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Generalized convexity: The case of lineally convex Hartogs domains

Christer Oscar Kiselman

Dedicated to the memory of Józef Siciak,
a great mathematician and
a faithful friend since 1974

Contents

Abstract
Inspired by mathematical morphology we study generalized convexity and prove that certain subsets of Hartogs domains are convex in a generalized sense.

1. Introduction

By the Hahn–Banach theorem, an open convex set in $\mathbf{R}^m$ is an intersection of open half-spaces; its complement a union of closed half-spaces. What if we replace the latter by balls? We shall study here a kind of generalized convexity where a set is called concave if it is a union of closed balls; its complement thus being an intersection of complements of closed balls. This will be done in particular for Hartogs domains which are lineally convex.

Lineal convexity is a kind of complex convexity intermediate between usual convexity and pseudoconvexity. More precisely, if A is a convex set in $\mathbf{C}^n$ which is either open or closed, then A is lineally convex (this is true also in the real category), and if Ω is a lineally convex open set in $\mathbf{C}^n$, then Ω is pseudoconvex.

There are several different notions of convexity related to lineal convexity. In increasing order of strength we have:

1. *Local weak lineal convexity in the sense of Yužakov & Krivokolesko* (Kiselman 2016: Definition 4.3);

2. *Local weak lineal convexity* (Kiselman 2016: Definition 4.1);

3. *Weak lineal convexity*, originally introduced as *Planarkonvexität* by Behnke & Peschl (1935:158, 162);

4. *Lineal convexity*, introduced as *convexité linéelle* by André Martineau (1966:73; 1977:228);

5. $\mathbf{C}$-*convexity,* originally introduced as *convexité linéelle forte* by Martineau (1967: 400; 1968; 1977:265, 325). Defined for subsets of $\mathbf{C}^n$ in Hörmander (1994: Definition 4.6.6) and (slightly differently) for subsets of projective space $\mathbf{P}^n$ in Andersson et al. (2004: Definition 2.2.1)

The main results are presented in Sections 8 and 10. It is shown there that certain subsets of Hartogs domains have convexity properties originating in mathematical morphology. We also study external tangent planes of sets that do not necessarily have a smooth boundary.

Notation

The inner product of two vectors in $\mathbf{R}^m$ or $\mathbf{C}^n$ shall be denoted by a dot:

$$x \cdot y = x_1 y_1 + \cdots + x_m y_m; \quad z \cdot w = z_1 w_1 + \cdots + z_n w_n, \qquad x, y \in \mathbf{R}^m,\ z, w \in \mathbf{C}^n.$$

The Euclidean norm will be written like this:

$$\|x\|_2 = \sqrt{x \cdot x}; \quad \|z\|_2 = \sqrt{z \cdot \bar{z}}, \qquad x \in \mathbf{R}^m,\ z \in \mathbf{C}^n.$$

We shall write

$$B_<(c, r) = \{x \in \mathbf{R}^m;\ \|x - c\|_2 < r\}; \quad B_\leqslant(c, r) = \{x \in \mathbf{R}^m;\ \|x - c\|_2 \leqslant r\}$$

for the open (strict) and closed (non-strict) balls, respectively, with center at c and of radius r. Similarly for balls in $\mathbf{C}^n$. When $n = 1$, we write instead $D_<(c, r)$ and $D_\leqslant(c, r)$ for the disks in $\mathbf{C}$.

When any norm can serve, we write just $\|x\|$.

For derivatives we write

$$f_{x_j} = \frac{\partial f}{\partial x_j}, \quad f_{y_j} = \frac{\partial f}{\partial y_j}, \qquad j = 1, \ldots, n,$$

and

$$f_{z_j} = \tfrac{1}{2}(f_{x_j} - i f_{y_j}), \quad f_{\bar{z}_j} = \tfrac{1}{2}(f_{x_j} + i f_{y_j}), \quad f_z = (f_{z_1}, \ldots, f_{z_n}).$$

For real-valued functions of one complex variable we have $\mathbf{grad}\, f = (f_x, f_y)$ with norm $\|\mathbf{grad}\, f\|_2 = 2|f_z|$.

2. Real and complex hyperplanes

Hyperplanes are affine subspaces with real or complex codimension 1, and they will play an important role in the sequel.

To any real hyperplane Y in $\mathbf{C}^n$ and every point $a \in Y$ there is a unique complex hyperplane $Y_{[a]}$ which contains a and is contained in Y. In fact

$$Y_{[a]} = Y \cap (i(Y - a) + a).$$

We note that $Y_{[a]}$ depends continuously on (Y, a) for the natural topology on hyperplanes and points.

Conversely, every complex hyperplane Z in $\mathbf{C}^n$ is contained in a real hyperplane, but there are now several choices. If a complex hyperplane Z is given and is defined by the equation $\beta \cdot (z - a) = 0$, then for any complex number θ with $|\theta| = 1$ the real hyperplane $Z^{[\theta]}$ defined by $\operatorname{Re} \theta(\beta \cdot (z - a)) = 0$ contains Z. The real hyperplane $Z^{[\theta]}$ does not depend on the choice of $a \in Z$ and satisfies $(Z^{[\theta]})_{[b]} = Z$ for every $b \in Z$.

If a real hyperplane Y and a point $a \in Y$ are given, then $(Y_{[a]})^{[\theta]} = Y$ for two values of θ with $|\theta| = 1$. Explicitly, if Y is given by the equation $\operatorname{Re} \beta \cdot (z - a) = 0$, then $Y_{[a]}$ is given by $\beta \cdot (z - a) = 0$ and $(Y_{[a]})^{[\theta]}$ by $\operatorname{Re} \theta(\beta \cdot (z - a)) = 0$; the two choices $\theta = \pm 1$ give us Y back.

Definition 2.1. Given an open subset Ω of $\mathbf{C}^n$ with boundary of class C^1, we denote by $T_{\Omega,\mathbf{R}}(b)$ the real tangent space at a boundary point b, and by $T_{\Omega,\mathbf{C}}(b)$ the complex tangent space at b (both containing the origin). The real or complex tangent planes which pass through b are $b + T_{\Omega,\mathbf{R}}(b)$ and $b + T_{\Omega,\mathbf{C}}(b)$, respectively. □

Clearly $T_{\Omega,\mathbf{C}}(b) = T_{\Omega,\mathbf{R}}(b)_{[0]}$; for the tangent planes, $b + T_{\Omega,\mathbf{C}}(b) = (b + T_{\Omega,\mathbf{R}}(b))_{[b]}$.

Definition 2.2. If A is a subset of $\mathbf{C}^n$, we shall denote by $\Gamma_A(a)$ the set of all complex hyperplanes Z which pass through the origin and are such that $a + Z$ does not intersect A. □

A mapping $F\colon X \to \mathscr{P}(Y)$ will be called a ***multifunction*** from X into Y and will be written $F\colon X \rightrightarrows Y$. This means that the value, image, or fiber $F(x)$ of F at a point x is a subset of Y, possibly empty. The ***graph*** of a multifunction F, denoted by $\mathbf{graph}(F)$, is the set $\{(x, y) \in X \times Y;\ y \in F(x)\}$.

If X and Y are topological spaces, we can equip $X \times Y$ with the Cartesian product topology. In all cases considered here, X is a T_1 space—equivalently, all singleton sets are closed. If so, for $\mathbf{graph}(F)$ to be a closed subset of $X \times Y$, it is necessary but not sufficient that the fiber

$$F(a) = (\{a\} \times Y) \cap \mathbf{graph}(F)$$

be a closed subset of Y for every $a \in X$.

Thus Γ_A is a multifunction $\Gamma_A\colon \mathbf{C}^n \rightrightarrows \mathrm{Gr}_{n-1}(\mathbf{C}^n) = M_{n,n-1}(\mathbf{C})$ with values in the Grassmann manifold of all complex hyperplanes in $\mathbf{C}^n$ passing through the origin. If Ω is open, $\Gamma_\Omega(a)$ is closed for every $a \in \mathbf{C}^n$. See also Proposition 7.2.

Lineal convexity of a set A means that $\Gamma_A(a)$ is nonempty for every $a \in \mathbf{C}^n \smallsetminus A$; weak lineal convexity of an open set Ω that $\Gamma_\Omega(b)$ is nonempty for every $b \in \partial\Omega$.

Let us agree to say that a topological space is ***connected*** if the only sets which are both open and closed are the empty set and the whole space (not necessarily distinct).[1] A subset of a topological space is said to be ***connected*** if it is connected as a topological subspace.

Zelinskiĭ (1981) has proved that a bounded lineally convex open set Ω is $\mathbf{C}$-convex if and only if $\Gamma_\Omega(b)$ is connected for every boundary point b. See also Andersson et al. (2004:46, Theorem 2.5.2) for the corresponding result on subsets of projective space.

3. Notions from mathematical morphology

Mathematical morphology is a branch of science which was created in the 1960s by Georges Matheron (1930–2000) and Jean Serra. It thrives in complete lattices, but here we shall need only those complete lattices which are the power set $\mathscr{P}(X)$ of some set X; most often either $\mathbf{R}^m$ or $\mathbf{C}^n$. All definitions in the sequel can be given for general complete lattices—it is enough to replace $\bigcap$ by $\bigwedge$ and $\bigcup$ by $\bigvee$.

3.1. Inverse and direct images

To any mapping $f\colon X \to Y$ we define two mappings on a higher level, $f^*\colon \mathscr{P}(Y) \to \mathscr{P}(X)$ and $f_*\colon \mathscr{P}(X) \to \mathscr{P}(Y)$. The first is defined by

$$f^*(B) = \{x \in X;\ f(x) \in B\}, \qquad B \in \mathscr{P}(Y). \tag{3.1}$$

Here $f^*(B)$ is called the ***inverse image*** of B. The second is defined by

$$f_*(A) = \{f(x);\ x \in A\}, \qquad A \in \mathscr{P}(X). \tag{3.2}$$

The set $f_*(A)$ is called the ***(direct) image*** of A.

