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Analytic continuation of fundamental solutions to differential equations with constant coefficients

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Abstract. If P is a polynomial in $\mathbf{R}^n$ such that $1/P$ integrable, then the inverse Fourier transform of $1/P$ is a fundamental solution E_P to the differential operator $P(D)$. The purpose of the article is to study the dependence of this fundamental solution on the polynomial P. For $n = 1$ it is shown that E_P can be analytically continued to a Riemann space over the set of all polynomials of the same degree as P. The singularities of this extension are studied.

Résumé. Prolongement analytique des solutions fondamentales des équations aux dérivées partielles à coefficients constants.
Si P est un polynôme dans $\mathbf{R}^n$ tel que $1/P$ soit sommable, alors la transformée inverse de Fourier de $1/P$ est une solution fondamentale E_P de l'opérateur $P(D)$. Le but de l'article est d'étudier la dépendence de cette solution fondamentale du polynôme P. Pour $n = 1$ on démontre que E_P peut être prolongée analytiquement à un espace de Riemann audessus de l'ensemble de tous les polynômes du même degré que P. Les singularités de ce prolongement sont étudiées.

1. Introduction

A fundamental solution for a partial differential operator $P(D)$ is a distribution E which satisfies $P(D)E = \delta$, where δ is the Dirac measure placed at the origin. (Here $D = (D_1, \dots, D_n)$ and $D_j = -i\partial/\partial x_j$.) This implies that $E * (P(D)\varphi) = P(D)(E*\varphi) = \varphi$ for every test function φ. In other words, E is a convolution inverse of the distribution $P(D)\delta$, which is supported by the origin: $E * (P(D)\delta) = \delta$. If E happens to be temperate, its Fourier transform is a multiplicative inverse to the polynomial $P(\xi)$: $\widehat{E} \cdot P = \widehat{\delta} = 1$.

If a polynomial P is such that $1/P$ is integrable, then this function is a multiplicative inverse of P, so its inverse Fourier transform E is a temperate fundamental solution:

$$E(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \frac{e^{ix\cdot\xi}}{P(\xi)} d\xi, \qquad x \in \mathbf{R}^n. \tag{1.1}$$

In particular E is a continuous function, and $P(D)E = \delta$ in the sense of distributions, i.e.,

$$\int_{\mathbf{R}^n} E(x)\Big(P(-D)\varphi\Big)(x)dx = \varphi(0), \qquad \varphi \in \mathscr{D}(\mathbf{R}^n).$$

The purpose of this paper is to study how the distribution E defined by (1.1) depends on P. In particular we shall study holomorphic extensions and singularities of this function. It is sometimes possible to define a fundamental solution by continuation from (1.1) even though the formula itself is no longer valid.

The ultimate goal is, given a polynomial P_0 of degree m and a fundamental solution E_{P_0}, to understand the complete structure of all fundamental solutions E_P that can be reached from E_{P_0} by analytic continuation on a Riemann domain over the vector space of all polynomials of degree $\leqslant m$. We are far from this goal. In fact, most of the results here are about ordinary differential operators; only in the last section do we give some fragments of results for partial differential operators.

If P has real zeros, the now classical method of Hörmander and Treves consists in replacing integration over $\mathbf{R}^n$ by integration over some suitable set in $\mathbf{C}^n$ (called Hörmander's staircase (Agranovič 1961: 34)), or by the more sophisticated integration in Hörmander (1990: 189–191). The viewpoint of this paper is different: we try to move P instead of the contour of integration.

In section 6 we shall consider one-dimensional subspaces in the space of all polynomials. Thus we consider a polynomial $P(\xi, z) = P(\xi) - zQ(\xi)$ of $n+1$ variables $\xi_1, \dots, \xi_n, z$. It may happen that for certain values of z we have an estimate

$$|P(\xi, z)| \geqslant c(1 + \|\xi\|)^\rho, \qquad \xi \in \mathbf{R}^n, z \in \Omega,$$

with $c > 0$ and $\rho > n$; this is for instance true if $P(\xi, z) = P(\xi) - zQ(\xi)$ with $Q(\xi) = \|\xi\|_2^{2k} + 1$ for some large integer k (Euclidean norm). Then the integral

$$F_z(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \frac{e^{ix\cdot\xi}}{P(\xi, z)} d\xi, \qquad x \in \mathbf{R}^n,$$

makes sense for these z. If the function $z \mapsto F_z(x)$ is holomorphic in Ω we can try to extend it by analytic continuation. More generally, it may happen that not $F_z(x)$ but its action on a test function φ possesses an extension:

$$\int_{\mathbf{R}^n} F_z(x)\varphi(x)dx = \Phi(z).$$

We can then ask whether $\Phi(0)$ defines a fundamental solution for the polynomial $P(\xi, 0) = P(\xi)$, which may be a given polynomial not satisfying the estimate at all.

2. On fundamental solutions

A sufficient condition for $1/P$ to be integrable is that P satisfies

$$|P(\xi)| \geqslant c(1 + \|\xi\|)^\rho, \qquad \xi \in \mathbf{R}^n, \tag{2.1}$$

for some constants $c > 0$ and $\rho > n$.

A little more generally, it may happen that P satisfies (2.1) for some $c > 0$ and some $\rho \in \mathbf{R}$. In fact, in view of the Tarski–Seidenberg theorem, this is true for any polynomial which does not vanish in $\mathbf{R}^n$.

Proposition 2.1. *Let P be a polynomial without real zeros. Then it satisfies* (2.1) *for some $c > 0$ and some real number ρ, and $1/P$ defines a temperate distribution. Its inverse Fourier transform is a fundamental solution in $\mathscr{S}'(\mathbf{R}^n)$ for the operator $P(D)$. It is the only temperate fundamental solution. It belongs to the space $B_{\infty,\tilde{P}}$ if and only if $P^{-1}\partial^\alpha P/\partial\xi^\alpha$ is bounded for every $\alpha \in \mathbf{N}^n$. It belongs to the local space $B^{\mathrm{loc}}_{\infty,\tilde{P}}$ if and only if $(\hat{\varphi} * P^{-1})\partial^\alpha P/\partial\xi^\alpha$ is bounded for every $\alpha \in \mathbf{N}^n$ and every test function $\varphi \in \mathscr{D}(\mathbf{R}^n)$.*

Proof. The hypothesis implies that φ/P is integrable for every function $\varphi \in \mathscr{S}(\mathbf{R}^n)$ and so defines a temperate distribution E by

$$E(\varphi) = (2\pi)^{-n}\int_{\mathbf{R}^n} \frac{\widehat{\varphi}(-\xi)}{P(\xi)}d\xi, \qquad \varphi \in \mathscr{S}(\mathbf{R}^n); \tag{2.2}$$

clearly $E(P(-D)\varphi) = \varphi(0)$.

The other statements follow from the definition of the spaces $B_{\infty,\tilde{P}}$ and $B^{\mathrm{loc}}_{\infty,\tilde{P}}$ (Hörmander 1983:7, 13). ☐

We recall that Hörmander proved that every non-zero partial differential operator with constant coefficients admits a fundamental solution in $B^{\mathrm{loc}}_{\infty,\tilde{P}}$ (Hörmander 1983: Theorem 10.2.1); in fact, a small exponential dampening at infinity is enough for the existence of a solution even in $B_{\infty,\tilde{P}}$: for every positive ε there exists a fundamental solution E_ε such that $E_\varepsilon/\psi_\varepsilon \in B_{\infty,\tilde{P}}$, where $\psi_\varepsilon(x) = \cosh\|\varepsilon x\|$. Moreover, he proved that this result is optimal: if a fundamental solution is in some space $B^{\mathrm{loc}}_{p,k}$, then this space contains $B^{\mathrm{loc}}_{\infty,\tilde{P}}$ (Hörmander 1983:17–18).

Lars Hörmander proved (personal communication 2010-11-08) that every operator of principal type (i.e., such that the gradient of its principal part does not vanish in $\mathbf{R}^n \setminus \{0\}$; Hörmander 1983: Definition 10.4.11) possesses a temperate fundamental solution with the best possible regularity: it is in $B^{\mathrm{loc}}_{\infty,\tilde{P}}$. In particular, if P is of principal type and does not have real zeros, the unique temperate fundamental solution defined by (2.2) has this regularity.

Even more recently, Hörmander (personal communication 2010-12-31) proved the stronger result that for an operator to possess a temperate fundamental solution of the best regularity, it is enough that, for some constants C and R, there is an estimate

$$\left|\frac{\partial^\alpha P}{\partial\xi^\alpha}(\xi)\right| \leqslant C(|P(\xi)| + \|\operatorname{grad} P(\xi)\|), \qquad \alpha \in \mathbf{N}^n,\ \xi \in \mathbf{R}^n,\ \|\xi\| \geqslant R. \tag{2.3}$$

(The estimate is of interest when $\|\alpha\|_1 \geqslant 2$.)

However, for operators that are not of principal type, there need not exist a temperate fundamental solution in $B^{\mathrm{loc}}_{\infty,\tilde{P}}$. Thus the local regularity condition and the global condition of being temperate are sometimes irreconcilable. We mention three examples:

Example 2.2. (Enqvist 1974:29.) The polynomial $P(\xi) = \xi_1^2\xi_2 + \xi_3 - i$ has no real zeros; it satisfies (2.1) for $c = 1$ and $\rho = 0$. For $\xi_1 \in \mathbf{R}$ and $\xi_2 = \xi_3 = 0$, $P^{-1}\partial P/\partial\xi_2 = i\xi_1^2$ is unbounded. Hence the fundamental solution defined by (2.2), which is the unique temperate fundamental solution, does not belong to $B_{\infty,\tilde{P}}(\mathbf{R}^3)$. Enqvist shows that it does not even belong to the larger space $B^{\mathrm{loc}}_{\infty,\tilde{P}}(\mathbf{R}^3)$. □

Example 2.3. (Hörmander 1983: Example 10.2.15.) The operator given by the polynomial $P(\xi) = \xi_1^2\xi_2^2+\xi_3^2+i\xi_4$ has no fundamental solution in $\mathscr{S}'(\mathbf{R}^4)\cap B^{\mathrm{loc}}_{\infty,\tilde{P}}(\mathbf{R}^4)$, for it does not satify the necessary condition given in Theorem 10.2.14. However, P has real zeros. □

There is now even an example which is minimal with respect to order as well as dimension (order three and dimension two):

Example 2.4. (Lars Hörmander, personal communication 2010-12-31.) For any nonzero complex number c, the polynomial $P(\xi) = \xi_1^2\xi_2 + c$ is such that the corresponding operator does not have a temperate solution in $B^{\mathrm{loc}}_{\infty,\tilde{P}}(\mathbf{R}^2)$. □

We contrast this example with the following, where the operator is of order two.

