}$ as

$$\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-i\xi x} dx, \qquad \xi \in \mathbf{R}.$$

The *inverse Fourier transform* of a function g is

$$\check{g}(x) = \frac{1}{2\pi} \int_{\mathbf{R}} g(\xi) e^{i\xi x} d\xi, \qquad x \in \mathbf{R}.$$

12.2. Under some hypotheses we have $(\hat{f})\check{} = f$. In particular this is true if both f and its transform $\hat{f}$ are integrable on the real axis.

12.3. If f, g are in $L^2(\mathbf{R})$, the space of all square-integrable functions, then $\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle$ (Parseval's relation); in particular $\|\hat{f}\|_2 = \sqrt{2\pi}\|f\|_2$ (Plancherel's formula).

[5] Ingrid Daubechies, b. 1954.

[6] Edward Charles Titchmarsh, 1899 06 01 – 1963 01 18.

13. Wavelets on the real axis

13.1. A *wavelet system for* $L^2(\mathbf{R})$ is a complete orthonormal system of the form $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$, for some $\psi \in L^2(\mathbf{R})$, where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $x \in \mathbf{R}$.

13.2. A *multiresolution analysis* with scaling function φ is an indexed family $(V_j)_{j\in\mathbf{Z}}$ of subspaces of $L^2(\mathbf{R})$ such that (i) $V_j \subset V_{j+1}$, $j \in \mathbf{Z}$; (ii) There exists a function $\varphi \in V_0$ (the *scaling function*) such that the family $(\varphi_{0,k})_{k\in\mathbf{Z}}$ is orthonormal and $V_0 = \big\{\sum_{k\in\mathbf{Z}} \varphi_{0,k}; z \in l^2(\mathbf{Z})\big\}$; (iii) $f \in V_0$ if and only if $x \mapsto f(2^j x)$ is in V_j; (iv) $\bigcap_j V_j = \{0\}$; (v) The closure of $\bigcup_j V_j$ is equal to all of $L^2(\mathbf{R})$.

13.3. *The Haar multiresolution analysis* consists of the spaces V_j of functions which are constant on the interval $\big[2^{-j}k, 2^{-j}(k+1)\big[$ for every $k \in \mathbf{Z}$. Its scaling function is the characteristic function of the interval $[0, 1[$.

13.4. The *scaling sequence* of a multiresolution analysis (V_j) is the sequence $u \in l^2(\mathbf{Z})$ such that $\varphi = \sum_{k\in\mathbf{Z}} u(k)\varphi_{1,k}$. It follows that $u(k) = \langle \varphi, \varphi_{1,k}\rangle$. The indexed family $(R_{2k}u)_{k\in\mathbf{Z}}$ is orthonormal in $l^2(\mathbf{Z})$.

13.5. Suppose $(V_j)_{j\in\mathbf{Z}}$ is a multiresolution analysis with scaling function φ and such that its scaling sequence u belongs to $l^1(\mathbf{Z})$. Define $v \in l^1(\mathbf{Z})$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$, $k \in \mathbf{Z}$, and $\psi = \sum_{k\in\mathbf{Z}} v(k)\varphi_{1,k}$. Then $(\psi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal family in $L^2(\mathbf{R})$. If we define $W_0 = \big\{\sum z(k)\psi_{0,k}; z \in l^2(\mathbf{Z})\big\}$, then $V_1 = V_0 + W_0$, and the two spaces V_0 and W_0 are orthogonal. Finally, $(\psi_{j,k})_{j,k\in\mathbf{Z}}$ is a wavelet system in $L^2(\mathbf{R})$. (Mallat's theorem.)

14. Orthogonal polynomials

14.1. *General.* We consider an inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\rho(x)dx,$$

defined on functions on an interval $[a, b]$, where $\rho \geqslant 0$ is called a *weight function.* A well-known example of an orthogonal sequence on $[0, \pi]$ with weight function $\rho(x) = 1$ is f_n defined by $f_n(x) = \cos nx$.