3.2. Increasing and co-increasing mappings

Let X and Y be any sets. A mapping $\varphi\colon \mathscr{P}(X) \to \mathscr{P}(Y)$ is said to be ***increasing*** if

$$\text{for all } A, B \in \mathscr{P}(X) \text{ with } A \subset B \text{ we have } \varphi(A) \subset \varphi(B).$$

It is called ***co-increasing*** if conversely

$$\text{for all } A, B \in \mathscr{P}(X) \text{ with } \varphi(A) \subset \varphi(B) \text{ we have } A \subset B.$$

A set $A \in \mathscr{P}(X)$ said to be a ***fixed point*** of a mapping $\varphi\colon \mathscr{P}(X) \to \mathscr{P}(X)$ if $\varphi(A) = A$. We denote by

$$\mathbf{invar}(\varphi) = \{A \in \mathscr{P}(X);\ \varphi(A) = A\},$$

the ***invariance set*** or the ***set of fixed points*** of a mapping $\varphi\colon \mathscr{P}(X) \to \mathscr{P}(X)$.

A mapping $\varphi\colon \mathscr{P}(X) \to \mathscr{P}(X)$ is said to be ***idempotent*** if $\varphi \circ \varphi = \varphi$. So idempotency means that φ maps $\mathscr{P}(X)$ into $\mathbf{invar}(\varphi)$ (in fact onto $\mathbf{invar}(\varphi)$).

[1] We follow here Bourbaki (1961: I: §11: 1) in that the empty space is defined to be connected. Adrien Douady (personal communication, 2000 June 26) argued for the empty space not to be connected. The difference is important in Definition 5.2, where $\mathbf{C}$-convexity is defined.

3.3. Inverses of mappings

Any mapping $\varphi\colon \mathscr{P}(X) \to \mathscr{P}(Y)$ has an ***upper inverse*** $\varphi^{[-1]}\colon \mathscr{P}(Y) \to \mathscr{P}(X)$ defined by

$$\varphi^{[-1]}(B) = \bigcap_{A\in\mathscr{P}(X)} (A;\ \varphi(A) \supset B), \qquad B \in \mathscr{P}(Y),$$

and a ***lower inverse*** $\varphi_{[-1]}\colon \mathscr{P}(Y) \to \mathscr{P}(X)$ defined by

$$\varphi_{[-1]}(B) = \bigcup_{A\in\mathscr{P}(X)} (A;\ \varphi(A) \subset B), \qquad B \in \mathscr{P}(Y).$$

So the upper inverse is obtained when considering values of φ containing a certain set B, and inversely for the lower inverse. The upper inverse is not always larger than the lower inverse; the names should rather be understood as approaching from above and approaching from below. We always have

$$\varphi^{[-1]} \circ \varphi \leqslant \mathbf{id}_{\mathscr{P}(X)} \leqslant \varphi_{[-1]} \circ \varphi,$$

so that in particular $\varphi^{[-1]} \leqslant \varphi_{[-1]}$ if φ is surjective.

If φ is co-increasing we have $\varphi_{[-1]} \circ \varphi = \mathbf{id}_{\mathscr{P}(X)} = \varphi^{[-1]} \circ \varphi$, so that in particular $\varphi^{[-1]} = \varphi_{[-1]}$ if φ is surjective and co-increasing.

3.4. Dilations and erosions: lattice-theoretical duality

Definition 3.1. A mapping $\delta\colon \mathscr{P}(X) \to \mathscr{P}(Y)$ is said to be a ***dilation*** if it commutes with the forming of unions, i.e., if

$$\delta(\textstyle\bigcup_{j\in J} A_j) = \displaystyle\bigcup_{j\in J} \delta(A_j), \qquad A_j \in \mathscr{P}(X). \qquad \square$$

It follows that $\delta(\varnothing) = \varnothing$, whereas $\delta(X) = \bigcup_{A\in\mathscr{P}(X)} \delta(A) \subset Y$.

Definition 3.2. A mapping $\varepsilon\colon \mathscr{P}(Y) \to \mathscr{P}(X)$ is said to be an ***erosion*** if it commutes with the forming of intersections, i.e., if

$$\varepsilon(\textstyle\bigcap_{j\in J} B_j) = \displaystyle\bigcap_{j\in J} \varepsilon(B_j), \qquad B_j \in \mathscr{P}(Y). \qquad \square$$

It follows that $\varepsilon(Y) = X$ and that $\varepsilon(\varnothing) = \bigcap_{B\in\mathscr{P}(Y)} \varepsilon(B) \supset \varnothing$.

The lower inverse $\delta_{[-1]}$ of a dilation δ is an erosion, and the upper inverse $\varepsilon^{[-1]}$ of an erosion ε is a dilation (Kiselman 2010, Theorem 6.13).

Example 3.3. The mappings f^* and f_* defined by (3.1) and (3.2) satisfy

$$f_*(A) \subset B \text{ if and only if } A \subset f^*(B), \qquad A \subset X,\ B \subset Y,$$

from which we deduce that $(f_*)_{[-1]} = f^*$ and $(f^*)^{[-1]} = f_*$. We note that f^* is both a dilation and an erosion, while f_* is a dilation but in general not an erosion. $\square$

3.5. Ethmomorphisms, anoiktomorphisms, and cleistomorphisms

Definition 3.4. A mapping $\eta\colon \mathscr{P}(X) \to \mathscr{P}(X)$ is called an ***ethmomorphism*** if it is increasing and idempotent.

An ethmomorphism is called an ***anoiktomorphism*** if it is smaller than the identity, and a ***cleistomorphism*** if it is larger than the identity. □

The set of all fixed points of a cleistomorphism κ has the property that the whole space X belongs to $\mathbf{invar}(\kappa)$ and that, if $A_j \in \mathbf{invar}(\kappa)$, $j \in J$, then also $\bigcap_{j\in J} A_j$ is a fixed point of κ. In other words, $\mathbf{invar}(\kappa) = \mathscr{F}$ is a ***Moore family***, and $\kappa(A)$ is the infimum of all supersets of A which belong to $\mathscr{F}$. Conversely, given a Moore family $\mathscr{F}$, the intersection of all supersets in $\mathscr{F}$ of an element A defines a cleistomorphism.

Classical examples are the operation of taking the interior in a topological space, $\alpha(A) = \mathbf{int}(A) = A^\circ$, and the operation of taking the closure, $\kappa(A) = \mathbf{clos}(A) = \overline{A}$. The fixed points of the former are the open sets; the fixed points of the latter, the closed sets.

For any dilation δ and any erosion ε, the compositions $\kappa_{[\delta]} = \delta_{[-1]} \circ \delta$ and $\kappa^{[\varepsilon]} = \varepsilon \circ \varepsilon^{[-1]}$ are cleistomorphisms in $\mathscr{P}(X)$, and the compositions $\alpha_{[\delta]} = \delta \circ \delta_{[-1]}$ and $\alpha[^{\varepsilon}] = \varepsilon^{[-1]} \circ \varepsilon$ are anoiktomorphisms in $\mathscr{P}(Y)$ (Kiselman 2010: Corollary 6.14).

Lemma 3.5. *The supremum of a family of anoiktomorphisms is an anoiktomorphism. Explicitly: if $\alpha_j\colon \mathscr{P}(X) \to \mathscr{P}(X)$, $j \in J$, is an arbitrary family of anoiktomorphisms, then $\alpha = \sup_{j\in J} \alpha_j$, defined by*

$$\alpha(A) = \bigcup_{j\in J} \alpha_j(A), \qquad A \in \mathscr{P}(X),$$

is an anoiktomorphism. By duality, the infimum any family of cleistomorphisms is a cleistomorphism.

The proof is easy.

Lemma 3.6. *The composition of an anoiktomorphism and a cleistomorphism in any order is an ethmomorphism.*

Proof. Clearly $\alpha \circ \kappa$ and $\kappa \circ \alpha$ are increasing mappings for any anoiktomorphism α and any cleistomorphism $\kappa\colon \mathscr{P}(X) \to \mathscr{P}(X)$. To prove idempotency we use the inequality $\alpha \leqslant \mathbf{id}_{\mathscr{P}(X)} \leqslant \kappa$ to conclude that

$$\alpha \circ \kappa = (\alpha \circ \mathbf{id}_{\mathscr{P}(X)}) \circ (\alpha \circ \kappa) \leqslant (\alpha \circ \kappa) \circ (\alpha \circ \kappa) \leqslant (\alpha \circ \kappa) \circ (\mathbf{id}_{\mathscr{P}(X)} \circ \kappa) = \alpha \circ \kappa.$$

By duality, also $\kappa \circ \alpha$ is idempotent. □

In general, the ethmomorphisms $\alpha \circ \kappa$ and $\kappa \circ \alpha$ are neither anoiktomorphisms nor cleistomorphisms.

When we take $\alpha = \mathbf{int}$ and $\kappa = \mathbf{clos}$ in Lemma 3.6, we obtain the ethmomorphisms $\mathbf{int} \circ \mathbf{clos}$ and $\mathbf{clos} \circ \mathbf{int}$. The fixed points of the former are the ***regular open sets***; those of the latter the ***regular closed sets***.[2]

These sets are those for which we do not lose information by taking the interior or the closure. In fact, if U is a regular open set, then $U^\circ = (\overline{U})^\circ = U$. Similarly,

[2] The expressions *regularly open* and *regularly closed* are also in use.

$\overline{C} = \overline{C^\circ} = C$ if C is a regular closed set. There are subsets A of $\mathbf{R}$ such that A as well as its complement $B = \mathbf{R} \smallsetminus A$ have the property that $A^\circ = B^\circ = \emptyset \subset \overline{A} = \overline{B} = \mathbf{R}$, for instance $A = \mathbf{Q}$, so these sets cannot be reconstructed from knowledge of their interiors and closures.

The complement of a regular open set is regular closed. Also, the closure of an open set is regular closed, and the interior of a closed set is regular open. The intersection of two regular open sets is regular open, but not always their union.

If $A^\circ \subset B \subset \overline{A}$ and A is regular open, then $B^\circ = A$. Similarly, if A is regular closed, then these inclusion relations imply that $\overline{B} = A$.

3.6. Operations on power sets

If G is an abelian group, it is well known that, for fixed sets $S, T \subset G$ (called *structuring elements*),

$$\mathscr{P}(G) \ni A \mapsto \delta_S(A) = A + S \in \mathscr{P}(G)$$

is a dilation and that

$$\mathscr{P}(G) \ni B \mapsto \varepsilon_T(B) = \{x;\ x + T \subset B\} \in \mathscr{P}(G)$$

is an erosion. If $0 \in S$, we have $\varepsilon_S \leqslant \mathbf{id} \leqslant \delta_S$.

We note that δ_S commutes with translations, while ε_S anticommutes with translations:

$$\delta_{S+c}(A) = \delta_S(A) + c, \qquad \varepsilon_{S+c}(A) = \varepsilon_S(A) - c, \qquad c \in G, \quad A \in \mathscr{P}(G). \tag{3.3}$$

We have $(\delta_S)_{[-1]} = \varepsilon_S$ and $(\varepsilon_S)^{[-1]} = \delta_S$. This is the lattice-theoretical duality already mentioned in Subsection 3.4.