Example 2.5. The polynomial $P(\xi) = \xi_1\xi_2 + i$ has no real zeros, and satisfies (2.1) for $c = 1$ and $\rho = 0$. But $P^{-1}\partial P/\partial\xi_2$ is unbounded: for $\xi_2 = 0$ it takes the value $-i\xi_1$. Hence the fundamental solution defined by (2.2) is not in $B_{\infty,\tilde{P}}(\mathbf{R}^2)$. But the operator is of principal type, so this fundamental solution is both temperate and in the local space $B^{\mathrm{loc}}_{\infty,\tilde{P}}(\mathbf{R}^2)$. □

In the first proofs that every nonzero operator with constant coefficients admits a fundamental solution one fixed P and constructed E without any considerations as to its dependence on P. This means that there is a function (with no special regularity) $G\colon \mathscr{P}_m(\mathbf{R}^n) \smallsetminus \{0\} \to \mathscr{D}'(\mathbf{R}^n)$ defined on the space of $\mathscr{P}_m(\mathbf{R}^n)$ of all polynomials in n variables and degree at most m with the origin removed. A more advanced construction is to establish the existence of a C^∞ smooth function $G\colon \mathscr{P}_m(\mathbf{C}^n) \smallsetminus \{0\} \to \mathscr{D}'(\mathbf{R}^n)$; this is what Hörmander does in his book (1983, Theorem 10.2.3).

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3. Polynomials of one variable with their zeros as parameters

We shall first study properties of fundamental solutions of ordinary differential operators whose corresponding polynomial has zeros $\tau_1, \dots, \tau_m$ and consider them as functions of $\tau = (\tau_1, \dots, \tau_m) \in \mathbf{C}^m$.

Theorem 3.1. *For $\tau = (\tau_1, \dots, \tau_m) \in \mathbf{C}^m$, $m \geqslant 2$, let $P(\zeta, \tau_1, ..., \tau_m)$ be the monic polynomial in ζ with zeros at the complex numbers τ_j, thus*

$$P(\zeta, \tau_1, ..., \tau_m) = \prod_1^m (\zeta - \tau_j), \qquad \zeta \in \mathbf{C}, \tag{3.1}$$

and define

$$F_\tau(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \frac{e^{ix\xi}}{P(\xi, \tau_1, ..., \tau_m)} d\xi, \qquad x \in \mathbf{R}, \ \tau \in \Omega = (\mathbf{C} \setminus \mathbf{R})^m. \tag{3.2}$$

Thus F_τ is defined when τ belongs to an open set Ω in $\mathbf{C}^m$ which has 2^m components

$$\Omega_\theta = \{\tau \in \mathbf{C}^m; \ \theta_j \mathrm{Im}\, \tau_j > 0\}, \qquad \theta \in \{-1, 1\}^m.$$

Then $F_\tau(x)$ and $\langle F_\tau, \varphi \rangle$ are holomorphic functions of $\tau \in \Omega$. Given θ, we define $M_\theta(\tau) = \prod(\tau_j - \tau_k)$, $\tau \in \mathbf{C}^m$, where the product is taken over all j and k such that $\theta_j > \theta_k$. Then the function

$$\Omega_\theta \ni \tau \mapsto M_\theta(\tau) F_\tau(x)$$

is the restriction to Ω_θ of an entire function. In particular, when we take all θ_j equal, $M_\theta(\tau) = 1$ and $F_\tau(x)$ is the restriction to $\Omega_{(1,\dots,1)}$ or $\Omega_{(-1,\dots,-1)}$ of an entire function.

The result also holds for $m = 1$ if we interpret the integral (3.2) as a generalized integral. The function $x \mapsto F_\tau(x)$ is no longer continuous at $x = 0$, but as a distribution-valued mapping it is holomorphic in τ, i.e., $\langle F_\tau, \varphi \rangle$ is holomorphic for every test function φ.

Proof. If $\tau \in \Omega$, we can number the τ_j so that $\mathrm{Im}\, \tau_j > 0$ for $j = 1, \dots, k$ and $\mathrm{Im}\, \tau_j < 0$ for $j = k+1, \dots, m$, for some $k = 0, \dots, m$, thus $\tau \in \Omega_\theta$ for $\theta = (1, \dots, 1, -1, \dots, -1)$. Then, if $x > 0$, we get using residue theory,

$$F_\tau(x) = i \sum_{s=1}^k R_s(\tau) e^{ix\tau_s},$$

where the R_s are rational functions,

$$R_s(\tau) = \prod_{j=1}^{s-1} (\tau_s - \tau_j)^{-1} \prod_{j=s+1}^{m} (\tau_s - \tau_j)^{-1}.$$

When we multiply the R_s by the polynomial

$$M_\theta(\tau) = \prod_{b=k+1}^{m} \prod_{a=1}^{k} (\tau_a - \tau_b),$$

we get new rational functions which still seem to have poles. However, when the denominator vanishes because two of the zeros coincide, also the numerator vanishes, so that the function $\tau \mapsto M_\theta(\tau) F_\tau(x)$ is entire.

As an example we calculate $M_\theta(\tau)F_\tau(x)$ for $m = 4$ and $k = 2$, thus with $\theta = (1, 1, -1, -1)$ and $x > 0$. So let $\operatorname{Im} \tau_1, \operatorname{Im} \tau_2 > 0$ and $\operatorname{Im} \tau_3, \operatorname{Im} \tau_4 < 0$. Then

$$\begin{aligned} M_\theta(\tau)F_\tau(x) &= (\tau_1 - \tau_3)(\tau_2 - \tau_3)(\tau_1 - \tau_4)(\tau_2 - \tau_4)F_\tau(x) \\ &= \frac{(\tau_2 - \tau_3)(\tau_2 - \tau_4)}{\tau_1 - \tau_2} i e^{ix\tau_1} + \frac{(\tau_1 - \tau_3)(\tau_1 - \tau_4)}{\tau_2 - \tau_1} i e^{ix\tau_2} \\ &\quad + \frac{(\tau_1 - \tau_4)(\tau_2 - \tau_4)}{\tau_3 - \tau_4} i e^{ix\tau_3} + \frac{(\tau_1 - \tau_3)(\tau_2 - \tau_3)}{\tau_4 - \tau_3} i e^{ix\tau_4}. \end{aligned}$$

The denominators vanish in the hyperplanes $\tau_1 = \tau_2$ and $\tau_3 = \tau_4$, but nevertheless the function is entire. □

We shall now see that the fundamental solutions F_τ found in this theorem can be extended from Ω_θ to larger regions.

Theorem 3.2. *We define open sets*

$$\Omega^{\{j\}} = \{\tau \in \mathbf{C}^m;\ \tau_s \neq \tau_j,\ \forall s \neq j\}, \qquad j = 1, \ldots, m,$$

and for $\tau \in \Omega^{\{j\}}$, $j = 1, \ldots, m$, *functions* $F_{j,\tau}$ *and* $G_{j,\tau}$ *on* $\mathbf{R}$ *by*

$$F_{j,\tau}(x) = \frac{1}{2\pi} \int_{\Gamma_j} \frac{e^{ix\zeta}}{P(\zeta, \tau_1, \ldots, \tau_m)} d\zeta, \quad x \geqslant 0; \qquad F_{j,\tau}(x) = 0, \quad x < 0; \text{ and} \tag{3.3}$$

$$G_{j,\tau}(x) = 0, \quad x \geqslant 0; \qquad G_{j,\tau}(x) = -\frac{1}{2\pi} \int_{\Gamma_j} \frac{e^{ix\zeta}}{P(\zeta, \tau_1, \ldots, \tau_m)} d\zeta, \quad x < 0, \tag{3.4}$$

where Γ_j *is a circle around* τ_j *of radius so small that it does not surround any other zero.*

Given any subset J *of* $\{1, \ldots, m\}$ *we define*

$$\Omega^J = \{\tau \in \mathbf{C}^m;\ \tau_j \neq \tau_s \text{ for all } j \in J \text{ and all } s \notin J\},$$

a connected open set in $\mathbf{C}^m$, *and*

$$E_\tau^J(x) = \sum_{j \in J} F_{j,\tau}(x) + \sum_{s \notin J} G_{s,\tau}(x), \qquad x \in \mathbf{R}. \tag{3.5}$$

This definition has a sense when τ belongs to $\Omega^{\{j\}}$ for all $j \in \{1, \ldots, m\}$, i.e., when all zeros are different. However, we can extend it to $\tau \in \Omega^J$. Then $\Omega^J \ni \tau \mapsto E^J_\tau(x)$ is a holomorphic function.

Given any $\theta \in \{-1, 1\}^m$, $\Omega_\theta \cup \Omega_{-\theta}$ is contained in Ω^J if we define $J = \{j;\ \theta_j = 1\}$, and we can extend the holomorphic function $\Omega_\theta \ni \tau \mapsto F_\tau$ as $\Omega^J \ni \tau \mapsto E^J_\tau$:

$$F_\tau(x) = E^J_\tau(x) \qquad \text{for } x \in \mathbf{R} \text{ and } \tau \in \Omega_\theta. \tag{3.6}$$

Conversely, given any $J \subset \{1, \ldots, m\}$, (3.6) holds if we define θ by $\theta_j = 1$ for $j \in J$, $\theta_j = -1$ for $j \notin J$.

Note that $\Omega^J = \Omega^K$ if K is the complement of J in $\{1, \ldots, m\}$; in particular $\Omega^\varnothing = \Omega^{\{1,\ldots,m\}} = \mathbf{C}^m$.