14.2. *Legendre*[7] *polynomials.* Here $a = -1$, $b = 1$, $\rho(x) = 1$. The polynomials are defined by

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n.$$

The first few Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_3(x) = \frac{1}{2}(5x^3-3x)$. A generating function is

$$w(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_0^\infty P_n(x)t^n.$$

The polynomial $u = P_n$ solves the equation

$$((1-x^2)u')' + n(n+1)u = 0.$$

[7]Adrien Marie Legendre, 1752–1833.

The inner product of P_n with itself is

$$\|P_n\|_\rho^2 = \langle P_n, P_n \rangle = \int_{-1}^{1} P_n^2 dx = \frac{2}{2n+1}.$$

A continuous function on $[-1, 1]$ can be expanded in a series $f(x) = \sum c_n P_n(x)$, where the coeffcients are given by $c_n = (n + \frac{1}{2})\langle f, P_n \rangle$. The Legendre polynomials can be used to define a solution to the Laplace equation in a ball in case the boundary values are independent of the longitude: the function

$$u(r, \theta) = \sum r^n a^{-n} f_n P_n(\cos\theta),$$

where the coefficients f_n are defined by the expansion $f(\theta) = \sum f_n P_n(\cos\theta)$, solves the equation $\Delta u = 0$ in the Euclidean ball $\|x\|_2 < a$ with boundary values $u = f$ on the sphere $\|x\|_2 = a$. Here we use spherical coordinates defined by the equations $x = (x_1, x_2, x_3) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$, so that $\pi/2 - \theta$ is the latitude, φ the longitude.

14.3. *Hermite*[8] *polynomials.* Here $a = -\infty$, $b = +\infty$, $\rho(x) = e^{-x^2}$, thus

$$\int_{-\infty}^{+\infty} H_m(x)H_n(x)e^{-x^2} dx = 0, \quad m \neq n.$$

The polynomials are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad n \in \mathbf{N}.$$

The first few Hermite polynomials are $H_0(x) = 1$, $H_1(x) = 2x$, $H_3(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$. A generating function is

$$w(x, t) = e^{2xt - t^2} = \sum_0^\infty \frac{1}{n!} H_n(x) t^n, \qquad t \in \mathbf{C}.$$

The Hermite polynomials solve the differential equation $u'' - 2xu' + 2nu = 0$. The function $v = e^{-x^2/2} H_n$ solves the equation $v'' + (2n + 1 - x^2)v = 0$. The Hermite polynomials can be used to solve the Laplace and Helmholtz equations in a cylinder bounded by a parabolic cylinder surface.

14.4. *Laguerre*[9] *polynomials.* Here $a = 0$, $b = +\infty$, $\rho(x) = x^\alpha e^{-x}$, thus

$$\int_0^{+\infty} L_m^\alpha(x) L_n^\alpha(x) x^\alpha e^{-x} dx = 0, \quad m \neq n.$$

The polynomials are defined by

$$L_n^\alpha(x) = e^x \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}), \qquad n \in \mathbf{N}, \quad \alpha > -1.$$

[8] Charles Hermite, 1822–1901.

[9] Edmond Nicolas Laguerre, 1834–1886.

The first few Laguerre polynomials are $L_0^\alpha(x) = 1$, $L_1^\alpha(x) = 1 + \alpha - x$, $L_3^\alpha(x) = \frac{1}{2}\big((1+\alpha)(2+\alpha) - 2(2+\alpha)x + x^2\big)$. The general formula is

$$L_n^\alpha(x) = \sum_{k=0}^{n} (n+\alpha)(n+\alpha-1)\cdots(k+1+\alpha)\frac{(-x)^k}{k!(n-k)!}.$$

A generating function is

$$w(x,t) = (1-t)^{-\alpha-1}e^{-xt/(1-t)} = \sum_0^\infty L_n^\alpha(x)t^n, \qquad |t| < 1.$$

The Laguerre polynomials solve the differential equation $xu'' + (\alpha+1-x)u' + nu = 0$. The Laguerre polynomials can be used to solve the problem of propagation of electromagnetic waves along a transmission line.