We define

$$\alpha_S(A) = \delta_S(\varepsilon_S(A)), \quad \kappa_S(A) = \varepsilon_S(\delta_S(A)).$$

In view of (3.3), α_S and κ_S are invariant under translation: $\alpha_{S+c} = \alpha_S$ and $\kappa_{S+c} = \kappa_S$.

Also, in view of (3.3) we may assume that $0 \in S$ as soon as S is nonempty. We can conclude that, when S contains 0,

$$\varepsilon_S \leqslant \alpha_S \leqslant \mathbf{id}_{\mathscr{P}(G)} \leqslant \kappa_S \leqslant \delta_S.$$

It may be convenient to put these mappings under a common roof: let us define

$$\varphi_{S,T,U}(A) = \bigcup_{x\in G} (x + S;\ x + T \subset A + U), \qquad A \in \mathscr{P}(G),$$

where S, T, U are three subsets of an abelian group G. Then

$$\delta_S = \varphi_{S,\{0\},\{0\}}, \quad \varepsilon_S = \varphi_{\{0\},S,\{0\}}, \quad \alpha_S = \varphi_{S,S,\{0\}}, \quad \kappa_S = \varphi_{\{0\},S,S}. \tag{3.4}$$

With the notation from Kiselman (2010: Definition 7.1), we have $\alpha_S = \delta_S /_\star \delta_S$, the lower quotient of δ_S and δ_S. Similarly, the cleistomorphism κ_S is the upper quotient $\kappa_S = \delta_S /^\star \delta_S$ of δ_S and δ_S.

When G is equal to $\mathbf{R}^m$ or $\mathbf{C}^n$ with the usual topologies, we can also form the composition $\mathbf{clos} \circ \alpha_S = \overline{\alpha_S}$, which is an ethmomorphism in view of Lemma 3.6, in order to get a closed set. When passing to complements we get an ethmomorphism ${}^\circ\kappa_S$ yielding open sets, and when applied to open sets we see that it is larger than the identity: $({}^\circ\kappa_S)(\Omega) = (\kappa_S(\Omega))^\circ \supset \Omega$.

It is convenient to express some of these properties in terms of accessibility:

Definition 3.7. If A is a subset of $\mathbf{R}^m$ or $\mathbf{C}^n$ and b a point in this space, we shall say that b is ***S-accessible from the outside*** if b belongs to the closure of $\alpha_S(\complement A)$. In particular we shall speak about accessibility from the outside by balls of radius r if S is equal to $B_{\leqslant}(0,r)$ or $B_{<}(0,r)$. □

Remark 3.8. If b is S-accessible from the outside of a certain class, then there is also a set T of the same class such that $\overline{A}\cap\overline{T}=\{b\}$. Indeed, if S satisfies

$$\{x;\ f(x)<0\}\subset S\subset\{x;\ f(x)\leqslant 0\},$$

then T can be taken as the set of all x such that $f(x)+\|x-b\|_2^2\leqslant 0$. □

We shall consider regularity classes $C^{k,\beta}$, where $k\in\mathbf{N}$ and $0\leqslant\beta\leqslant 1$, meaning that the functions considered are of class C^k and all derivatives of order k are Hölder continuous of order β, with the understanding that $C^{k,0}=C^k$.

Definition 3.9. If $b\in\partial A$ is accessible from the outside by a structuring element S having boundary of class $C^{k,\beta}$ with $k\geqslant 1$, then we shall say that the unique tangent plane to S at b is an ***external tangent space*** of A at b. The set of all external tangent spaces at a point b, a subset of the Grassmann manifold $\mathrm{Gr}_{m-1}(\mathbf{R}^m)=M_{m,m-1}(\mathbf{R})$ of all real hyperplanes passing through the origin, will be denoted by $\Theta^{k,\beta}_{A,\mathbf{R}}(b)$, and the corresponding multifunction $\partial A\rightrightarrows\mathrm{Gr}_{m-1}(\mathbf{R}^m)$ by $\Theta^{k,\beta}_{A,\mathbf{R}}$.

If Ω is an open subset of $\mathbf{C}^n$, we shall denote by $\Theta^{k,\beta}_{\Omega,\mathbf{C}}(b)$ the set of all complex hyperplanes through the origin contained in planes in $\Theta^{k,\beta}_{\Omega,\mathbf{R}}(b)$; we call them ***complex external tangent spaces***. It is the set of all complex hyperplanes $Z=Y_{[0]}$, $Y\in\Theta^{k,\beta}_{\Omega,\mathbf{R}}(b)$.

When the class is clear from the context or is unimportant, we shall omit the superscripts ${}^{k,\beta}$. □

It is easy to see that $\Theta^{1,1}_{\Omega,\mathbf{R}}=\Theta^{2}_{\Omega,\mathbf{R}}=\Theta^{\infty}_{\Omega,\mathbf{R}}$.

The relation between $\Gamma_\Omega(b)$ and $\Theta^2_{\Omega,\mathbf{C}}(b)$, $b\in\partial\Omega$, seems to be of interest.

Definition 3.10. Let us say that Ω is ***tangentially lineally convex at*** $b\in\partial\Omega$ if no complex external tangent plane of class C^2 at b meets Ω, i.e., if $\Theta^2_{\Omega,\mathbf{C}}(b)\subset\Gamma_\Omega(b)$. □

Proposition 3.11. *Let $b\in A\subset\mathbf{R}^m$ be accessible from the outside by balls of radius $r>0$. Then $\Theta^{k,\beta}_{A,\mathbf{R}}(b)$ is connected.*

Proof. Take $b=0$ and assume that $\overline{A}\cap\overline{U_j}=\{0\}$, $j=0,1$, where U_j is the set of all points x such that $f_j(x)<0$, and f_j is a function of a given regularity and with nonvanishing gradient wherever it is zero. This is justified by Remark 3.8. We now form $f_s=(1-s)f_0+sf_1$, $0\leqslant s\leqslant 1$, and claim that the set where f_s is negative defines an open set U_s which serves to prove that all gradients

$$(\mathbf{grad}\, f_s)(0)=(1-s)(\mathbf{grad}\, f_0)(0)+s(\mathbf{grad}\, f_1)(0)$$

can occur, implying that there is a curve connecting the hyperplane defined by f_0 to that defined by f_1. We note that the gradient of f_s is nonzero at the origin except in the case when $(\mathbf{grad}\, f_1)(0)$ is a negative multiple of $(\mathbf{grad}\, f_0)(0)$. In that case, however, the hyperplanes defined by the two gradients are the same, so there is nothing to prove.

We modify f_s outside a neighborhood of the origin if necessary to make sure that it satisfies the requirement that its gradient be nonzero everywhere where the function itself vanishes.

If $x \in \overline{A} \smallsetminus \{0\}$, then $x \notin \overline{U_j}$, $j = 0, 1$, so that $f_j(x) > 0$, $j = 0, 1$. This implies that $f_s(x) > 0$, so that $x \notin \overline{U_s}$. Thus we have proved that $\overline{A} \cap \overline{U_s} \subset \{0\}$; obviously $\overline{A} \cap \overline{U_s} \supset \{0\}$. In conclusion, we have proved that the tangent plane of U_s at $b = 0$ belongs to $\Theta^{k,\beta}_{A,\mathbf{R}}(0)$ for all s with $0 \leqslant s \leqslant 1$. $\square$

Example 3.12. Let us define a cleistomorphism $\kappa_r \colon \mathscr{P}(\mathbf{C}^n) \to \mathscr{P}(\mathbf{C}^n)$ or $\mathscr{P}(\mathbf{R}^m) \to \mathscr{P}(\mathbf{R}^m)$ as the cleistomorphism with structuring element $U = \complement B_<(0, r)$ for some positive radius r. It follows that $\kappa_r(A)$ is closed for any set A, perhaps most easily seen by observing that its complement, denoted by $\alpha_r(\complement A)$, is the union of all open balls $B_<(x, r)$ which are contained in $\complement A$.

Thus $\kappa_r(A)$ is the smallest invariance set containing A whose boundary points are all accessible by balls of radius r, and we see that the boundary points of a closed set F are accessible by such balls if and only if $\kappa_r(F) = F$.

To treat open sets, we define $\lambda_r(A)$ as the interior of $\kappa_r(A)$. In view of Lemma 3.6 the operation $A \mapsto \lambda_r(A) = (\kappa_r(A))^\circ$ is an ethmomorphism. If we restrict it to open sets, it is larger than the identity, i.e., $\lambda_r(\Omega) \supset \Omega$ for all open sets Ω. So accessibility for open sets is defined by the fixed points of λ_r.

The infimum of all the κ_r, $r > 0$, is just the topological closure. $\square$

3.7. Set-theoretical duality

In addition to the lattice-theoretical duality, there are also a set-theoretical duality. It can be proved that $\delta_U(\complement A) = \complement \varepsilon_{-U}(A)$, where $-U$ is the set of all points $-x$ with $x \in U$.

Forming compositions, we obtain, $\alpha_U = \delta_U \circ \varepsilon_U$, an anoiktomorphism, and $\kappa_U = \varepsilon_U \circ \delta_U$, a cleistomorphism. They are related by the formula $\kappa_U(\complement A) = \complement \alpha_{-U}(A)$, $A \in \mathscr{P}(G)$. This is the set-theoretical duality defined by passing to the complement of the sets to which the operations are applied.

4. Concavity and convexity with respect to a structuring element or a family of structuring elements

Just as it is sometimes easier to look at lineally concave sets rather than lineally convex sets, it can be more convenient to define accessibility from the inside than from the outside. We shall do this in terms of concavity and convexity with respect to a structuring element, treating both properties in parallel:

Definition 4.1. Given a subset S (called ***structuring element***) of an abelian group G, we shall say that a subset A of G is S-***concave*** if it is a union of translates $x + S$ with x in some subset X of G. We shall say that it is S-***convex*** if its complement is S-concave.

We define the S-***kernel*** of a set A, denoted by $\alpha_S(A)$, as the union of all translates $x + S$ contained in A. We define the S-***hull*** of a set B, denoted by $\kappa_S(B)$, as the complement of the S-kernel of $\complement B$. $\square$

Obviously $\complement\alpha_S(A) = \kappa_S(\complement A)$.