Proof. Each term $F_{j,\tau}$ can be defined if τ_j is different from the τ_s, $s \neq j$. However, when we add over all $j \in J$, the sum can easily be defined in Ω^J: we just use a curve in the integral corresponding to (3.3) which surrounds all the τ_j, $j \in J$, but none of the τ_s, $s \notin J$. There is now no danger in letting some of the τ_j, $j \in J$, coincide as long as they are different from the τ_s, $s \notin J$. ☐

4. Polynomials of one variable with their coefficients as parameters

Theorem 4.1. *We consider polynomials $P(\xi) = A_0 + A_1\xi + \cdots + A_m\xi^m$ of degree $m \geqslant 2$. We identify the set $\mathscr{P}_m(\mathbf{R}^n) \smallsetminus \mathscr{P}_{m-1}(\mathbf{R}^n)$ of these polynomials with a pseudoconvex open subset*

$$U = \{(A_0, \ldots, A_m) \in \mathbf{C}^{m+1};\ A_m \neq 0\} \tag{4.1}$$

of $\mathbf{C}^{m+1}$. Those which have no real zero form a subset W which is open in U and which has $m+1$ components W_k defined by the requirement that there are exactly k roots with positive imaginary part, $k = 0, \ldots, m$.

We define

$$L_k(A_0, \ldots, A_m)(x) = A_m^{-1} F_\tau(x), \qquad x \in \mathbf{R},\ (A_0, \ldots, A_m) \in W_k, \tag{4.2}$$

where F_τ is defined by (3.2), where P and τ are related by the formula

$$P(\xi) = A_m \prod (\xi - \tau_j),$$

and where we have numbered the τ_j so that $\operatorname{Im} \tau_j > 0$ for $j = 1, \ldots, k$ and $\operatorname{Im} \tau_j < 0$ for $j = k+1, \ldots, m$. Then L_k is a holomorphic function of the coefficients $(A_0, \ldots, A_m) \in W$, thus in W_k for each k. Two of these functions, viz. for $k = 0$ and $k = m$, are restrictions of meromorphic functions of $(A_0, \ldots, A_m) \in \mathbf{C}^{m+1}$ with *singularities only on the hyperplane $A_m = 0$; more precisely, $A_m L_m$ is the restriction of an entire function in $\mathbf{C}^{m+1}$. For $k = m$, $A_m \neq 0$, the extended solution is*

$$L_m(x) = E_\tau^{\{1,\ldots,m\}}(x) = \sum_{k=1}^{m} F_{j,\tau}(x) = \frac{1}{2\pi}\int_\Gamma \frac{e^{ix\zeta}}{P(\zeta)}d\zeta,\ x > 0; \qquad L_m(x) = 0,\ x \leqslant 0,$$

where Γ is a circle surrounding all the zeros $\tau_1, \ldots, \tau_m$ of P.

Proof. The vector of coefficients $(A_0, \ldots, A_m)$ determines the set of zeros but not the numbering. The function F_τ is however symmetric under the possible numberings, which implies that L_k is well-defined and holomorphic in the coefficients of P. There are $\binom{m}{k}$ different sets Ω_θ of Theorem 3.1 that are mapped onto W_k, where k is the cardinality of $\{j;\ \theta_j = 1\}$. □

To study the extensions of L_k from W_k for $0 < k < m$ it is necessary to consider multivalued functions, in other words Riemann domains.

Lemma 4.2. *Let U be the set of all polynomials of degree $m \geqslant 2$ identified with the open set $U \subset \mathbf{C}^{m+1}$ defined in (4.1). Let X denote the algebraic set*

$$X = \Big\{(P, \tau_1, \ldots, \tau_m) \in U \times \mathbf{C}^m; P(\zeta) = \textstyle\prod_{j=1}^m (\zeta - \tau_j)\Big\}.$$

We have two projections defined on X, viz.,

$$\pi\colon (P, \tau_1, \ldots, \tau_m) \mapsto P, \text{ and}$$
$$\rho\colon (P, \tau_1, \ldots, \tau_m) \mapsto (\tau_1, \ldots, \tau_m),$$

which enable us to use functions defined on the coefficients of polynomials as well as functions defined on the zeros of polynomials. Let M be the set of those polynomials which possess a multiple zero, and define the Riemann domain Y over $U \smallsetminus M$ as

$$Y = \{(P, \tau, \ldots, \tau_m) \in X;\ P \notin M\},$$

the set of all elements $(P, \tau_1, \ldots, \tau_m)$ where $P \in U \smallsetminus M$ and the τ_j are the zeros of the polynomial P. Thus Y has $m!$ sheets over $U \smallsetminus M$. Then Y is connected.

Proof. Let two elements of Y, $(P^{(s)}, \tau_1^{(s)}, \ldots, \tau_m^{(s)})$, $s = 0, 1$, be given, thus with $P^{(s)}(\xi) = \prod(\xi - \tau_j^{(s)})$. We shall construct a curve in Y connecting them. We construct first the straight lines between the zeros:

$$[0,1] \ni t \mapsto \tau_j(t) = (1-t)\tau_j^{(0)} + t\tau_j^{(1)}, \qquad j = 1, \ldots, m.$$

Most of the time this will give us what we want, i.e., a curve in Y. However, it may happen that two roots agree for a certain value of t so that, for example, $\tau_1(t) = \tau_2(t)$ for some t. We can then modify as follows.

(1) If $\tau_j^{(0)} = \tau_j^{(1)}$ for both $j = 1$ and $j = 2$, nothing needs to be done.

(2) If $\tau_1^{(0)} = \tau_1^{(1)}$ while $\tau_2^{(0)} \neq \tau_2^{(1)}$, we modify the segment $\left[\tau_2^{(0)}, \tau_2^{(1)}\right]$ to a curve

$$\tau_2(t) = (1-t)\tau_2^{(0)} + t\tau_2^{(1)} + ct(1-t)(\tau_2^{(1)} - \tau_2^{(0)}), \tag{4.3}$$

where $c = i\varepsilon$ for a small positive number ε.
(3) If both $\tau_1^{(0)}$ and $\tau_1^{(1)}$ are on the line through $\tau_2^{(0)}$ and $\tau_2^{(1)}$ with $\tau_2^{(1)} \neq \tau_2^{(0)}$, then again we use the curve (4.3) with $c = i\varepsilon$ for a small positive ε.
(4) In the remaining cases we use the curve defined in (4.3) but now with $c = \varepsilon$ for a small positive ε.

With these modified curves, $\tau_1(t)$ and $\tau_2(t)$, $t \in [0,1]$, never meet. This shows how to avoid the set M of multiple zeros and proves the connectedness of Y. □

As a preparation for the next theorem we shall see how to approximate any polynomial by polynomials with distinct zeros:

Lemma 4.3. *Let Q be any polynomial with zeros $\sigma_1, \dots, \sigma_m$, let $\omega \in \mathbf{R}^n$ be a real vector such that $\omega_j \neq \omega_k$ for $j \neq k$. Then*

$$Q_\varepsilon(\zeta) = \prod_{j=1}^{m} (\zeta - \sigma_j - \varepsilon\omega_j)$$

has distinct zeros $\sigma_j + \varepsilon\omega_j$ for

$$0 < \varepsilon < \varepsilon_0 = \min_{j,k} \Big(|\sigma_j - \sigma_k| / |\omega_j - \omega_k|; \sigma_j \neq \sigma_k \Big).$$

The number of zeros with positive imaginary part is the same for Q_ε as for Q.

Proof. If $\sigma_j = \sigma_k$ and $\sigma_j + \varepsilon\omega_j = \sigma_k + \varepsilon\omega_k$, then we must have $\varepsilon = 0$. Hence, as soon as $\varepsilon \neq 0$, this cannot happen. If on the other hand $\sigma_j \neq \sigma_k$ and $\sigma_j + \varepsilon\omega_j = \sigma_k + \varepsilon\omega_k$, then

$$\varepsilon = \frac{|\sigma_j - \sigma_k|}{|\omega_j - \omega_k|} \geqslant \varepsilon_0,$$

where ε_0 is the positive number defined in the statement of the lemma.

Finally we only need to note that $\operatorname{Im}(\sigma_j + \varepsilon\omega_j) = \operatorname{Im}\sigma_j$ if ε and ω_j are real, so that the number of zeros with imaginary part of a certain sign is preserved. □

The holomorphic extensions of the fundamental solutions defined in W_0 and W_m in Theorem 4.1 are defined in all of U, so for them we do not need to define any Riemann domain (they can of course be lifted to Y). However, for $0 < k < m$ this is not so:

Theorem 4.4. *Let Y be the Riemann domain defined in Lemma 4.2 and define $Y_k = \pi^{-1}(W_k \smallsetminus M)$. The fundamental solution L_k in Theorem 4.1, defined originally in each W_k, can be lifted from $W_k \smallsetminus M$ to Y_k and then extended to all of Y. If $0 < k < m$, they explode as we approach M. More precisely, for any point $(Q, \sigma_1, \dots, \sigma_m) \in X \smallsetminus Y$, thus with $Q \in M$, there exist a sheet of Y and a curve in that sheet such that the extension along the curve of the fundamental solution originating from W_k explodes as we move along the curve on that sheet and the base point approaches Q.*

Proof. We recall the definitions of $F_{j,\tau}$, $G_{j,\tau}$ and E^J_τ from Theorem 3.1.

When $(P, \tau_1, \ldots, \tau_m) \in Y$, we have $(\tau_1, \ldots, \tau_m) \in \Omega^J$ for any J, so $E^J_\tau(x)$ is well defined and defines an extension of L_k.

Let now Q be any given polynomial in M and let its zeros be σ_j, $j = 1, \ldots, m$, with σ_1 a zero of multiplicity $s \geqslant 2$. We may number the zeros so that $\sigma_1 = \sigma_2 = \cdots = \sigma_s$ while $\sigma_j \neq \sigma_1$ for $j = s+1, \ldots, m$.