14.5. *Chebyshev*[10] *polynomials.* Here $a = -1$, $b = 1$, $\rho(x) = 1/\sqrt{1-x^2}$, thus

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = 0, \quad m \neq n.$$

The polynomials are defined by

$$T_n(x) = \cos(n \arccos x), \qquad x \in [-1,1], \ n \in \mathbf{N}.$$

(Note that the T_n are polynomials and therefore can be extended to all $x \in \mathbf{R}$.) A generating function is

$$w(x,t) = \frac{1-t^2}{1-2xt+t^2} = T_0 + 2\sum_1^\infty T_n(x)t^n, \qquad |t| < r,$$

where $r = \min(r_1, r_2)$, r_j being the two roots of the equation $1 - 2xt + t^2 = 0$, $t = r_1, r_2$. Chebyshev proved that every continuous function on an interval $[a,b]$ has a unique best approximant in the supremum norm among the polynomials of degree $< m$. If $a = -1$, $b = 1$, then the best approximant to $f(x) = x^m$ among the polynomials of degree $< m$ is $2^{-m+1}T_m$; more precisely

$$x^m - p_{m-1}(x) = 2^{-m+1}\cos(m \arccos x) = 2^{-m+1}T_m(x).$$

(See my lecture notes *Approximation by polynomials,* Uppsala University, Department of Mathematics, Lecture Notes 1999:LN1.)

14.6. *Jacobi*[11] *polynomials.* Here $a = -1$, $b = 1$, $\rho(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, thus

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = 0, \quad m \neq n.$$

[10] Pafnutiĭ L′vovič Čebyšëv, 1821–1894.
[11] Carl Gustav Jacob Jacobi, 1804–1851.

The polynomials are defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right), \ n \in \mathbf{N}.$$

The function $u = P_n^{(\alpha,\beta)}$ solves the differential equation

$$(1-x^2)u'' + (\beta - \alpha - (\alpha+\beta+2)x)u' + n(n+\alpha+\beta+1)u = 0.$$

A generating function is

$$w(x,t) = 2^{\alpha+\beta}R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta} = \sum_0^\infty P^{(\alpha,\beta)}(x)t^n, \qquad |t| < r,$$

where $R = (1-2xt+t^2)^{-1/2}$, and $r = \min(r_1, r_2)$, r_j being the two roots of the equation $1 - 2xt + t^2 = 0$, $t = r_1, r_2$.

15. The Radon transformation

15.1. *History.* Johann Radon (1887–1956) published a remarkable paper in 1917:

> Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. *Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Nat. Kl.* **69** (1917), 262–277.

Forty-six years later Alan M. Cormack (b. 1924) published the paper

> Representation of a function by its line integrals, with some radiological applications I, II. *J. Appl. Phys.* **34** (1963), 2722–2727; **35** (1964), 2908–2912.

Cormack and Godfrey N. Hounsfield (b. 1919) produced the first picture of a brain using this method in 1972.

15.2. *Definitions.* The *Radon transform* of a function f defined in $\mathbf{R}^2$ is

$$\mathcal{R}f(L) = \int_L f,$$

where L is any straight line in the plane. We assume that the function f tends to zero at infinity in such a way that the integral converges.

More generally, the *Radon transform* of f is:

$$\mathcal{R}f(\omega, p) = \int\limits_{\omega\cdot x = p} f(x)\,dm(x), \qquad (\omega, p) \in S^{n-1} \times \mathbf{R},$$

where f is defined on $\mathbf{R}^n$ and $(\omega, p) \in S^{n-1} \times \mathbf{R}$ defines a hyperplane

$$\{x \in \mathbf{R}^n;\ \omega \cdot x = p\}.$$

Here, again, we must assume that the function tends to zero at infinity sufficiently rapidly.

The *dual Radon transform* $\mathcal{R}^\sharp$ of a function φ defined on the set of all hyperplanes is

$$\mathcal{R}^\sharp\varphi(x) = \int_{\xi\ni x} \varphi(\xi)d\mu(\xi), \qquad x \in \mathbf{R}^n.$$

The integral is defined to yield the mean value of φ over all hyperplanes passing through x. If we let ξ be defined by $(\omega, p) \in S^{n-1} \times \mathbf{R}$, then this definition takes the form

$$\mathcal{R}^\sharp g(x) = \int\limits_{S^{n-1}} g(\omega, \omega \cdot x)d\omega, \qquad x \in \mathbf{R}^n,$$

where g is defined on $S^{n-1} \times \mathbf{R}$ and the integral is normalized to yield the mean value over the $(n-1)$-dimensional sphere.