The anoiktomorphism α_S, defined above in Subsection 3.6 as

$$\alpha_S(A) = \bigcup_{x\in G}(x+S;\ x+S\subset A), \qquad A\in\mathscr{P}(G),$$

has as fixed points the S-concave sets. We have $\alpha_S(A)\subset A\subset\kappa_S A)$ for all nonempty sets $S\in\mathscr{P}(G)$ and all A.

We can consider the sets $x+S$ as voxels or pixels, and see that no smaller sets are allowed to build up a S-concave set. Or we can think of elements x as atoms and sets $x+S$ as molecules—no free atoms are allowed; they must all be part of a molecule.

What we have done so far is define concavity and convexity with respect to a single set S. Let us also consider families $\mathscr{S}$ of structuring sets:

Definition 4.2. Given a family $\mathscr{S}$ of subsets of an abelian group G, we shall say that a subset A of G is ***$\mathscr{S}$-concave*** if it is a union of translates $x+S$ with $x\in X\subset G$, $S\in\mathscr{S}$. We shall say that B is ***$\mathscr{S}$-convex*** if its complement is $\mathscr{S}$-concave.

We define

$$\alpha_{\mathscr{S}}(A) = \bigcup_{\substack{x\in G\\ S\in\mathscr{S}}}(x+S;\ x+S\subset A),$$

called the ***$\mathscr{S}$-kernel*** of A, and $\kappa_{\mathscr{S}}(B) = \complement\alpha_{\mathscr{S}}(\complement B)$, called the ***$\mathscr{S}$-hull*** of B. □

Thus $\{S\}$-concavity is the same as S-concavity.

Classical examples are when we take $\mathscr{S}$ as the family $\mathscr{U}$ of all open half-spaces in $\mathbf{R}^m$, defined by an inequality $\xi\cdot x>c$, or the the family $\mathscr{C}$ of all closed half-spaces in $\mathbf{R}^m$, defined by an inequality $\xi\cdot x\geqslant c$, with $\xi\in\mathbf{R}^m\smallsetminus\{0\}$, $c\in\mathbf{R}$. We can also consider the set of all real or complex hyperplanes, or intersections of complex hyperplanes with balls.

Example 4.3. The set $A =]0,1[^2\cup\{(0,0)\}\subset\mathbf{R}^2$ (an open square with a vertex added) is convex, but is not an intersection of open half planes, nor of closed half planes. Here we obtain

$$A^\circ =]0,1[^2 \subsetneqq A \subsetneqq \kappa_{\mathscr{U}}(A) = [0,1[^2 \subsetneqq \kappa_{\mathscr{C}}(A) = [0,1]^2 = \overline{A},$$

indicating that more general half planes are needed. □

In view of the above example we now define more general half-spaces, called here ***refined half-spaces***, by which we mean convex sets Y such that

$$\{x\in\mathbf{R}^m;\ \xi\cdot x<c\}\subset Y\subset\{x\in\mathbf{R}^m;\ \xi\cdot x\leqslant c\}$$

for some $\xi\in\mathbf{R}^m\smallsetminus\{0\}$ and $c\in\mathbf{R}$. Let us denote by $\mathscr{Y}$ the family of all such sets Y.

Obviously $\kappa_{\mathscr{C}}(A)$ is always a closed set. In view of the Hahn–Banach theorem it is equal to the closed convex hull of A. The mapping $\kappa_{\mathscr{U}}$ takes an open set to its convex hull (which is open) and a compact set to its convex hull (which is closed).

This is convexity viewed from the outside. We can also work with convexity from the inside: We define the ***convex hull*** of a set $A\subset\mathbf{R}^m$ as

$$\mathbf{cvxh}(A) = \left\{\textstyle\sum_{j=1}^{m+1}\lambda_j a^{(j)};\ \lambda_j\geqslant 0,\ \sum_{j=1}^{m+1}\lambda_j = 1,\ a^{(j)}\in A\right\}, \qquad A\in\mathscr{P}(\mathbf{R}^m).$$

Actually $\mathbf{cvxh} = \kappa_{\mathscr{Y}}$. This operation maps any set to its convex hull, which need not be closed even if A is closed. The composition $\mathbf{clos}\circ\mathbf{cvxh}$ takes any set to its closed convex hull. (The composition $\mathbf{cvxh}\circ\mathbf{clos}$ is not idempotent if $m\geqslant 2$.)

Definition 4.4. We shall say that an open subset of $\mathbf{R}^m$ or $\mathbf{C}^n$ is r***-concave*** if it is a union of open balls of radius r. A closed subset is called r***-concave*** if it is a union of closed balls of radius r. A set is called r***-convex*** if its complement is r-concave. $\square$

This definition agrees for open sets in $\mathbf{C}$ with that of Sergey Favorov and Leonid Golinskii (2015:3). They defined the *r-convex hull* of a set $E \subset \mathbf{C}$, denoted by $\mathbf{conv}_r(E)$, as the set

$$\mathbf{conv}_r(E) = \bigcap \left(\complement D_<(z,r);\ E \subset \complement D_<(z,r)\right), \qquad E \subset \mathbf{C},\ r > 0.$$

Thus $\complement\,\mathbf{conv}_r(E)$ is a union of open disks. They call a set *r-convex* if $\mathbf{conv}_r(E) = E$. Such a set is always closed. The generalization to $\mathbf{R}^m$ or $\mathbf{C}^n$ is obvious, and we see that $\mathbf{conv}_r(E)$ is exactly the set $\kappa_{B_<(0,r)}(E)$ with the notation from Definition 4.1. When r tends to $+\infty$, we get the closed convex hull $\overline{\mathbf{cvxh}(E)}$ as a limiting case.[3]

5. Lineal convexity viewed from mathematical morphology

Definition 5.1. A subset A of $\mathbf{C}^n$ is said to be ***lineally concave*** if it is a union of affine complex hyperplanes. It is said to be ***lineally convex*** if its complement is lineally concave. $\square$

Thus we have here an example of $\mathscr{S}$-concavity, with $\mathscr{S}$ equal to the family $\mathscr{Z}$ of all complex hyperplanes in $\mathbf{C}^n$ containing the origin. Weak lineal convexity means that $\kappa_{\mathscr{Z}}(\Omega)$ does not meet the boundary of Ω.

There are also local variants of these definitions: we take $\mathscr{S} = \mathscr{Z}_r$ as the family of all intersections $Z \cap B_\leqslant(0,r)$, where Z is a complex hyperplane passing through the origin. The corresponding $\mathscr{Z}_r$-convexity, for some positive r, can be called ***uniform local lineal convexity***.

Definition 5.2. A subset A of $\mathbf{C}^n$ is said to be $\mathbf{C}$ ***convex*** if $A \cap L$ is a connected and simply connected subset of L for every affine complex line L (Hörmander 1994: Definition 4.6.6).

A subset A of $\mathbf{P}$, the complex projective space of dimension 1, is called $\mathbf{C}$***-convex*** if $A \neq \mathbf{P}$ and both A and $\mathbf{P} \smallsetminus A$ are connected (Andersson et al. 2004: Definition 2.2.1).

A subset B of n-dimensional projective space $\mathbf{P}^n$ is called $\mathbf{C}$***-convex*** if all its intersections with complex lines are $\mathbf{C}$-convex (Andersson et al. 2004: Definition 2.2.1). $\square$

Since by definition the empty set is connected, it follows that it is $\mathbf{C}$ convex (Hörmander) as well as $\mathbf{C}$-convex (Andersson et al.) The whole space $\mathbf{C}^n$ is $\mathbf{C}$ convex in Hörmander's sense, wheras the whole space $\mathbf{P}^n$ is not $\mathbf{C}$-convex in the sense of Andersson et al.

[3] The notion of r-convex closed sets is used by these authors as a hypothesis in results on Blaschke-type conditions for the Riesz measure of a subharmonic function, thus in a context quite different from the one studied here. Since I worked on generalized convexity during the period 1996–2001 (see for example Proposition 4.9 in my paper (1996)) and then again since 2014, and with quite different problems, our respective studies are independent.

Let us take again the family $\mathscr{S}$ of structuring elements in Definition 4.2 as the set $\mathscr{Z} \subset \mathscr{P}(\mathscr{P}(\mathbf{C}^n))$ of all complex affine hyperplanes in $\mathbf{C}^n$. We define a dilation $\psi\colon \mathscr{P}(\mathscr{Z}) \to \mathscr{P}(\mathbf{C}^n)$ by

$$\psi(\mathscr{B}) = \bigcup_{Z\in\mathscr{B}} Z, \qquad \mathscr{B} \in \mathscr{P}(\mathscr{Z}). \tag{5.1}$$

Its lower inverse $\psi_{[-1]}\colon \mathscr{P}(\mathbf{C}^n) \to \mathscr{P}(\mathscr{Z})$ is defined by

$$\psi_{[-1]}(A) = \bigcup_{\mathscr{B}\in\mathscr{Z}} (\mathscr{B};\ \psi(\mathscr{B}) \subset A) = \{Z \in \mathscr{Z};\ Z \subset A\}, \qquad A \in \mathscr{P}(\mathbf{C}^n). \tag{5.2}$$

We note that $\varepsilon = \psi_{[-1]}$ is an erosion—as the lower inverse of a dilation, but also easily seen directly. There is a relation between Γ_A and ε:

$$\Gamma_A(b) = \{Z \in \varepsilon(\complement A);\ b \in Z\}.$$

The upper inverse $\varepsilon^{[-1]}\colon \mathscr{P}(\mathscr{Z}) \to \mathscr{P}(\mathbf{C}^n)$ of ε is a dilation defined by

$$\varepsilon^{[-1]}(\mathscr{B}) = \bigcap_{A\in\mathscr{P}(\mathbf{C}^n)} (A; \varepsilon(A) \supset \mathscr{B}) = \bigcup_{Z\in\mathscr{B}} Z = \psi(\mathscr{B}), \qquad \mathscr{B} \in \mathscr{P}(\mathscr{Z}). \tag{5.3}$$

By composition we obtain an anoiktomorphism $\alpha_{\mathscr{Z}}\colon \mathscr{P}(\mathbf{C}^n) \to \mathscr{P}(\mathbf{C}^n)$:

$$\alpha_{\mathscr{Z}}(A) = (\varepsilon^{[-1]} \circ \varepsilon)(A) = (\psi \circ \psi_{[-1]})(A) = \bigcup (Z;\ Z \subset A), \qquad A \in \mathscr{P}(\mathbf{C}^n),$$

the union of all complex affine hyperplanes contained in A. We can also form

$$\kappa_{\mathscr{Z}}(\mathscr{B}) = (\varepsilon \circ \varepsilon^{[-1]})(\mathscr{B}) = (\psi_{[-1]} \circ \psi)(\mathscr{B}), \qquad \mathscr{B} \in \mathscr{P}(\mathscr{Z}).$$

We have $\alpha_{\mathscr{Z}}(A) = A$ (equivalently $\alpha_{\mathscr{Z}}(A) \supset A$) if and only if A is lineally concave, which happens if and only if $\complement A$ is lineally convex. If Ω is open, it is lineally convex if and only if $\alpha_{\mathscr{Z}}(\complement\Omega) \supset \complement\Omega$, and weakly lineally convex if and only if $\alpha_{\mathscr{Z}}(\complement\Omega) \supset \partial\Omega$.