We now define a curve $\gamma\colon [0,1] \to X$ with $\gamma(t) \in Y$ when $t \in [0,1[$, starting for $t = 0$ from any point $\gamma(0) = (P_0, \tau_1(0), \ldots, \tau_m(0))$ in Y_k. We suppose that the zeros are numbered so that $\operatorname{Im}\tau_1(0) > 0$ and $\operatorname{Im}\tau_m(0) < 0$. Since $k \neq 0, m$, this is possible. The curve shall end for $t = 1$ at the point $\gamma(1) = (P_1, \tau_1(1), \ldots, \tau_m(1)) = (Q, \sigma_1, \ldots, \sigma_m) \in X$, where Q is the given polynomial.

We have to distinguish two cases. The curve from $\gamma\left(\frac{1}{2}\right)$ to $\gamma(1)$ is the same in both cases, whereas the curve from $\gamma(0)$ to $\gamma\left(\frac{1}{2}\right)$ depends on whether $s \leqslant k$ or $s > k$. So let us first describe how to go from $\gamma\left(\frac{1}{2}\right)$ to $\gamma(1)$.

We define $\gamma\left(\frac{1}{2}\right)$ as the point $(P_{1/2}, \sigma_1 + \varepsilon\omega_1, \ldots, \sigma_m + \varepsilon\omega_m)$, where $P_{1/2} = Q_\varepsilon$, Q being the given polynomial, and Q_ε as well as the vector ω and the positive number $\varepsilon < \varepsilon_0$ are as in Lemma 4.3. Then the curve from $\gamma\left(\frac{1}{2}\right)$ to $\gamma(1)$ is given by the straight line

$$\tau_j(t) = \sigma_j + (2-2t)\varepsilon\omega_j, \qquad t \in \left[\tfrac{1}{2}, 1\right], \quad j = 1, \ldots, m,$$

which gives the approach to the multiple zero in $X \smallsetminus Y$.

We now construct the curve from $\gamma(0)$ to $\gamma\left(\frac{1}{2}\right)$ in the two cases.

Case 1: $s \leqslant k$. In this case we map $\tau_j(0)$ to $\sigma_{j+1} + \varepsilon\omega_{j+1}$, $j = 1, \ldots, m-1$, and $\tau_m(0)$ to $\sigma_1 + \varepsilon\omega_1$. In view of Lemma 4.2 it is possible to do this by moving along a curve in Y. This means that $\tau_1(0)$ in the upper half plane is moved to $\sigma_2 + \varepsilon\omega_2 = \sigma_1 + \varepsilon\omega_2$ close to σ_1, and that $\tau_m(0)$ in the lower half plane is moved to $\sigma_1 + \varepsilon\omega_1$, also close to σ_1.

Since we start in W_k, the set $J = \{1, \ldots, k\}$ will give rise to a fundamental solution

$$E^J_{\tau(t)} = \sum_{j=2}^{k+1} F_{j,\tau(t)}.$$

The solutions $\sum_{j=1}^{k+1} F_{j,\tau(t)}$ and $\sum_{j=s+2}^{k+1} F_{j,\tau(t)}$ are holomorphic as functions of t in a neighborhood of $t \in \left[\frac{1}{2}, 1\right]$, so to prove the explosion it is enough to prove that $E^{\{1\}}_\tau$ explodes.

We therefore now study the behavior of the fundamental solution $E^{\{1\}}_\tau$ on the curve. It is defined on $\Omega^{\{1\}}$.

We factorize $P_t(\zeta)$ as $(\zeta - \tau_1(t)) \cdots (\zeta - \tau_s(t)) R_t(\zeta)$ and then obtain, if Γ_1 is a circle surrounding $\tau_1(t)$ but none of the other zeros,

$$E^{\{j\}}_{\tau(t)}(x)=\frac{1}{2\pi}\int_{\Gamma_1}\frac{e^{ix\zeta}}{P_t(\zeta)}d\zeta=\frac{ie^{ix\tau_1(t)}}{(\tau_1(t)-\tau_2(t))\cdots(\tau_1(t)-\tau_s(t))R_t(\tau_1(t))}$$
$$=\frac{ie^{ix\tau_1(t)}}{(2-2t)^{s-1}\varepsilon^{s-1}(\omega_1-\omega_2)\cdots(\omega_1-\omega_s)R_t(\tau_1(t))},$$

which certainly explodes as $t\to 1$. More precisely,

$$(2-2t)^{s-1}E^{\{1\}}_{\tau(t)}(x)\to\frac{ie^{ix\sigma_1}}{\varepsilon^{s-1}(\omega_1-\omega_2)\cdots(\omega_1-\omega_s)R_1(\sigma_1)}.$$

Case 2: $k<s$. In this case we map $\tau_j(0)$ to $\sigma_j+\varepsilon\omega_j$ for all j. Now $J=\{1,\dots,k\}$ involves the points $\sigma_1+\varepsilon\omega_1,\dots,\sigma_k+\varepsilon\omega_k$, which are fewer than the s points $\sigma_j+\varepsilon\omega_j$ used for the convergence. We can calculate the residues as follows.

We factorize again $P_t(\zeta)$ as $(\zeta-\tau_1(t))\cdots(\zeta-\tau_s(t))R_t(\zeta)$ and then obtain, if Γ_k is a curve surrounding $\tau_1(t),\dots,\tau_k(t)$ but none of the other zeros,

$$E^{J}_{\tau(t)}(x)=\frac{1}{2\pi}\int_{\Gamma_k}\frac{e^{ix\zeta}}{P_t(\zeta)}d\zeta=\frac{1}{2\pi}\int_{\Gamma_k}\frac{ie^{ix\zeta}}{(\zeta-\tau_1(t))\cdots(\zeta-\tau_s(t))R_t(\zeta)}d\zeta.$$

This integral can be calculated using residues; it is equal to

$$\sum_{j=1}^{k}\frac{ie^{ix\tau_j(t)}}{R_t(\tau_j(t))\prod_l(\tau_j(t)-\tau_l(t))},$$

where

$$\prod_l(\tau_j(t)-\tau_l(t))=\prod_{\substack{l=1\\ l\neq j}}^{s}(\tau_j(t)-\tau_l(t))$$

is the product over all $l\in[1,s]_{\mathbf{Z}}\smallsetminus\{j\}$ for a fixed $j\in[1,k]_{\mathbf{Z}}$. This sum is equal to

$$\sum_{j=1}^{k}\frac{ie^{ix\tau_j(t)}}{R_t(\tau_j(t))\prod_l(2-2t)\varepsilon(\omega_j-\omega_l)}=\sum_{j=1}^{k}\frac{ie^{ix\tau_j(t)}}{(2-2t)^{s-1}\varepsilon^{s-1}R_t(\tau_j(t))\prod_l(\omega_j-\omega_l)},$$

which explodes as $t\to 1$; more precisely

$$(2-2t)^{s-1}E^{J}_{\tau(t)}(x)\to\sum_{j=1}^{k}\frac{ie^{ix\sigma_1}}{\varepsilon^{s-1}R_1(\sigma_1)\prod_l(\omega_j-\omega_l)}=C_{s,k}\frac{ie^{ix\sigma_1}}{\varepsilon^{s-1}R_1(\sigma_1)},$$

where the constant

$$C_{s,k}=\sum_{j=1}^{k}\prod_l(\omega_j-\omega_l)^{-1}$$

is nonzero when $k < s$. When $k = s$ we have $C_{s,s} = 0$; then there is no explosion. As an example we may take $\omega_j = j$; then we have

$$\prod_l (j-l)^{-1} = \frac{(-1)^{s-j}}{(j-1)!(s-j)!},$$

so that the sum over $1 \leqslant j \leqslant k$ can be written as

$$\frac{1}{(s-1)!}\sum_{q=0}^{k-1}\tbinom{s-1}{q}(-1)^{s-1-q}.$$

When $k = s$ this is equal to

$$\frac{1}{(s-1)!}(1-1)^{s-1} = 0,$$

while the partial sums over $j \in [1,k]_{\mathbf{Z}}$, $1 \leqslant k \leqslant s-1$, are easily seen to be nonzero. In fact, because of the alternating signs we can estimate

$$\left|\sum_{q=0}^{k-1}\tbinom{s-1}{q}(-1)^{s-1-q}\right| \geqslant \tbinom{s-1}{k-1} - \left|\sum_{q=0}^{k-2}\tbinom{s-1}{q}(-1)^{s-1-q}\right| > \tbinom{s-1}{k-1} - \tbinom{s-1}{k-2} \geqslant 0$$

as soon as $2 \leqslant k \leqslant \frac{1}{2}(s-2)$. For $k = 1$ the result is obvious, and for $\frac{1}{2}(s-2) < k \leqslant m-1$ the result follows by symmetry. We are done. □

5. Zeros converging to real zeros

We have studied polynomials with non-real zeros and now want to investigate what happens if some of the zeros converge to the reals. First some examples.

Example 5.1. We define $P(\xi,\tau_1,\tau_2) = (\xi-\tau_1)(\xi-\tau_2)$ and study the convergence of

$$F_\tau(x) = \frac{1}{2\pi}\int\frac{e^{ix\xi}}{(\xi-\tau_1)(\xi-\tau_2)}d\xi, \qquad \operatorname{Im}\tau_j \neq 0,$$

as $\tau_j \to \alpha_j$, where α_1 and α_2 are two given complex numbers. If $x \operatorname{Im}\tau_j < 0$, then $F_\tau(x) = 0$. Next suppose that $x > 0$ and that $\operatorname{Im}\tau_j > 0$, $\tau_1 \neq \tau_2$. Then

$$F_\tau(x) = i\frac{e^{i\tau_1 x}}{\tau_1-\tau_2} + i\frac{e^{i\tau_2 x}}{\tau_2-\tau_1} = i\frac{e^{i\tau_1 x}-e^{i\tau_2 x}}{\tau_1-\tau_2},$$

which is the restriction of an entire function to the set of (τ_1,τ_2) satisfying $\operatorname{Im}\tau_j > 0$. When for instance $\tau_j \to 0$ under this condition, $F_\tau(x) \to -xH$, which is a fundamental solution for $P(D,0,0) = D^2 = -d^2/dx^2$ (here H is the Heaviside function).