15.3. *Rules of calculus.* We note that

$$\mathcal{R}(\partial_j f) = \omega_j \frac{\partial}{\partial p}\mathcal{R}f;$$

and that the Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

intertwines with the operator $\Box = \partial^2/\partial p^2$ as follows:

$$\mathcal{R} \circ \Delta = \Box \circ \mathcal{R}, \qquad \Delta \circ \mathcal{R}^\sharp = \mathcal{R}^\sharp \circ \Box.$$

15.4. *Relation to the Fourier transformation.* Let $\mathcal{F}_n$ denote the Fourier transformation in $\mathbf{R}^n$:

$$\mathcal{F}_n f(\xi) = \int_{\mathbf{R}^n} f(x)e^{-i\xi\cdot x}dx, \qquad \xi \in \mathbf{R}^n.$$

Then the *Fourier slice theorem* says that $\mathcal{F}_n = \mathcal{F}_1 \circ \mathcal{R}$, where $\mathcal{F}_1$ is the one-dimensional Fourier transformation in the variable p.

15.5. *Relation to convolution.* We have $\mathcal{R}(f * g) = \mathcal{R}f *_p \mathcal{R}g$, where $*_p$ denotes convolution in the p variable only.

15.6. *Inversion via the Fourier transformation.* From the Fourier slice theorem $\mathcal{F}_n = \mathcal{F}_1 \circ \mathcal{R}$ we deduce that $\mathrm{Id} = \mathcal{F}_n^{-1} \circ \mathcal{F}_n = \mathcal{F}_n^{-1} \circ \mathcal{F}_1 \circ \mathcal{R}$, so that f can be recovered from its Radon transform φ as $f = \mathcal{F}_n^{-1}(\mathcal{F}_1(\varphi))$. This formula is actually used in numerical computations.

When $n = 2$, we can recover f from $\mathcal{R}f$ by the formula:

$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{dg_x(q)}{q}, \qquad x \in \mathbf{R}^2,$$

a Stieltjes integral, where $g_x(q)$ is the mean value of $\mathcal{R}f(L)$ over all lines L with distance q to x.

For $n = 3$:

$$f(x) = -\frac{1}{8\pi^2}\Delta \int_{S^2} \mathcal{R}f(\omega, \omega \cdot x)\,d\omega, \qquad x \in \mathbf{R}^3.$$

The *inversion formula* for any dimension is:

$$f = c_n \Delta^{(n-1)/2} \mathcal{R}^\sharp \mathcal{R} f, \qquad f \in \mathcal{S}(\mathbf{R}^n).$$

Note: When n is odd, we have an integer power of the Laplacian: the operator is local. When n is even, the operator is not local; it has to be interpreted via the Fourier transformation. Recall that $-\Delta$ corresponds to multiplication by $\|\xi\|_2^2 = \langle \xi, \xi \rangle$. Therefore any power $(-\Delta)^\alpha$ can be interpreted as multiplication by $\|\xi\|_2^{\alpha/2}$ on the Fourier transform side.

15.7. *The Helgason*[12] *support theorem.* If $f \in C(\mathbf{R}^n)$ decreases so fast that $\sup_{x \in \mathbf{R}^n} \|x\|_2^m |f(x)|$ is finite for all $m \geqslant 0$ and $\mathcal{R}f$ has support in $\{(\omega, p);\, |p| \leqslant r\}$, then f has support in the ball $\{x;\, \|x\|_2 \leqslant r\}$. More generally, if $\mathcal{R}f(\omega, p) = 0$ for all (ω, p) such that $\{x;\, \omega \cdot x = p\}$ does not meet a convex compact set K, then $\operatorname{supp} f \subset K$.

To reconstruct f outside K we need to know $\mathcal{R}f$ only for hyperplanes that do not meet K.

Application: Let K contain the heart (which is pounding all the time)...

Note: The function f must decrease rapidly, faster than any power of $\|x\|_2$.

[12] Sigurdur Helgason, b. 1927.