6. Exterior accessibility of Hartogs domains

We shall now study Hartogs domains in $\mathbf{C}^n \times \mathbf{C}$, where we write coordinates as $(z,t) \in \mathbf{C}^n \times \mathbf{C}$.

Definition 6.1. A subset A of $\mathbf{C}^n \times \mathbf{C}$ is said to be a ***Hartogs set*** if $(z,t) \in A$, $|s| = |t|$ implies $(z,s) \in A$. It is said to be a ***complete Hartogs set*** if $(z,t) \in A$, $|s| \leqslant |t|$ implies $(z,s) \in A$. □

An open complete Hartogs domains is thus given in the space $\mathbf{C}^n \times \mathbf{C}$ by an inequality $|t| < R(z)$, where $R\colon \mathbf{C}^n \to [-\infty, +\infty]$, as

$$\Omega = \{(z,t) \in \mathbf{C}^n \times \mathbf{C};\ |t| < R(z)\} = \{(z,t) \in \omega \times \mathbf{C};\ |t| < R(z)\}, \tag{6.1}$$

where we have defined ω as the set where R is positive or equal to $+\infty$. The fact that Ω is open implies that $R|_\omega$ is lower semicontinuous.

To define complete Hartogs sets, we may use either the function R, the function $h = R^2$, or the function $f = -\log R$. An open complete Hartogs set is then defined equivalently by $|t| < R(z)$; $|t|^2 < h(z)$; $|t| < \mathrm{e}^{-f}$, and we are free to choose whichever is convenient for a specific calculation. We note that if f is plurisubharmonic, then Ω, defined by $\log|t| + f(z) < 0$, is pseudoconvex.

Complex hyperplanes in $\mathbf{C}^n \times \mathbf{C}$ are of three kinds:

1. A hyperplane can be given by an equation $\beta\cdot(z-z^0)=0$ for some $\beta\in\mathbf{C}^n\smallsetminus\{0\}$ and some point $z^0\in\mathbf{C}^n$ (we shall call it a ***vertical hyperplane***).
2. It can have the equation $t=c$ for some complex constant c (we shall call it a ***horizontal hyperplane***).
3. Finally it can have the equation $t=\beta\cdot(z-z^0)$, where β is nonzero. Such a hyperplane intersects the hyperplane $t=0$ in a hyperplane in $\mathbf{C}^n$ containing z^0.

The projection $\mathbf{C}^n\times\mathbf{C}\ni(z,t)\mapsto(z,|t|)\in\mathbf{C}^n\times\mathbf{R}$ can be used to visualize the set. Equivalently, we can look at the intersection of Ω with the set $\{(z,t);\ z\in\mathbf{C}^n, t\geqslant 0\}$. A hyperplane is then represented in $\mathbf{C}^n\times\mathbf{R}$ by either

1. a vertical plane;
2. a horizontal plane $|t|=|c|$; or
3. a cone $|t|=|\beta\cdot(z-z^0)|$ with vertices at all the points z satisfying $\beta\cdot(z-z^0)=0$; when $n=1$ just the unique point z^0.

If $b=(z^0,t^0)$ is a boundary point with $t^0=0$, then there is a complex line of equation $z=z^0$ in the complement of Ω, and there may or may not exist a hyperplane in $\Gamma_\Omega(b)$—if the set ω in $\mathbf{C}^n$ where R is positive is lineally convex, of course there is such a hyperplane. If on the other hand $b=(z^0,t^0)$ is a boundary point satisfying $|t^0|=R(z^0)>0$, then a hyperplane $Z\in\Gamma_\Omega(b)$ is given by an equation $t/t^0=\beta\cdot z$; the parallel hyperplane $b+Z$ passing through b has the equation $t/t^0=1+\beta\cdot(z-z^0)$. It may happen that all real hyperplanes containing $b+Z$ cuts Ω, but if this is not the case, the only real hyperplane containing $b+Z$ and not cutting Ω is that of equation $\operatorname{Re} t/t^0=1+\operatorname{Re}\beta\cdot(z-z^0)$.

Theorem 6.2. *Let a function $R\colon\mathbf{C}^n\to[-\infty,+\infty]$ be given and consider the complete Hartogs set Ω defined by* (6.1)*. Assume that Ω is open and weakly lineally convex. Then R is continuous at every point where it is finite and positive, and all boundary points of Ω satisfying (z^0,t^0) with $|t^0|=R(z^0)>0$ are accessible from the outside of class C^2. In fact, every complex hyperplane which passes through a boundary point (z^0,t^0) with $|t^0|=R(z^0)>0$ and does not meet Ω is contained in a real external tangent plane. In particular $\Gamma_\Omega(b)\subset\Theta_{\Omega,\mathbf{C}}(b)$ for all points $b=(z^0,t^0)$ with $|t^0|=R(z^0)>0$ ($\Theta_{\Omega,\mathbf{C}}(b)$ is defined in Definition 3.9).*

Proof. Any point (z^0,t^0) with $|t^0|=R(z^0)>0$ belongs to the boundary of Ω, so there exists by hypothesis a vector $\beta\in\mathbf{C}^n$ such that the complex hyperplane defined by $t/t^0=1+\beta\cdot(z-z^0)$ lies entirely in the complement of Ω. We shall prove that there is a real external tangent plane of class C^2 containing it.

That the complex hyperplane does not meet Ω means that

$$\frac{R(z)}{|t^0|}\leqslant|1+\beta\cdot(z-z^0)|,\qquad z\in\mathbf{C}^n.$$

Now

$$|1+z|\leqslant\tfrac12+\tfrac12|1+z|^2=1+\operatorname{Re} z+\tfrac12|z|^2,\qquad z\in\mathbf{C},$$

with equality if and only if $|1+z|=1$. It follows that for any $\gamma>\frac12$,

$$|1+z|\leqslant 1+\operatorname{Re} z+\gamma|z|^2,\qquad z\in\mathbf{C},$$

with equality only when $z=0$. Hence

$$|1+\beta\cdot(z-z^0)| \leqslant 1+\operatorname{Re}\beta\cdot(z-z^0)+\gamma|\beta\cdot(z-z^0)|^2 \leqslant 1+\operatorname{Re}\beta\cdot(z-z^0)+\gamma\|\beta\|_2^2\|z-z^0\|_2^2,$$

with equality between the first and last expression only when $z=z^0$ or $\beta=0$. Therefore, if we choose $c>\frac{1}{2}\|\beta\|_2^2$,

$$R(z)/|t^0| \leqslant 1+\operatorname{Re}\beta\cdot(z-z^0)+c\|z-z^0\|_2^2, \qquad z\in\mathbf{C}^n,$$

with equality only when $z=z^0$.

So the set

$$U=\{(z,t);\ \operatorname{Re}(t/t^0)>1+\operatorname{Re}\beta\cdot(z-z^0)+c\|z-z^0\|_2^2\},$$

taking $c>\frac{1}{2}\|\beta\|_2^2$, is a set with smooth boundary and satisfies the requirement in Definition 2.1; the real hyperplane defined by $\operatorname{Re}t/t^0=1+\operatorname{Re}\beta\cdot(z-z^0)$ is an external tangent plane of class C^2 of Ω at (z^0,t^0).

From what we just proved it follows in particular that R is upper semicontinuous where positive. On the other hand, Ω is open by hypothesis, which, as we noted, implies that the restriction $R|_\omega$ is lower semicontinuous. □

7. Unions of increasing squences of domains

If an increasing family $(V_j)_{j\in\mathbf{N}}$ of open sets in $\mathbf{R}^m$ is given with union V and if $b\in\partial V$, let us denote by $\limsup\Theta_{V_j,\mathbf{R}}(b)$, understood as $(\limsup\Theta_{V_j,\mathbf{R}})(b)$, all limits of real hyperplanes $Y_j\in\Theta_{V_j,\mathbf{R}}(b^{(j)})$ at points $b^{(j)}\in\partial V_j$ such that $b^{(j)}\to b$ as $j\to\infty$. We shall use a similar notation for the complex hyperplanes: $\limsup\Theta_{\Omega_j,\mathbf{C}}(b)$ when Ω_j increases to Ω, and also $\limsup\Gamma_{\Omega_j}(b)$.