However, if $x > 0$, $\operatorname{Im}\tau_1 > 0$ and $\operatorname{Im}\tau_2 < 0$, then

$$F_\tau(x) = i\frac{e^{i\tau_1 x}}{\tau_1 - \tau_2}$$

which does not converge when $\tau_j \to 0$. Thus the lesson is that if we dissolve the double zero at the origin of the polynomial ξ^2 as $(\xi - \tau_1)(\xi - \tau_2)$ with $\operatorname{Im}\tau_j$ of the same sign, then we get very good convergence, but not when the imaginary parts have different signs. The same phenomenon can of course appear even if we use only one parameter: $(\xi - \tau)(\xi - 2\tau)$ compared with $(\xi - \tau)(\xi + \tau)$. □

Let P be a polynomial with zeros $\alpha_1, \ldots, \alpha_m$, possibly real. We want to investigate what happens as non-real zeros converge to the α_j.

If $\operatorname{Im}\alpha_k \neq 0$ we can let $\tau_k \to \alpha_k$ arbitrarily, but if we have a multiple real zero, say $\alpha_1 = \alpha_2 = \cdots = \alpha_k \in \mathbf{R}$, then, as we have seen in Example 5.1, we must require either $\operatorname{Im}\tau_j > 0$ or $\operatorname{Im}\tau_j < 0$ for all $j = 1, \ldots, k$ to obtain convergence.

We can also use only one parameter τ, as follows. Let P be any polynomial in one variable of degree at least 2. Then it grows sufficiently fast at infinity, but it may of course have real zeros. Let its zeros be $\alpha_1, \ldots, \alpha_m$. We define

$$P(\xi, \tau) = \prod (\xi - \alpha_j - \tau).$$

Then $P(\xi, 0) = P(\xi)$ and there is a positive ε such that $P(\xi, \tau)$ has no real zeros for $0 < \operatorname{Im}\tau < \varepsilon$ and $-\varepsilon < \operatorname{Im}\tau < 0$. If P has a double zero, say $\alpha_1 = \alpha_2$, then we consider instead

$$P(\xi, \tau) = (\xi - \alpha_1 - \tau)(\xi - \alpha_2 - 2\tau) \prod_3^m (\xi - \alpha_j + \tau);$$

if P has a triple zero $\alpha_1 = \alpha_2 = \alpha_3$, we instead replace the first three factors by $(\xi - \alpha_1 - \tau)(\xi - \alpha_2 - 2\tau)(\xi - \alpha_3 - 3\tau)$ for example. It is therefore clear how to define a polynomial $P(\xi, \tau)$ which has only simple and non-real zeros $\alpha_k(\tau)$ for $0 < \operatorname{Im}\tau < \varepsilon$ and $-\varepsilon < \operatorname{Im}\tau < 0$. The residues are easy to calculate, and we get for $x > 0$,

$$F_\tau(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{ix\xi}}{P(\xi, \tau)} d\xi = i\sum_k \frac{e^{ix\alpha_k(\tau)}}{\prod_j (\alpha_k(\tau) - \alpha_j(\tau))},$$

where the product is over all $j \neq k$ for a fixed k, and the sum is over all k such that $\operatorname{Im}\alpha_k(\tau) > 0$. If all zeros are simple, we can take $\alpha_k(\tau) = \alpha_k + \tau$ which makes the differences $\alpha_k(\tau) - \alpha_j(\tau)$ independent of τ and the convergence as $\tau \to 0$ is easy. If we choose $\operatorname{Im}\tau > 0$ we get

$$F_\tau(x) = ie^{i\tau x}\sum_k \frac{e^{i(\alpha_k + \tau)x}}{\prod_j (\alpha_k - \alpha_j)} \longrightarrow i\sum_k \frac{e^{i\alpha_k x}}{\prod_j (\alpha_k - \alpha_j)}, \qquad x > 0,$$

where the sum is over all k with $\operatorname{Im}\alpha_k \geqslant 0$, while $F_\tau(x) = 0$ for $x < 0$. When $\operatorname{Im}\tau < 0$ we get instead

$$F_\tau(x) \longrightarrow -i\sum_k \frac{e^{i\alpha_k x}}{\prod_j (\alpha_k - \alpha_j)}, \qquad x < 0,$$

where the sum is now over all k such that $\operatorname{Im}\alpha_k \leqslant 0$, while $F_\tau(x) = 0$ for $x > 0$. Therefore the limit of F_τ from above or from below is the usual combination of exponential functions to the left and right of the origin. When the zeros are not simple the whole thing is not really more difficult. We note that the convergence here is very good: all functions are restrictions of entire functions of τ to strips like $0 < \operatorname{Im}\tau < \varepsilon$.

6. One-dimensional slices of fundamental solutions

The ultimate goal is to understand the dependence of E_P on P globally, but it is easier first to see what happens on a straight line in the space of polynomials. Let us first look at two examples.

Example 6.1. Take $n = 1$ and define, with $P(\xi, z) = P(\xi) - zQ(\xi) = \xi^2 + z$,

$$F_z(x) = \frac{1}{2\pi}\int \frac{e^{ix\xi}}{\xi^2 + z} d\xi = i\frac{e^{-|x|\sqrt{z}}}{2i\sqrt{z}}, \qquad z \in \mathbf{C} \smallsetminus \,]-\infty, 0]\,,$$

where we have written F_z for E_{P-zQ}. This explodes when $z \to 0$, but if we subtract $(e^{x\sqrt{z}} + e^{-x\sqrt{z}})/4\sqrt{z}$ (which solves the homogeneous equation $(P(D) - zQ(D))u = 0$) we get

$$\frac{e^{-|x|\sqrt{z}}}{2\sqrt{z}} - \frac{e^{x\sqrt{z}} + e^{-x\sqrt{z}}}{4\sqrt{z}} \to -\tfrac{1}{2}|x|, \qquad z \to 0, z \in \mathbf{C} \smallsetminus \,]-\infty, 0]\,,$$

which is a fundamental solution for $P(D, 0) = P(D) = D^2 = -d^2/dx^2$. □

Example 6.2. A slight variation of the last example gives a function which is holomorphic outside a compact set. Define $P(\xi, z) = P(\xi) - zQ(\xi) = \xi^2 - z(\xi^2 + 1)$. Then

$$F_z(x) = \frac{1}{2\pi}\int \frac{e^{ix\xi}}{\xi^2 - z(\xi^2+1)} d\xi = \frac{1}{2\pi(1-z)}\int \frac{e^{ix\xi}}{\xi^2 - z/(1-z)} d\xi = -\frac{e^{-|x|\sqrt{z/(z-1)}}}{2\sqrt{z(z-1)}},$$

defined first for $|z| > 1$, and then by analytic continuation to the complement of $[0, 1]$, or to the complement of any curve connecting 0 and 1. The solution explodes at $z = 0$. To get convergence we can subtract the value at $x = 0$, i.e., $-1/2\sqrt{z(z-1)}$.

When $z \to 1$ the solution does not explode but converges to $-\delta$, which is a fundamental solution to $P(D, 1) = -1$. The point $z_* = 1$ is the point where the degree of $P(\xi, z)$ drops. □

We shall thus look at one-dimensional slices of the function $P \mapsto E_P$. This means that we consider a complex line $z \mapsto P - zQ$ passing through P and $P - Q$, and study the singularities of $F_z = E_{P-zQ}$. The situation is particularly simple if Q is chosen with all its zeros in the upper or lower half plane:

Theorem 6.3. *Let P and Q be polynomials in one variable of the same degree $m \geqslant 2$ and assume that all zeros of Q have positive (resp. negative) imaginary part. Then there is a constant R such that $|P(\xi)| \leqslant R|Q(\xi)|$ for all real ξ. The function*

$$F_z(x) = E_{P-zQ}(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{ix\xi}}{P(\xi)-zQ(\xi)}d\xi, \qquad x \in \mathbf{R}, \quad z \in \mathbf{C}, \quad |z| > R,$$

is holomorphic for $|z| > R$. It has a holomorphic extension to $S^2 \smallsetminus \{z_*\}$, where S^2 is the Riemann sphere $\mathbf{C} \cup \{\infty\}$ and $z_* = \lim_{\xi\to\infty} P(\xi)/Q(\xi) \neq 0$. Its value at $z = 0$ is a fundamental solution for $P(D)$.

Proof. With a z satisfying $|z| > R$ we have

$$|P - zQ| \geqslant |z||Q| - |P| \geqslant (|z| - R)|Q| > 0$$

on the real axis. The integral defining $F_z(x)$ is convergent for $|z| > R$. For very large $|z|$, the zeros of $P - zQ$ are close to those of Q; hence the zeros of $P - zQ$ have positive imaginary part also for all z with $|z| > R$, since they cannot pass the real line. By Theorem 4.1 the function $z \mapsto F_z(x)$ has only one singularity, viz. the point where the degree drops, which is the point $z_* \neq 0$. The extension to $z = \infty$ follows because F_z is bounded for $|z| \geqslant R + 1$. ☐

If we choose a Q with zeros in both the upper and lower half-planes, then the function $F_z(x)$ in Theorem 6.3 will in general have more than one singular point, and its analytic extension cannot be defined in the complex plane. Therefore we shall now discuss extensions defined on a Riemann surface.

7. Meromorphic functions over the Riemann sphere

Let f and g be meromorphic in $\mathbf{C}$. If α and β are two points and γ a curve connecting them, we study the function

$$h(z) = \int_\gamma \frac{f(\zeta)}{g(\zeta)-z}d\zeta.$$

It is holomorphic on $S^2 \smallsetminus g(\gamma)$ if γ avoids the poles of f. Indeed, for any given point $z_0 \in \mathbf{C} \smallsetminus g(\gamma)$ we can write

$$h(z) = \int_\gamma \frac{1}{1 - \dfrac{z-z_0}{g(\zeta)-z_0}} \cdot \frac{f(\zeta)}{g(\zeta)-z_0}d\zeta = \sum_0^\infty (z-z_0)^k \int_\gamma \frac{f(\zeta)}{(g(\zeta)-z_0)^{k+1}}d\zeta,$$

which converges nicely when $|z - z_0|$ is less than the distance from z_0 to $g(\gamma)$ and γ avoids the poles of f.