Proposition 7.1. *Let $(V_j)_{j\in\mathbf{N}}$ be an increasing family of open subsets of $\mathbf{R}^m$. Define $\Theta_{V,\mathbf{R}}$ as in Definition 3.9 using as structuring element a set S with boundary of class $C^{k,\beta}$ with $k\geqslant 1$. Then $\Theta_{V,\mathbf{R}}(b)\subset\limsup\Theta_{V_j,\mathbf{R}}(b)$ for all points $b\in\partial V$. A similar result holds for the complex external tangent planes $\Theta_{\Omega,\mathbf{C}}(b)$ of an open subset Ω of $\mathbf{C}^n$. Here the inclusion can be strict. The limit superior is always nonempty.*

Proof. Take $b=0$ and let U be an open set with boundary of the class in question such that $\overline{V}\cap\overline{U}=\{0\}$, defined as the set of all points x where $\varphi(x)$ is negative, φ being of the right class and with nonvanishing gradient where it is zero. Let φ_s, $s>0$, be the function

$$\varphi_s(x)=\varphi(x)-s+\|x\|_2^2, \qquad x\in\mathbf{R}^m,$$

and let U_s be the set where φ_s is negative. We note that when $x\in V$, then $x\notin U$, so that $\varphi(x)\geqslant 0$. If $x\in V\cap U_s$, then $\varphi(x)\geqslant 0$ while $\varphi_s(x)<0$. So $\|x\|_2^2<s-\varphi(x)\leqslant s$. Since φ is of class C^1, its gradient at any point in $V\cap U_s$ is close to its gradient at the origin. For every large enough j there is a smallest s_j such that U_{s_j} and V_j have a common boundary point $b^{(j)}$. Necessarily, then, $\|b^{(j)}\|_2^2\leqslant s$. For large j, s_j is small, so small that the external tangent plane of U_{s_j} at $b^{(j)}$ is as close as we like to the tangent plane of U at the origin. This shows that any hyperplane in $\Theta_{V,\mathbf{R}}(0)$ can be approximated by hyperplanes in $\Theta_{V_j,\mathbf{R}}(b^{(j)})$. □

Proposition 7.2. *Let $(\Omega_j)_{j\in\mathbf{N}}$ be an increasing family of lineally convex open subsets of $\mathbf{C}^n$ and denote their union by Ω. Then* $\limsup \Gamma_{\Omega_j}(b) = \Gamma_\Omega(b)$. *In particular the graph of Γ_Ω is closed.*

Proof. If $Z \notin \Gamma_\Omega(b)$, then $b+Z$ intersects Ω. Take a compact ball K in Ω which contains a point of $b+Z$ in its interior. Then for all suffiently large j, Ω_j contains K. All hyperplanes which are close enough to $b+Z$ intersect K and hence also Ω_j for these j. Therefore, if Z_j tends to Z and $b^{(j)} \in \partial\Omega_j$ tends to b, then $b^{(j)} + Z_j$ intersects Ω_j for large j. This means that hyperplanes $b^{(j)} + Z_j$ with $Z_j \in \Gamma_{\Omega_j}(b^{(j)})$ cannot approach $b+Z$. So we have $\limsup \Gamma_{\Omega_j}(b) \subset \Gamma_\Omega(b)$.

The opposite inclusion is trivially true. □

Lemma 7.3. *If A is a closed set in $\mathbf{R}^m$ and $b \in \partial A$, then $\overline{\Theta^2_{A,\mathbf{R}}}(b)$, where we use a Euclidean ball as structuring element, is nonempty.*

Proof. Given $b \in \partial A$ and a positive number s, take $c \notin A$ with $\|c-b\| < s$. Take then $r > 0$ maximal so that $B_<(c,r)$ does not cut A. Clearly $r \leqslant s$. On the boundary of this ball, there must exist a point $p \in A$. Then $\|p-b\| \leqslant r+s \leqslant 2s$, and p is accessible from the outside of class C^2, which means that $\Theta^2_{A,\mathbf{R}}(p)$ is nonempty. Since s is arbitrarily small, the closure of the the graph of $\Theta^2_{A,\mathbf{R}}$ has a nonempty fiber over b. □

Theorem 7.4. *Let Ω be an open subset of $\mathbf{C}^n$ which is equal to the interior of its closure. If Ω is tangentially lineally convex at all points b in some open subset B of $\partial\Omega$ (see Definition 3.10), then $\Theta^1_{\Omega,\mathbf{C}}(b) \subset \overline{\Theta^2_{\Omega,\mathbf{C}}}(b) \subset \Gamma_\Omega(b)$, and $\Gamma_\Omega(b)$ is nonempty for all $b \in B$. In particular, tangential lineal convexity at all points $b \in \partial\Omega$ implies weak lineal convexity.*

Proof. We apply Lemma 7.3 to $A = \overline{\Omega}$. Then the interior of A is equal to Ω. Moreover, $\mathbf{graph}(\Gamma_\Omega)$ is closed, see Proposition 7.2. □

If Ω is lineally convex, $\Gamma_\Omega(b)$ is not necessarily connected, not even when Ω is a Hartogs set, as is shown by the example below as well as by Example 8.2 in the next section.

Example 7.5. Let Ω be the Cartesian product of an annulus and a disk,

$$\Omega = \{(z_1,z_2) \in \mathbf{C}^2;\ 1 < |z_1| < 2,\ |z_2| < 1\},$$

a lineally convex set. We define complex hyperplanes Z_β passing through 0 by the equations $\beta z_1 = (1-\beta)z_2$, $z \in \mathbf{C}^2$, $\beta \in [0,1]$. Use a ball $B_\leqslant(0,r)$ with $0<r<1$ as structuring element. Then $\Theta^2_{\Omega,\mathbf{C}}(b)$, where $b=(1,1)$, consists of all the Z_β, $\beta \in [0,1]$, whereas $\Gamma_\Omega(b)$ consists of Z_0 and Z_β, $\frac{1}{2} \leqslant \beta \leqslant 1$. Thus $\Gamma_\Omega(b)$ does not contain $\Theta^2_{\Omega,\mathbf{C}}(b)$. We also note that $\Gamma_\Omega(b)$ is not connected; it has two components, $\{Z_0\}$ and $\left\{Z_\beta;\ \frac{1}{2} \leqslant \beta \leqslant 1\right\}$.

If $0 < \beta < \frac{1}{2}$, then there are points $z = (-1-s, z_2) \in Z_\beta \cap \Omega$ far from $b=(1,1)$ (take $s>0$, $s = -1-z_1 < 2/\beta - 4$) as well as points in $Z_\beta \cap \Omega$ arbitrarily close to b.

The set Ω is lineally convex, but if we approximate it from the inside by a set with boundary of class C^1 containing all points in Ω with distance to $\partial\Omega$ at least equal to $\varepsilon > 0$, then we get a set which is $\mathscr{Z}_1$-convex but not $\mathscr{Z}_r$-convex for $r \geqslant 1+\varepsilon > 1$. (For $\mathscr{S}$-convexity, se Definition 4.2; for $\mathscr{Z}_r$-convexity, see the beginning of Section 5.) □

8. Convexity properties of superlevel sets

Definition 8.1. Given any function f on a set X and with values in the set $[-\infty, +\infty]$ of extended real numbers and an element c of $[-\infty, +\infty]$, we define its (non-strict) ***superlevel set*** as $\{x \in X;\ f(x) \geqslant c\}$. Analogously we define its (non-strict) ***sublevel set*** as $\{x \in X;\ f(x) \leqslant c\}$. □

Given a complete Hartogs set with radius function R, we shall denote by M_c the superlevel set $\{z \in \mathbf{C}^n;\ R(z) \geqslant c\}$.

Example 8.2. Consider the lineally convex Hartogs set $\Omega \subset \mathbf{C} \times \mathbf{C}$ defined by the radius function

$$R(z) = \min(|z-2|, |z+2|), \quad |z| < 1; \qquad R(z) = 0, \quad |z| \leqslant 1.$$

Then $(0, 2)$ belongs to the boundary of Ω and $\Gamma_\Omega((0, 2))$ consists of precisely two elements, the hyperplanes defined by $t = -z$ and $t = z$, respectively; thus it is not connected. (This shows that Ω is not $\mathbf{C}$-convex in view of Zelinskiĭ's criterion mentioned near the end of Section 2.) However, the union of all the $\Gamma_\Omega(b)$ with $b \in \partial\Omega$ is connected. We note that $\Gamma_\Omega((i, \sqrt{5}))$ is connected and contains $\Gamma_\Omega((0, 2))$. (This example is Example 3.1 in Kiselman (1996).)

The boundary points of Ω are accessible from the outside by balls of a not too large radius, and $\Theta_{\Omega,\mathbf{C}}((0, 2))$ consists of all hyperplanes $t = \lambda z$, with $\lambda \in [-1, 1]$. We also note that the intersection of Ω with the complex line $t = c$ has two components if $2 \leqslant |c| < \sqrt{5}$.

Also, for $a = s + i(1 - s/2)$ with a small positive number s, the superlevel set $M_{R(a)}$ is $B_<(0, r)$-convex for r slightly smaller than $\sqrt{5}$, whereas for $s = 0$, $a = i$, the superlevel set $M_{R(i)}$, now equal to $\{i, -i\}$, is $B_\leqslant(0, r)$-convex for any r but not convex. (This is a warning that r-convexity is not so meaningful for sets that are not regular open or regular closed.) □

For simplicity we shall assume below that $n = 1$.

Theorem 8.3. *Let $\Omega \subset \mathbf{C} \times \mathbf{C}$ be a lineally convex Hartogs domain defined as in* (6.1) *with $n = 1$. Assume that a point $a \in \omega$ is such that $R(a) < \sup R$. Then there exists an $r > 0$ such that $a \in \varepsilon_{D_<(0,r)}(\omega)$ implies $a \in \alpha_{D_\leqslant(0,r)}(\complement M_{R(a)})$. In other words, since a belongs to the open set ω, the distance r to $\complement\omega$ is positive, and a is exterior accessible in $M_{R(a)}$ by disks of radius r.*

Proof. There is a complex hyperplane (thus a complex line in the present situation) in the complement of Ω which passes through $(a, R(a))$. It cannot be vertical since $a \in \omega$ and it cannot be horizontal since $R(a) < \sup R$, so it must have an equation of the form

$$\frac{t}{R(a)} = 1 + \beta(z - a) = \beta(z - a_\beta),$$

where $\beta \neq 0$ and $a_\beta = a - 1/\beta$ is the point where the line hits the line $t = 0$.

This implies that the cone in $\mathbf{C} \times \mathbf{R}$ defined by $|t|/R(a) \geqslant |\beta(z - a_\beta)|$ does not meet any point $(z, |t|) \in \Omega$, in particular that a belongs to the disk $D_\leqslant(a_\beta, s)$ with center at a_β and radius $s = |a - a_\beta| = 1/|\beta|$. As noted, this disk does not meet ω, so $a \in \alpha_{D_\leqslant(0,s)}(\complement M_{R(a)})$. We note finally that $s = |a - a_\beta| \geqslant d(a, \complement\omega) = r$. □

There is no uniformity here: r depends on a. But if Ω is bounded and we restrict attention to points a in a compact subset of ω and with $R(a) \geqslant c > 0$, we can choose a fixed $r > 0$. Thus $M_{R(a)}$ is $D_<(0, r)$-convex.

There may be several lines of the form $t = \beta(z - a_\beta)$ as mentioned in the proof. Then among all the possible values of $\beta \in \mathbf{C}$ we can take the infimum of their absolute values, and any limit of these numbers must also define a line in the complement of Ω, since the complement is closed. This gives the largest possible value to $r = 1/|\beta|$.

In Example 8.2 we see that, for a real such that $0 < a < 1$,

$$r = 2 - a > d(a, \complement\omega) = 1 - a,$$

implying that the number r obtained in the proof can be smaller than it is in an actual situation.