We can extend h to a Riemann surface Y as follows. Let σ_j be the zeros of g', the critical points of g. For simplicity we suppose that there are only finitely many of those; $g(\sigma_j)$ are then the finitely many critical values of g. Denote by Z the finite set $\{g(\alpha), g(\beta), g(\sigma_j)\}$. Then Y shall be the universal covering surface of $S^2 \smallsetminus Z$, where again we write $S^2 = \mathbf{C} \cup \{\infty\}$ for the Riemann sphere. In other words, Y consists of ∞ and the space obtained from the universal covering surface of $\mathbf{C} \smallsetminus Z$ by identifying all points over a point in $\mathbf{C}$ of sufficiently large modulus—if we go around all the points we do not come to a new sheet. The elements of Y can be described as pairs (z, C), where C is a curve avoiding Z, starting at infinity, ending at z, and considered modulo homotopic curves in $S^2 \smallsetminus Z$. We shall refer to Y simply as *the Riemann surface over* $S^2 \smallsetminus Z$.

Proposition 7.1. *Let f and g be meromorphic functions in $\mathbf{C}$. We assume that f has finitely many poles and g' finitely many zeros. Let γ_0 be a curve connecting a point α with a point β and avoiding the poles of f. Then*

$$h(z) = \int_{\gamma_0} \frac{f(\zeta)}{g(\zeta) - z} d\zeta, \tag{7.1}$$

defined originally for $z \in S^2 \smallsetminus g(\gamma_0)$, has a meromorphic extension to the Riemann surface Y over $S^2 \smallsetminus Z$, where Z is the image under g of the finite set consisting of α, β and the zeros of g'. The poles of h are at points $g(\tau)$, where τ is a pole of f which is not a pole of g.

Example 7.2. Let $f(\zeta) = 1$, $g(\zeta) = \zeta^2$. Zero is a critical value of g. We get

$$h(z) = \int_\alpha^\beta \frac{1}{\zeta^2 - z} d\zeta = \frac{1}{2\sqrt{z}} \left(\log \frac{\beta - \sqrt{z}}{\alpha - \sqrt{z}} - \log \frac{\beta + \sqrt{z}}{\alpha + \sqrt{z}} \right)$$

when $z \in S^2 \smallsetminus [0, \max(\alpha^2, \beta^2)]$, which can be extended to the Riemann surface over $S^2 \smallsetminus \{0, \alpha^2, \beta^2\}$. Here the choice of square root does not influence the value of $h(z)$, but we have to be careful with the logarithm.

If $\alpha < 0 < \beta$, there is a singularity at the origin, and we have $|h(z)| \to +\infty$ as $\operatorname{Re} z < 0$, $z \to 0$. More precisely $h(z) \approx i\pi/\sqrt{z}$ as $\operatorname{Re} z < 0$, $z \to 0$, if we define $\sqrt{z}$ in $\mathbf{C} \smallsetminus i\,[0, -\infty[$. This shows that there can actually appear singularities at the critical values of g.

Example 7.3. Let $f(\zeta) = 1/\zeta$, $g(\zeta) = \zeta$. Here g has no critical values, but f has a pole at the origin. Define

$$h(z) = \int_\gamma \frac{1}{\zeta(\zeta - z)} d\zeta = \frac{1}{z} \left(\log \frac{\beta - z}{\alpha - z} - \log \frac{\beta}{\alpha} \right), \qquad z \in S^2 \smallsetminus T,$$

where γ is a curve from $\alpha < 0$ to $\beta > 0$ passing under the origin and under z if $\alpha < \operatorname{Re} z < \beta$, and where

$$T = \{z = \alpha + it;\ t \leqslant 0\} \cup \{z = \beta + it;\ t \leqslant 0\}.$$

Then there is no singularity of h at the origin. But if we let z make one revolution around β, there appears a pole. So poles of h can appear at points $g(\tau)$, τ a pole of f, although they do not necessarily appear on every sheet. □

Proof of Proposition 7.1. We shall define the extensions of h using the formula

$$h(z) = \int_\gamma \frac{f(\zeta)}{g(\zeta) - z} d\zeta, \tag{7.2}$$

where γ is the sum of a curve connecting α and β, and finitely many circles around $g(\tau_k)$, τ_k being the poles of f that are not poles of g. We shall always use curves of class C^1 avoiding the poles of f. When γ is moved across a point $g(\tau)$, with τ a pole of f, we have to add or subtract a term

$$\int_\Gamma \frac{f(\zeta)}{g(\zeta) - z} d\zeta, \tag{7.3}$$

where Γ is a small circle around $g(\tau) \in \mathbf{C}$; *small* here means so small that it does not contain other images of poles of f under g, and that z is outside $g(\Gamma)$. The contribution of (7.3) is

$$2\pi i \operatorname*{\mathbf{res}}_{\zeta=\tau} \left(\frac{f(\zeta)}{g(\zeta) - z} \right).$$

If τ is a simple pole of f and $g(\tau)$ is the only image of a pole of f, then the residue is

$$\operatorname*{\mathbf{res}}_{\zeta=\tau} \left(\frac{f(\zeta)}{g(\zeta) - z} \right) = \frac{\mathbf{res}_{\zeta=\tau} f(\zeta)}{g(\tau) - z},$$

giving rise to a simple pole of h at the point $g(\tau)$. More generally, if τ is a pole of f of order s,

$$f(\zeta) = \sum_1^s \frac{A_j}{(\zeta - \tau)^j} + O(1), \qquad \zeta \to \tau,$$

where the A_j are some constants with $A_s \neq 0$, then, assuming that $\tau = 0$ and $g(\tau) = g(0) = 0$ to simplify the formulas,

$$\frac{f(\zeta)}{g(\zeta) - z} = -f(\zeta) \sum_1^\infty \frac{g(\zeta)^{k-1}}{z^k} \qquad \text{when } |\zeta/z| \text{ is small.}$$

The residue is

$$\operatorname*{\mathbf{res}}_{\zeta=0} \frac{f(\zeta)}{g(\zeta) - z} = -\operatorname*{\mathbf{res}}_{\zeta=0} \sum_{j=1}^s \sum_{k=1}^\infty \frac{A_j g(\zeta)^{k-1}}{\zeta^j z^k} = -\operatorname*{\mathbf{res}}_{\zeta=0} \sum_{j=1}^s \sum_{k=1}^s \frac{A_j g(\zeta)^{k-1}}{\zeta^j z^k} = \sum_1^s \frac{B_k}{z^k},$$

where

$$B_k = -\operatorname*{\mathbf{res}}_{\zeta=0} \sum_{j=1}^s \frac{A_j g(\zeta)^{k-1}}{\zeta^j}.$$

This shows that the order of the pole is at most s. In fact, since $\tau = 0$ is a simple zero of g by assumption, it is precisely s, since

$$B_s = -\operatorname*{\mathbf{res}}_{\zeta=0} \sum_1^s \frac{A_j g(\zeta)^{s-1}}{\zeta^j} = -A_s g'(0)^{s-1} \neq 0.$$

In spite of this, as Example 7.3 shows, there need not appear a pole of h at $g(\tau)$ on every sheet. This is because (7.3) expresses the difference between h on different sheets over the point $g(\tau)$.

If τ is, say, a quadruple pole of f, we get the residue

$$\operatorname*{\mathbf{res}}_{\zeta=\tau} \frac{f(\zeta)}{g(\zeta)-z} = -\frac{A_4 g'(\tau)^3}{(g(\tau)-z)^4} + \frac{A_4 g'(\tau)g''(\tau) + A_3 g'(\tau)^2}{(g(\tau)-z)^3} - \frac{A_4 g'''(\tau) + 3A_3 g''(\tau) + 6A_2 g'(\tau)}{6(g(\tau)-z)^2} + \frac{A_1}{g(\tau)-z},$$

which gives rise to a quadruple pole of h at $g(\tau)$. From the formula we can of course also see what happens if the pole of f is of order two or three.

In general it can happen that $g(\tau_1) = g(\tau_2) = \cdots = g(\tau_r)$ for a finite number of poles of f, thus giving rise to a finite sums of this form.

The poles of f which are also poles of g cause no trouble. Indeed, if τ is such a pole, then $g(\tau) = \infty$, and h is originally defined and equal to 0 at ∞.

Suppose we have a curve γ which defines a function h by (7.2), and let (z, C) represent a point of the Riemann surface Y. If C does not intersect γ, we already have a definition of h for all points of C, in particular at z; if not, we follow the curve C from ∞ towards $g(\gamma)$. There is a point w which is the first point where we hit $g(\gamma)$. This means that $g^{-1}(w) \cap \gamma$ is nonempty; by compactness it must consist of finitely many points α_j. By construction w is not in Z, so $g'(\alpha_j) \neq 0$ and g defines a holomorphism of a neighborhood of α_j onto a neighborhood of w. Let C be parametrized by $t \in [0,1]$ so that $C(0) = \infty$ and $C(1) = w$. We now deform the curve γ near every α_j so that the image under g of the new curve γ' does not meet the curve C for parameter values $t \leqslant 1$. Therefore the integral with γ replaced by γ' defines an extension of h defined in a neighborhood of w. When we let the curve pass over a pole of f, we get a pole of h as described above. This procedure can go on à la Weierstraß until we cover all of Y. There is never any pole of f on the curves we use, and g is a local holomorphism on all points of all curves except possibly the endpoints α and β. ☐

8. Fundamental solutions defined on a Riemann surface over the Riemann sphere

Theorem 8.1. *Let P and Q be polynomials of one variable of degree at least two satisfying $|P| \leqslant |Q|$ and $Q \neq 0$ on the real axis. Define, for any test function φ on $\mathbf{R}$, a holomorphic function h for $|z| > 1$ by*

$$h(z) = \frac{1}{2\pi} \int_{\mathbf{R}} \frac{\widehat{\varphi}(-\xi)}{P(\xi) - zQ(\xi)} d\xi.$$

Denote by g the rational function P/Q. Then h has a meromorphic extension to the Riemann surface over $S^2 \smallsetminus Z$ with $Z = \{g(\infty), g(\sigma_j)\}$, where the σ_j are the zeros *of g'. The extension is holomorphic except for possible poles at points over $g(\tau_k)$, where τ_k is a common zero of P and Q such that its order as a zero of P is at least as high as its order as a zero of Q. Assume now that $g(\sigma_j), g(\tau_k) \neq 0$.*

(A). *If $\deg Q = \deg P$, then $z_* = g(\infty) \neq 0$ and there is no problem with the convergence as $z \to 0$. The value $h(0)$, for any admissible choice of the curve C in the proof of Proposition 7.1, defines a fundamental solution of $P(D)$.*

(B). *If $\deg Q = \deg P + 1$, then g has a simple zero at infinity, and we see that $g([-\infty, +\infty] \smallsetminus [-\alpha, \alpha])$ divides a typical neighborhood of 0 into two components V^+ and V^-. We have convergence of $h(z)$ as $z \in V^+$, $z \to 0$, and also as $z \in V^-$, $z \to 0$. Moreover h is of class C^∞ on the closures of V^+ and V^-.*

(C). *If $\deg Q \geqslant \deg P + 2$, then g has a multiple zero at ∞, and the function h may explode as $z \to 0$.*

If $g(\sigma_j) = 0$ for some j we see that σ_j is a multiple zero of g and we do not necessarily have convergence as $z \to 0$. Similarly if $g(\tau_k) = 0$ for some k.