Remark 8.4. In the other direction, if a closed r-convex set M in $\mathbf{C}$ is given, then there exists a lineally convex open set in $\mathbf{C}^n \times \mathbf{C}$ with radius function R such that $M_{\sup R} = M$; see Proposition 4.9 in my paper (1996). $\square$

Corollary 8.5. *If Ω is lineally convex and bounded, and its boundary is of class C^1 at the set where $R > 0$, then a point $a \in \omega$ belongs to $\alpha_{D_\leqslant(0,r)}(\complement M_{R(a)})$ if*

$$r \leqslant \frac{1}{\|(\mathbf{grad}\, R)(a)\|_2}.$$

This is the case for all points z with $R(z) = R(a)$ if

$$r \leqslant \frac{1}{\sup_{z\in\omega}\Big(\|(\mathbf{grad}\, R)(z)\|_2;\ R(z) = R(a)\Big)}.$$

We see that $r \nearrow +\infty$ when $R(a) \nearrow \sup R$, meaning that the superlevel set becomes more and more convex. We shall make this precise in Theorem 10.1.

Proof. In this situation there is only one line in the complement of Ω passing through $(a, R(a))$, and the absolute value of the coefficient β is $\|(\mathbf{grad}\, R)(a)\|_2 = 2|R_z|$. The radius r depends on a and may vary, but among all the points z with $R(z) = R(a)$ its lower bound is positive. $\square$

We now consider a situation with two levels, $R(a)$ and $R(a) + s \geqslant R(a)$.

Theorem 8.6. *Let $\Omega \subset \mathbf{C} \times \mathbf{C}$ be a lineally convex Hartogs domain defined as in* (6.1) *with $n = 1$ and take a point $a \in \omega \subset \mathbf{C}$ with $R(a) < \sup R$. Then there exists a number $r > 0$ such that for all $s \geqslant 0$,*

$$d(a, M_{R(a)+s}) \geqslant \frac{sr}{R(a)},$$

where the inequality means that any point w with $R(w) = R(a) + s$ is outside the disk $D_<(a_\beta, r_1)$ with $r_1 = r + sr/R(a)$.

It follows that w is accessible with disks of radius r_1 in the complement of the superlevel set $M_{R(a)+s}$.

Proof. As in the proof of Theorem 8.3, we see that the cone defined by $|t|/R(a) \geqslant |\beta(z - a_\beta)|$, where β is the coefficient in the equation of the line in the complement of Ω passing through $(a, R(a))$, viz. $t/R(a) = 1 + \beta(z - a)$, does not contain any point of the form $(z, |t|)$ in Ω. We take $r = 1/|\beta|$. In particular the disk $D_<(a_\beta, r_1)$ with $r_1 = r + sr/R(a)$ for any w with $R(w) = R(a) + s$ does not meet $M_{R(a)+s}$.

Since also $(w, R(w))$ admits a line $t/R(w) = 1 + \gamma(z - w)$ in the complement of Ω, we must have $|\gamma| \leqslant |\beta|$, so the corresponding radius $r_2 = 1/|\gamma|$ is not smaller than r_1. □

9. Admissible multifunctions

Definition 9.1. Let Ω be an open subset of $\mathbf{C}^n$ and $\gamma\colon B \rightrightarrows \mathrm{Gr}_{n-1}(\mathbf{C}^n)$ a multifunction defined on a subset B of the boundary of Ω and with values in the Grassmann manifold of all hyperplanes through the origin. We shall say that γ is ***admissible*** if

(9.1.1). $\gamma(b) \subset \Gamma_\Omega(b)$ for all $b \in B$;

(9.1.2). The graph of γ is closed; and

(9.1.3). $\gamma(b)$ is nonempty and connected for every $b \in B$. □

It follows that $\mathbf{graph}(\gamma)$ is connected if $\partial\Omega$ is connected; see Lemma 9.3 below.

An example of an admissible multifunction is $\Gamma_\Omega\colon \partial\Omega \rightrightarrows \mathrm{Gr}_{n-1}(\mathbf{C}^n)$ provided $\Gamma_\Omega(b)$ is connected for every $b \in \partial\Omega$. (In particular, this is the case if the boundary is of class C^1.) The graph is then automatically closed in view of Proposition 7.2. It is easy to see that in Examples 7.5 and 8.2, there is no admissible multifunction γ in any neighborhood of the points $(1, 1)$ and $(0, 2)$, respectively.

If Ω is tangentially lineally convex, a candidate for γ might be the closure of $\Theta_{\Omega,\mathbf{C}}$. Then property (9.1.1) holds by hypothesis, (9.1.2) by construction, and (9.1.3) may hold if the boundary of Ω is sufficiently regular.

Example 9.2. Let Ω be a convex open set in $\mathbf{C}^n$. If Ω is empty or equal to the whole space, then its boundary is empty. If Ω is a slice, then its boundary has two components.

In all other cases, $\partial\Omega$ is connected, and we know that the set of all real hyperplanes passing through a fixed boundary point b and not intersecting Ω is connected. Then also the set of all complex hyperplanes containing b and contained in such a real hyperplane is connected—the mapping $Y \mapsto Y_{[b]}$ is continuous as we noted in Section 2. Thus Γ_Ω is an admissible multifunction except in the first-mentioned cases, even if the boundary is not of class C^1. □

If a lineally convex open set Ω has a C^1 boundary, $\Gamma_\Omega(b)$, a singleton set, depends continuously on b. When $\Gamma_\Omega(b)$ is no longer a singleton, the following result will serve instead of the continuity.

Lemma 9.3. *Let Ω be an open set in $\mathbf{C}^n$ and $\gamma\colon A \rightrightarrows \mathrm{Gr}_{n-1}(\mathbf{C}^n)$ an admissible multifunction on a subset A of $\partial\Omega$. Then the graph of γ over B,*

$$\mathbf{graph}_B(\gamma) = \{(b, Z);\ b \in B \text{ and } Z \in \gamma(b)\},$$

is connected for every connected subset B of A. In particular the graph of γ is connected if the boundary of Ω is connected and γ is defined on all of it.

Proof. Assume that $\mathbf{graph}_B(\gamma) = V_0 \cup V_1$, where the V_j are disjoint and closed relative to $\mathbf{graph}_B(\gamma)$. Define B_j as the set of all points b such that some hyperplane in $\gamma(B)$ belongs to V_j, $j = 0, 1$. Then B_0 and B_1 are disjoint, since by hypothesis every $\gamma(b)$ is connected. Moreover B_0 and B_1 are closed relative to B, since the graph of γ is closed and the manifold $\mathrm{Gr}_{n-1}(\mathbf{C}^n)$ is compact. By hypothesis B is connected, so either B_0 or B_1 must be empty. Hence V_0 or V_1 is empty, proving that the graph of γ over B is connected.

We note that $\gamma_*(B)$ is connected as a continuous image of the graph (it is the projection of the graph on the target space $\mathrm{Gr}_{n-1}(\mathbf{C}^n)$). □

Proposition 9.4. *Let Ω be an open subset of $\mathbf{C}^n$ and F an affine subspace of $\mathbf{C}^n$. Denote by Ω_F the set $\Omega \cap F$ considered as an open subset of F. Every complex hyperplane Z in $\mathbf{C}^n$ which does not contain F gives rise to a complex hyperplane $\psi(Z) = Z \cap F$ in F. Let an admissible multifunction $\gamma\colon B \rightrightarrows \mathrm{Gr}_{n-1}(\mathbf{C}^n)$ be given and define a multifunction γ_F on $B \cap \partial\Omega_F$ by $\gamma_F(b) = \{\psi(Z);\ Z \in \gamma(b)\}$. Then γ_F is an admissible multifunction on $B \cap \partial\Omega_F$.*

Proof. Since $\psi(Z) \subset Z$, it is clear that $\gamma_F(b) \subset \Gamma_{\Omega_F}(b)$; thus (9.1.1) in Definition 9.1 holds. The graph of γ over any compact subset of $\partial\Omega$ is compact; hence the graph of γ_F over any compact subset of $\partial\Omega_F$ is compact, thus closed: property (9.1.2) holds. Finally (9.1.3) follows since ψ is continuous and thus maps connected subsets onto connected subsets. □

The proposition can in particular be applied to Γ_Ω if $\Gamma_\Omega(b)$ is connected for all $b \in \partial\Omega$.

10. Links to ordinary convexity

Theorem 10.1. *Let R be a continuous real-valued function defined on $\mathbf{C}^n$ and define Ω by (6.1). Assume that Ω is connected and that its boundary is of class C^1 (at least in a neighborhood of $M_{\sup R}$). Then the set $M_{\sup R}$ where R attains its maximum,*

$$M_{\sup R} = \{z \in \mathbf{C}^n;\ R(z) = \sup R\}, \tag{10.1}$$

is convex.

Proof. A set is convex if and only if its intersection with every one-dimensional complex affine subspace is convex. Therefore it is enough to prove the theorem for $n = 1$.

So let $n = 1$ and let a belong to the boundary of $M_{\sup R}$. We shall prove that there is an open half plane with a on its boundary which does not meet $M_{\sup R}$, proving the convexity of that set.

We have $(\mathbf{grad}\, R)(a) = 0$, and near a there are points c with $(\mathbf{grad}\, R)(c)$ nonzero and arbitrarily small. In view of Corollary 8.5 this means that there is a disk of arbitrarily large radius with c on its boundary. The disk is of the form $D_{\leqslant}(c_\beta, r)$, where $r = |c - c_\beta|$, $c_\beta = c - 1/\beta$ being the point where the line $t/R(c) = 1 + \beta(z - c)$ hits the plane $t = 0$. The normalized vectors $(c - c_\beta)/|c - c_\beta|$ have an accumulation point, and this proves that the union of all the disks $D_{\leqslant}(c_\beta, r)$ when c varies in an arbitrarily small neighborhood of a contains an open half plane with a on its boundary. We are done. □

The assumption that the boundary be of class C^1 can be weakend, as we shall now show.

Theorem 10.2. *Let R be a continuous real-valued function defined on $\mathbf{C}^n$ and define Ω by* (6.1)*. Assume that Ω is bounded and connected and that there exists an admissible multifunction γ defined at all boundary points (z^0, t^0) with $|t^0| = R(z^0) > 0$ (see Definition 9.1). Then the set $M_{\sup R}$ where R attains its maximum is convex.*

For a Hartogs domain Ω we always have $\Gamma_\Omega(b) \subset \Theta_{\Omega,\mathbf{C}}(b)$ when $b = (z^0, t^0)$ with $|t^0| = R(z^0) > 0$ (Theorem 6.2); if the domain is tangentially lineally convex, we have $\Gamma_\Omega(b) = \Theta_{\Omega,\mathbf{C}}(b)$. For such domains we therefore have an admissible multifunction $\gamma = \Gamma_\Omega = \Theta_{\Omega,\mathbf{C}}$: (9.1.1) is obvious; (9.1.2) follows from Proposition 7.2; (9.1.3) follows from Proposition 3.11.

We note that the hypothesis is satisfied in particular if Ω is lineally convex and R is of class C^1. In Kiselman (1996, Theorem 4.8) the result was proved under this hypothesis, and even under the weaker one that R can be approximated from below by C^1 functions.

In view of Zelinskiĭ's characterization of $\mathbf{C}$-convex sets mentioned near the end of Section 2, the hypotheses are satisfied for $\mathbf{C}$-convex sets, again taking $\gamma = \Gamma_\Omega$. There are easy examples which show that $M_{\sup R}$ need not be convex if we drop the hypothesis of connectedness; see Example 8.2.

Proof of Theorem 10.2. Again, the set $M_{\sup R}$ is convex if its intersection with every one-dimensional complex affine subspace is convex. Proposition 9.4 shows that if we have an admissible multifunction on a subset of $\partial\Omega$, then there is one also on a corresponding subset of $\partial\Omega_F$, F being any affine subspace of $\mathbf{C}^n \times \mathbf{C}$. Therefore, taking F as the Cartesian product of a complex line in $\mathbf{C}^n$ and the line $z = 0$, we see that it is enough to prove the theorem for $n = 1$.

So let $n = 1$. To prove that $M_{\sup R}$ is convex means to prove that the segment $[s_0, s_1]$ is contained in $M_{\sup R}$ if $s_0, s_1 \in M_{\sup R}$. There is no loss in generality if we assume that $s_0 = -1$ and $s_1 = 1$.

A non-vertical and non-horizontal complex line through (a, t^0) with $t^0 \neq 0$ has the equation

$$\frac{t}{t^0} = 1 + \beta(z - a) = \beta(z - a_\beta), \qquad z \in \mathbf{C},$$

where $a_\beta = a - 1/\beta$ is the point where the line hits the plane $t = 0$. We define

$$q(a, \beta) = \begin{cases} a - 1/\beta \text{ if } \beta \neq 0, \\ \infty \text{ if } \beta = 0. \end{cases}$$

In case R is differentiable at the point a, β is uniquely determined if we require that the line be in $\Gamma_\Omega((a, t^0))$.

We denote as before by ω the set of all points $z \in \mathbf{C}$ such that $R(z) > 0$. In general the external tangent is not unique and we shall denote by $Q(a)$ the set of all points $a - 1/\beta$ that can be obtained from complex lines in $\gamma((a, t^0))$, thus

$$Q(a) = \{q(a, \beta);\ \beta \in \gamma(a, R(a))\} \subset S^2 = \mathbf{C} \cup \{\infty\}, \qquad a \in \omega. \tag{10.2}$$

We define $Q(a) = \{a\}$ when $a \notin \omega$. Thus Q is a multifunction, $Q\colon S^2 \rightrightarrows S^2 \smallsetminus \omega$; its images $Q(a)$ are compact and connected.

The radius can always be estimated by

$$R(z) \leqslant R(a)|\beta| \cdot |z - a_\beta|, \qquad z \in \mathbf{C},\ a \in \omega,\ \beta \in \gamma(a, R(a)),\ a_\beta = q(a, \beta),$$

with equality for $z = a$, assuming $\beta \neq 0$. In particular, if $w \in M_{\sup R}$, then

$$R(a)|\beta| \cdot |a - q(a, \beta)| = R(a) \leqslant R(w) \leqslant R(a)|\beta| \cdot |w - q(a, \beta)|.$$

If $a_\beta \in Q(a) \smallsetminus \{\infty\}$, then necessarily $\beta \neq 0$, so that

$$|a - a_\beta| \leqslant |w - a_\beta|, \qquad a \in \omega,\ w \in M_{\sup R},\ a_\beta \in Q(a) \smallsetminus \{\infty\}. \tag{10.3}$$

Assume that -1 and 1 belong to $M_{\sup R}$; we shall then prove that any point $c \in [-1, 1]$ belongs to $M_{\sup R}$. Consider $Q(c + iy)$ for real y. We know from Lemma 9.3 that the set $Q_*(c + i\mathbf{R})$ is connected. If ω is bounded and y or $-y$ is very large, then $Q(c + iy) = \{c + iy\}$. In general we can prove that $\operatorname{Im} a > 1$ implies that $\operatorname{Im} b > 0$ for all $b \in Q(a)$, and similarly $\operatorname{Im} a < -1$ implies $\operatorname{Im} b < 0$ for all $b \in Q(a)$. This follows from the following lemma.

Lemma 10.3. *If Ω is a complete Hartogs domain in $\mathbf{C}^2$ with radius function R and if $\pm 1 \in M_{\sup R}$, then for all $b \in \mathbf{C}$ with $|\operatorname{Re} a| \leqslant 1$ and all $b \in Q(a) \smallsetminus \{\infty\}$ we have*

$$\operatorname{Im} a \geqslant 1 \textit{ implies } \operatorname{Im} b \geqslant \tfrac{1}{2}(\operatorname{Im} a - 1) \quad \textit{and} \quad \operatorname{Im} a \leqslant -1 \textit{ implies } \operatorname{Im} b \leqslant \tfrac{1}{2}(\operatorname{Im} a + 1).$$

Proof. We know from (10.3) that $|a - b| \leqslant |\pm 1 - b|$. Expanding $|\pm 1 - b|^2 - |a - b|^2 \geqslant 0$, we get

$$2(\operatorname{Re} b)(\operatorname{Re} a \mp 1) + 1 - (\operatorname{Im} a)(\operatorname{Im} a - 2\operatorname{Im} b) \geqslant (\operatorname{Re} a)^2 \geqslant 0,$$

from which we deduce that $1 \geqslant (\operatorname{Im} a)(\operatorname{Im} a - 2\operatorname{Im} b)$, an inequality which implies those in the lemma. $\square$

Proof of Theorem 10.2, cont'd. So $Q(c + iy)$ must pass from the upper half plane to the lower half plane when y goes from large positive values to large negative values, c being fixed. But it can never pass the real axis at points with $x \geqslant 1$ or $x \leqslant -1$. Indeed, if b is real and larger than or equal to 1, we get from (10.3), taking $a = c + iy$,

$$|a - b| \leqslant |1 - b| = b - 1,$$

implying $\operatorname{Re} a \geqslant 1$, so that $c \geqslant 1$ contrary to assumption. Likewise, $Q(c + iy)$ cannot pass the real axis at a point with $x \leqslant -1$.

However, $Q(c + iy)$ cannot pass from numbers with arbitrarily large positive imaginary part to numbers with large negative imaginary part in the strip $-1 < \operatorname{Re} z < 1$ either. In fact, ω is connected, so there exists a curve contained in ω connecting -1 to 1, and $Q(c + iy)$ cannot cross that curve.

Hence it is impossible for $Q(c + iy)$ to pass from the upper half plane to the lower half plane if it has only finite values. So it must have an infinite value, which means that $c + iy^0 \in M_{\sup R}$ for at least one y^0.

We thus know that there is a y^0 such that $c + iy^0 \in M_{\sup R}$; without loss of generality we may assume that it is nonnegative. Choose y^0 as small as possible. If $y^0 = 0$ we are done: $c \in M_{\sup R}$. Let us assume that $y^0 > 0$ and try to reach a contradiction.

By (10.3) any point $b \in Q(a) \smallsetminus \{\infty\}$ must lie in each of the three half planes

$$|a - b| \leqslant |1 - b| \qquad |a - b| \leqslant |-1 - b|, \qquad |a - b| \leqslant |c + iy^0 - b|.$$

The intersection of these three half planes is a triangle, and the union of these triangles when $a = c + iy$ with $y \in [\frac{1}{2}y^0, y^0]$ is bounded. Thus the possible finite values for b

when a varies as indicated is bounded, and for $a = c + iy$ with $\frac{1}{2}y^0 \leqslant y < y^0$ the point b cannot be infinity. On the other hand, when $a = c + iy^0 \in M_{\sup R}$, then $Q(a)$ must contain ∞. This means that the set of all points $b \in Q(a)$ originating from points $a = c + iy$ with $y \in [\frac{1}{2}y^0, y^0]$ consists of ∞ and a nonempty bounded set; it is not connected, in contradiction to Lemma 9.3. This contradiction shows that we must have $c \in M_{\sup R}$ and proves the theorem. □

It is easy to modify Theorem 10.2 using Möbius mappings, at least if $n = 1$. In fact, any mapping

$$\mathbf{C} \times \mathbf{C} \ni (z, t) \mapsto \left(\frac{a + bz}{c + dz}, \frac{t}{c + dz} \right) = (z', t') \in \mathbf{C} \times \mathbf{C}$$

preserves lineal convexity, as was shown in Kiselman (1996: Lemma 8.1). Denote by a_β the point where a line $t/t^0 = 1 + \beta(z - z^0)$ intersects the z-plane. The line can be mapped by a Möbius mapping to a line $t' =$ constant. This mapping takes the point a_β to infinity, and all circles in the z-plane which pass through a_β are mapped onto straight lines. Convex sets are transformed accordingly:

Definition 10.4. Let b be a complex number or ∞. Let us say that a subset A of the Riemann sphere $\mathbf{C} \cup \{\infty\}$ is b-***convex*** if

(10.4.1). $b \notin A$; and

(10.4.2). $\varphi_*(A)$ is convex if φ is a Möbius mapping which maps b to infinity. □

Corollary 10.5 (to Theorem 10.2)**.** *Let Ω and γ be as in Theorem 10.2, assume that $n = 1$ and let π denote the projection defined by $\pi(z, t) = z$. Consider a line $Z \in \gamma(a)$, where $a = (z^0, t^0)$, $|t^0| = R(z^0) > 0$, and let b be the point such that $(b, 0) \in a + Z$. Then the set $\pi_*((a + Z) \cap \overline{\Omega})$ is b-convex.*

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Author's address: Uppsala University, Department of Information Technology,
P. O. Box 337, SE-751 05 Uppsala, Sweden.
Amber addresses: `kiselman@it.uu.se`, `christer@kiselman.eu`
URL: `http://www.cb.uu.se/~kiselman`