If P has no multiple zeros, then $g(\sigma_j) \neq 0$ (for if $g(\sigma_j) = 0$, then g has a multiple root, and therefore also $P = gQ$). Also $g(\tau_k) \neq 0$ (for if this is not the case, then again $P = gQ$ has a multiple zero). Therefore, for polynomials P without multiple zeros, we can take any Q of the same degree as P and with the property $|Q| \geqslant |P|$, $Q \neq 0$ on the real axis, and conclude that we have the simple case (A). More generally, if P has no real double roots, then we can take Q of the same degree as P, without real zeros, and with a zero of the same multiplicity as that of P wherever P has a zero of multiplicity at least two. This also gives case (A). If P is without double roots we can also take Q of degree $\deg P + 1$, and get case (B).

When Q has all its roots in the upper or lower half plane, Theorem 6.3 says that E_{P-zQ} has only one singular point z_*. So it is not always necessary to avoid the points $g(\sigma_j)$ or $g(\tau_k)$.

In case (B) g has a simple zero at infinity, and as noted every sufficiently small neighborhood of the origin is divided by $g([-\infty, +\infty] \smallsetminus [-\alpha, \alpha])$. If g has a double zero (case (C)) this can also be so, but not necessarily. Indeed, if $g(1/\eta) \approx \eta^2 + i\eta^3$, then every sufficiently small neighborhood of the origin is divided by the image under g of $[-\infty, +\infty] \smallsetminus [-\alpha, \alpha]$, but, on the other hand, if g is an even function and near infinity satisfies $g(1/\eta) \approx \eta^2 + i\eta^4$, then $g([-\infty, +\infty] \smallsetminus [-\alpha, \alpha])$ does not divide a connected neighborhood. That both cases can occur is shown by the following simple examples, both belonging to case (C).

Example 8.2. If $P = \xi + i$, $Q = \xi^3 + 2i$, then

$$g(1/\eta) = \frac{\eta^2 + i\eta^3}{1 + 2i\eta^3} \approx \eta^2 + i\eta^3.$$

If $P = \xi^2 + i$, $Q = \xi^4 + 2$, then g is even and

$$g(1/\eta) = \frac{\eta^2 + i\eta^4}{1 + 2\eta^4} \approx \eta^2 + i\eta^4. \quad \square$$

Proof of Theorem 8.1. To prove the theorem we shall apply Proposition 7.1 to $f(\zeta) = \widehat{\varphi}(-\zeta)/Q(\zeta)$ and $g = P/Q$. Certainly g' has finitely many zeros σ_j, and f has finitely many poles τ_k. If τ_k is a pole of f which is not a zero of P, then τ_k is a zero of Q and a pole of g, so it will cause no trouble according to Proposition 7.1. But if τ_k is a common zero of Q and P, then it need not be a pole of g, and so must be considered. More precisely, there can be poles of h only at points over those points $g(\tau_k)$ where τ_k is a zero of Q of order less than or equal to its order as a zero of P, for otherwise $g(\tau_k) = \infty$, which is harmless.

We divide the integral defining h into three pieces:

$$\int_{\mathbf{R}} = \int_{-\infty}^{-\alpha} + \int_{-\alpha}^{\alpha} + \int_{\alpha}^{+\infty},$$

where α has to be determined. Here the middle term is holomorphic on the Riemann surface Y over $S^2 \smallsetminus Z$, where Z is $\{g(-\alpha), g(\alpha), g(\sigma_j)\}$. In order to define its extension we have to move z on a curve C from infinity to a neighborhood of the origin and then choose, for each such z, curves γ_z connecting $-\alpha$ to α and such that $g(\xi) \neq z$ when ξ is on γ_z.

(A). If $g(\infty), g(\sigma_j), g(\tau_k) \neq 0$ we just choose a large α to keep $g(\pm\alpha)$ away from the origin. It is easy to see that the integral over $[\alpha, +\infty[$ tends to zero at a point $z \neq z_* = g(\infty)$ as $\alpha \to +\infty$, uniformly for $\varepsilon \leqslant |z - z_*| \leqslant 1/\varepsilon$. In fact, $|P(\xi) - zQ(\xi)| \geqslant 1$ when ξ is large, $|\xi| \geqslant R_z$, $z \neq z_*$, and uniformly when z is in a compact set. Thus in this case h is holomorphic in Y and the behavior at the origin is no problem, except of course that there may be several sheets.

(B). If on the other hand $g(\infty) = 0$, we have to consider a little more carefully the integral

$$h_\alpha(z) = \frac{1}{2\pi} \int_{\alpha}^{+\infty} \frac{\widehat{\varphi}(-\xi)}{P(\xi) - zQ(\xi)} d\xi.$$

In fact, the theorem of dominated convergence is not applicable, because z can be very close to $g([\alpha, +\infty[)$, so that $P(\xi) - zQ(\xi)$ becomes very small. We shall prove that

$$|h_\alpha(z) - h_\alpha(0)| \leqslant R|z|$$

for some constant R. Consider

$$\frac{\widehat{\varphi}(-\xi)}{P(\xi) - zQ(\xi)} - \frac{\widehat{\varphi}(-\xi)}{P(\xi)} = z \frac{\widehat{\varphi}(-\xi)}{P(\xi)(g(\xi) - z)}.$$

Clearly it suffices to show that

$$\int_{\alpha}^{+\infty} \frac{\widehat{\varphi}(-\xi)}{P(\xi)(g(\xi) - z)} d\xi$$

is bounded when $z \in V^+$. Since in this case ∞ is a simple zero of g, we can use $t = g(\xi)$ as the variable of integration and write the integral as

$$\int_0^{\beta} \frac{\psi(t)}{t - z} dt,$$

where integration is along a curve defined by g. That this integral is bounded when $z \in V^+$ follows from the next proposition.

Proposition 8.3. *Let γ be a C^∞ smooth curve starting at the origin and ending at $\beta \neq 0$. Let ψ be holomorphic in a punctured neighborhood of the origin and assume that its restriction to γ has a zero of high order at 0. Then*

$$\Psi(z) = \int_0^\beta \frac{\psi(t)}{t-z} dt,$$

defined for z not on γ, is bounded near the origin. Moreover, if $\psi|_\gamma$ has a zero of infinite order at 0, then Ψ is C^∞ up to the boundary in the complement of γ.

Proof. We integrate by parts twice:

$$\begin{aligned}\Psi(z) = \int_0^\beta \frac{\psi(t)}{t-z} dt = \Big[\psi(t)\log(t-z)\Big]_\gamma - \Big[\psi'(t)(t-z)(\log(t-z)-1)\Big]_\gamma \\ + \int_\gamma \psi''(t)(t-z)(\log(t-z)-1)dt \\ = \psi(\beta)\log(\beta-z) - \psi'(\beta)(\beta-z)(\log(\beta-z)-1) \\ + \int_\gamma \psi''(t)(t-z)(\log(t-z)-1)dt.\end{aligned}$$

Now $(t-z)\log(t-z)$ is bounded for all $t \in \gamma$, for both $t-z$ and its argument are bounded ($t-z$ does not wind around a lot as $t \to 0$ along γ).

If ψ has a zero of infinite order on γ we see that Ψ is of class C^∞ up to γ. In fact,

$$\Psi^{(k)}(z) = c_k \int_0^\beta \frac{\psi(t)}{(t-z)^{k+1}} dt,$$

and we can decrease the order of the singularity using integration by parts:

$$\int_0^\beta \frac{\psi(t)}{(t-z)^{k+1}} dt = -\frac{\psi(\beta)}{k(\beta-z)^k} + \frac{1}{k}\int_0^\beta \frac{\psi'(t)}{(t-z)^k} dt, \qquad k = 1, 2, 3, \ldots .$$

This formula shows that all derivatives of Ψ are bounded in the complement of γ. This concludes the proof of Proposition 8.3 and consequently that of Theorem 8.1. □

Maybe the following example exhibits the phenomena in case (B) more clearly.

Example 8.4. Let $P(\xi) = \xi$ and $Q(\xi) = \xi^2 + 1$; thus $P(D) = D = -id/dx$. Then we study

$$\frac{1}{2\pi}\int \frac{e^{ix\xi}}{\xi + z(\xi^2+1)} d\xi = \frac{1}{2\pi z}\int \frac{e^{ix\xi}}{\xi^2 + \xi/z + 1} d\xi, \qquad z \in \mathbf{C} \smallsetminus \mathbf{R}$$

There are poles at

$$\xi = \frac{-1 \pm \sqrt{1-4z^2}}{2z}.$$

It converges to $i(H-1)$ for $\operatorname{Im} z > 0$ and to iH for $\operatorname{Im} z < 0$, both of which are fundamental solutions for $-id/dx$. Calculations become a bit easier if we use instead

$$P(\xi, z) = \xi + z(\xi^2 - z\xi - 1) = z(\xi - z)(\xi + 1/z).$$

(This is the image of a straight line under a certain local biholomorphism in the space of polynomials.) Then

$$F_z(x) = E_{P-zQ} = \frac{1}{2\pi}\int \frac{e^{ix\xi}}{P(\xi,z)}d\xi = \frac{1}{2\pi z}\int \frac{e^{ix\xi}}{(\xi - z)(\xi + 1/z)}d\xi = 0$$

when $x \operatorname{Im} z < 0$. When $x > 0$ and $\operatorname{Im} z > 0$ we get instead

$$F_z(x) = \frac{1}{2\pi z}\int \frac{e^{ix\xi}}{(\xi - z)(\xi + 1/z)}d\xi = \frac{i}{1+z^2}e^{ixz} - \frac{i}{1+z^2}e^{-ix/z}.$$

We now apply this to a test function $\varphi \in \mathscr{D}$:

$$\int F_z(x)\varphi(x)dx = \frac{i}{1+z^2}(\widehat{\varphi}_0(-z) - \widehat{\varphi}_0(1/z)), \tag{8.1}$$

denoting by $\varphi_0 = H\varphi$ the function which is zero for $x < 0$ and $\varphi(x)$ for $x > 0$. When $z \to 0$, the last expression tends to $i\widehat{\varphi}_0(0)$, which means that F_z tends to iH weakly. ☐

The phenomenon in case (B) of Theorem 8.1 can now be expressed as a certain regularity of $\varphi_0(1/z)$ in the upper half plane:

Proposition 8.5. *Let $\varphi \in \mathscr{S}(\mathbf{R})$ and define*

$$G(z) = \int_0^{+\infty} e^{-it/z}\varphi(t)dt, \qquad z \in U = \{z \in \mathbf{C};\ \operatorname{Im} z > 0\}.$$

Then G is holomorphic in U and has a C^∞ extension to all of $\mathbf{C}$.

Proof. We have a formula for the derivatives of G:

$$G^{(k)}(z) = -iQ_{k-1}(t, d/dt)\varphi\Big|_{t=0} + \int_0^{+\infty} e^{-it/z}Q_k(t, d/dt)\varphi(t)dt, \qquad \operatorname{Im} z > 0,$$

where Q_k is a differential operator of order $2k$ with polynomial coefficients:

$$Q_{-1} = 0, \qquad Q_0 = 1, \qquad Q_k = -i(2d/dt + td^2/dt^2)Q_{k-1}(t, d/dt), \quad k \geqslant 1.$$

This formula is proved by induction, integrating by parts twice for each step and using the formula

$$\frac{\partial}{\partial z}e^{-it/z} = -it\frac{\partial^2}{\partial t^2}e^{-it/z}.$$

Because $|e^{-it/z}|$ is bounded by 1 for $\operatorname{Im} z > 0$, $t \geqslant 0$, we see that all derivatives of G in the open upper half plane are bounded, which implies that G has a C^∞ extension to the closed upper half plane as well as to $\mathbf{C}$.

This implies that F_z, defined by (8.1) and considered as a distribution-valued function for $\operatorname{Im} z > 0$, has a smooth extension to $\operatorname{Im} z \geqslant 0$. But it is not a holomorphic extension. The value for $z = 0$ is a fundamental solution for the operator we started with, $P(D) = -id/dx$. □

Example 8.6. A two-parameter variant of this is

$$P(\xi, z) = P(\xi, z_1, z_2) = z_2(\xi - z_1)(\xi + 1/z_2) = (\xi - z_1)(z_2\xi + 1).$$

Then for $x > 0$, $\operatorname{Im} z_j > 0$,

$$F_z(x) = \frac{1}{2\pi z_2}\int \frac{e^{ix\xi}}{(\xi - z_1)(\xi + 1/z_2)}d\xi = \frac{i}{1 + z_1 z_2}(e^{ixz_1} - e^{-ix/z_2});$$

$$\langle F_z, \varphi\rangle = \frac{i}{1 + z_1 z_2}(\widehat{\varphi}_0(-z_1) - \widehat{\varphi}_0(1/z_2)).$$

This function, defined for $|z_j| < 1/2$ and $\operatorname{Im} z_j > 0$, has a C^∞ extension to $|z_j| < 1/2$, $\operatorname{Im} z_j \geqslant 0$, $j = 1, 2$. □

9. What about two variables?

Example 9.1. Let R denote a rectangle in $\mathbf{R}^2$. The function

$$\int_R \frac{1}{\xi_1 + i\xi_2 - z}d\xi$$

is defined for all z, is holomorphic for $z \notin R$, and its restriction to the complement of R has a holomorphic extension to the Riemann surface defined by the corners of R. (This is true for all polygons R.) □

Therefore I guess that an integral of the form

$$h(z) = \int_{\mathbf{R}^2} \frac{\widehat{\varphi}(-\xi)}{P(\xi) - zQ(\xi)}d\xi, \qquad |z| > 1, \tag{9.1}$$

has a holomorphic extension to $\mathbf{C} \smallsetminus \{0\}$, provided $|P| < |Q|$ in $\mathbf{R}^2$ and P/Q tends to zero at infinity. Maybe the proof is not so different: we consider first integrals over large squares and prove that they define functions on the universal covering surface over the plane minus the corners of the square, and then prove some estimate for the integral over the complement of a large square. The first result proves that the integral (9.1) admits a holomorphic extension to $\mathbf{C} \smallsetminus \{0\}$, but its limit as $z \to 0$ is now more difficult to study.

The part of the integral (9.1) where $g = P/Q$ has maximal rank should cause no problem. The critical values form a set of Lebesgue measure zero in view of Sard's theorem. Such a set could be a curve, which would be very bad. But I believe the set where the rank is one is not dangerous.

Let us look at a couple of examples.

Example 9.2. The function

$$\int_\alpha^\beta \frac{1}{\xi_1 - z} d\xi_1 = \log \frac{z-\beta}{z-\alpha}, \qquad z \in S^2 \smallsetminus [\alpha, \beta],$$

has no singularity at the origin; it can be extended to the Riemann surface over $S^2 \smallsetminus \{\alpha, \beta\}$. Similarly

$$\int_\alpha^\beta \frac{1}{\xi_1^2 - z} d\xi_1 = \frac{1}{2\sqrt{z}}\left(\log \frac{\beta - \sqrt{z}}{a - \sqrt{z}} - \log \frac{\beta + \sqrt{z}}{a + \sqrt{z}}\right), \qquad z \in S^2 \smallsetminus [0, \max(\alpha^2, \beta^2)],$$

can be extended to the Riemann surface over $S^2 \smallsetminus \{0, \alpha^2, \beta^2\}$. If $0 < \alpha < \beta$, however, there is actually no singularity at the origin. But if $\alpha < 0 < \beta$, then the function does have a singularity at the origin: it equals $i\pi/\sqrt{z}$ for $z < 0$ if we choose $\sqrt{z}$ with positive imaginary part. This shows that the critical values of g in Proposition 7.1 can very well be singularities. But when we integrate such a singularity, it can disappear! □

Example 9.3. In two variables we study

$$\int_R \frac{1}{\xi_1^2 + i\xi_2 - z} d\xi = -i \int_{a_1}^{b_1} \log\left(\frac{\xi_1^2 + ib_2 - z}{\xi_1^2 + ia_2 - z}\right) d\xi_1$$

where R is the rectangle $\{\xi \in \mathbf{R}^2;\ a_j \leqslant \xi_j \leqslant b_j,\ j = 1, 2\}$. Here there is no singularity at the points with $\operatorname{Re} z = 0$, $a_2 < \operatorname{Im} z < b_2$, so that the fact that g has lower rank at these points causes no difficulty. We regard g here as a mapping from $\mathbf{R}^2$ into $\mathbf{R}^2$: $g(\xi_1, \xi_2) = (\xi_1^2, \xi_2)$. As such it can have rank 0, 1 or 2. When $\xi_1 \neq 0$ the rank is 2 and we have no problem; when $\xi_1 = 0$ the rank is 1 and we might have a problem, but this example shows that there is none! The remaining case, with rank zero, must be taken into account, but then again the images of such points form a finite set. □

The discussion around the three examples leads to two conjectures.

Conjecture 9.4. *Let P be a polynomial in n variables, not identically zero, and let Q be a polynomial without real zeros. Assume that $|P| \leqslant |Q|$ in $\mathbf{R}^n$ and consider, writing $g = P/Q$,*

$$h(z) = \int_{\mathbf{R}^n} \frac{\widehat{\varphi}(-\xi)}{P(\xi) - zQ(\xi)} d\xi = \int_{\mathbf{R}^n} \frac{\widehat{\varphi}(-\xi)}{Q(\xi)(g(\xi) - z)} d\xi, \qquad |z| > 1.$$

Then h is holomorphic for $|z| > 1$. The equation $g'(\zeta) = 0$ defines a variety M in $\mathbf{C}^n$, and $g(M)$ is its image in $\mathbf{C}$, the critical values of g. Now $g(M)$ consists *of finitely many points. Then $h(z)$ can be extended to the Riemann surface over $S^2 \smallsetminus (g(M) \cup g(\infty))$.*

Here $g(\infty)$ denotes the set of all limits of $g(\zeta)$ as $|\zeta| \to +\infty$. We can of course choose Q so that $g(\infty)$ is $\{0\}$.

Since we want to study the limit of h as z tends to zero, we will have some trouble if $0 \in g(M)$ or if $g(\infty) = 0$ is a double point. There may be an explosion at such a point. If $n = 1$, one can avoid this as we have seen, but I do not know if this is possible for $n \geqslant 2$.

Conjecture 9.5. *Define*

$$E_P(x) = (2\pi)^{-n} \int \frac{e^{ix\cdot\xi}}{P(\xi)} d\xi, \qquad x \in \mathbf{R}^n,$$

and

$$\langle E_P, \varphi \rangle = (2\pi)^{-n} \int \frac{\widehat{\varphi}(-\xi)}{P(\xi)} d\xi, \qquad \varphi \in \mathscr{S}(\mathbf{R}^n),$$

for any polynomial P of degree m such that

$$\inf_{\xi \in \mathbf{R}^n} |P(\xi)|(1 + \|\xi\|)^{-\rho} > 0.$$

In the first case we require that $\rho > n$, in the second this is not necessary. This defines a holomorphic function of P in an open subset of $\mathbf{C}^N$. It has a holomorphic extension to the Riemann domain spread over the set of all polynomials of degree m without multiple zeros.

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