

Duality of functions defined in lineally convex sets

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Resumo: *Dualeco de funkcioj difinitaj en linie konveksaj aroj*

Ni enkondukas dualecon, similan al la transformo de Fenchel, inter funkcioj difinitaj en aroj linie konveksaj.

Abstract: We introduce a duality, similar to the Fenchel transformation, of functions that are defined in lineally convex sets.

1. Introduction

Lineal convexity, a kind of complex convexity intermediate between usual convexity and pseudoconvexity, appears naturally in the study of Fantappiè transforms of analytic functionals. A set is called lineally convex if its complement is a union of complex hyperplanes. This property can be most conveniently defined in terms of the notion of dual complement: the dual complement of a set in \mathbf{C}^n is the set of all hyperplanes that do not intersect the set. It is natural to add a hyperplane at infinity and consider \mathbf{C}^n as an open subset of \mathbf{P}^n , complex projective space of dimension n . The definition of dual complement is then the same, and somewhat more natural: the set of all hyperplanes is again a projective space. (In this setting the dual complement is often called the projective complement. Indeed Martineau [1966] called it *le complémentaire projectif*; the term *dual complement* used here was introduced by Andersson, Passare and Sigurdsson in the 1991 version of their survey [1995].)

We can now simply define a lineally convex set as a set which is the dual complement of its dual complement (here it becomes obvious that we should identify the hyperplanes in the space of all hyperplanes with the points in the original space). So this duality works well for sets. What about functions?

In convexity theory, a convenient dual object of a set is its supporting function. Now a set A can be identified with its indicator function I_A (which takes the value 0 in A and $+\infty$ outside). The supporting function H_A of A is then the Fenchel

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transform of the indicator function of A , so the duality for sets is a special case of the Fenchel transformation, which expresses the duality for functions in convexity theory.

Is there a duality for functions which generalizes the duality for sets defined by the dual complement? In this note we shall study such a duality. We call it the logarithmic transformation. It has of course many properties in common with the Fenchel transformation. However, there are some striking differences. The effective domain of a Fenchel transform is always convex, but the effective domain of a logarithmic transform need not be lineally convex (Example 3.7). This is connected with the fact that the union of an increasing sequence of lineally convex sets is not necessarily lineally convex (Example 3.8). However, the interior of the effective domain of a logarithmic transform is always lineally convex (Theorem 3.5), and the transform is plurisubharmonic there (Theorem 3.9).

Working with functions defined on \mathbf{P}^n is the same as working with functions defined on $\mathbf{C}^{1+n} \setminus \{0\}$ which are constant on complex lines, i.e., homogeneous of degree zero. For instance a plurisubharmonic function on an open subset of \mathbf{P}^n can be pulled back to an open cone in $\mathbf{C}^{1+n} \setminus \{0\}$ and the pull-back is plurisubharmonic for the $1+n$ coordinates there. However, I cannot define a duality for such functions. I have been led to consider instead functions defined on subsets of $\mathbf{C}^{1+n} \setminus \{0\}$ which are homogeneous in another sense: they satisfy $f(tz) = -\log |t| + f(z)$. Such functions are of course not pull-backs of functions on projective space, but the duality works for them. In a coordinate patch like $z_0 = 1$ we can identify them with functions on a subset of \mathbf{P}^n . Given any function F on \mathbf{C}^n , we can define a function f on $\mathbf{C}^{1+n} \setminus \{0\}$ by $f(z) = F(z_1/z_0, \dots, z_n/z_0) + c \log |z_0|$ when $z_0 \neq 0$ and $f(z) = +\infty$ when $z_0 = 0$, where c is an arbitrary real constant; this function is homogeneous in the sense that $f(tz) = c \log |t| + f(z)$, so we can choose any type of homogeneity. In other words, locally all kinds of homogeneity are equivalent, and there is no restriction in imposing the homogeneity we have here ($c = -1$).

There are several other notions related to lineal convexity. The property called *Planarkonvexität* in German (see Behnke & Peschl [1935]) or *weak lineal convexity* is weaker than lineal convexity: an open connected set is called weakly lineally convex if through any boundary point there passes a complex hyperplane which does not intersect the set. Aizenberg [1967] proved that these domains are precisely the components of Ω^{**} (for notation see (2.1) below).

Strong lineal convexity was defined by Martineau [1966, Definition 2.2] as a topological property of the space of holomorphic functions in a domain. Martineau [1966, Theorem 2.2] and Aizenberg [1966] proved independently that convex sets are strongly lineally convex. The property was given a geometric characterization by Znamenskij [1979]. This geometric property is now called *\mathbf{C} -convexity*. Its relation to lineal convexity has been studied by Zelinskij [1988] and others. For these two properties we refer also to the survey by Andersson, Passare and Sigurdsson [1995] and the monograph by Hörmander [1994].

Another generalization is the notion of *k -lineal convexity*. A set is said to be *k -lineally convex* if its complement is a union of affine subspaces of codimension k ($1 \leq k \leq n$). This concept, which makes sense also in an infinite-dimensional space, was studied in Kiselman [1978]. Thus *n -lineal convexity* is no condition at all,

whereas 1-lineal convexity is the lineal convexity studied here; the other notions are intermediate.

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2. Lineal convexity

Let A be a subset of $\mathbf{C}^{1+n} \setminus \{0\}$, where $n \geq 1$. We shall say that A is **homogeneous** if $tz \in A$ as soon as $z \in A$ and $t \in \mathbf{C} \setminus \{0\}$. To any homogeneous subset A of $\mathbf{C}^{1+n} \setminus \{0\}$ we define its **dual complement** A^* as the set of all hyperplanes passing through the origin which do not intersect A . Since any such hyperplane has an equation $\zeta \cdot z = \zeta_0 z_0 + \cdots + \zeta_n z_n = 0$ for some $\zeta \in \mathbf{C}^{1+n} \setminus \{0\}$, we can define

$$(2.1) \quad A^* = \{\zeta \in \mathbf{C}^{1+n} \setminus \{0\}; \zeta \cdot z \neq 0 \text{ for every } z \in A\}.$$

Strictly speaking, we should have two copies of $\mathbf{C}^{1+n} \setminus \{0\}$ (a Greek and a Latin one), and consider A^* as a subset of the dual (i. e., Greek) space. A homogeneous set is called **lineally convex** if $\mathbf{C}^{1+n} \setminus A$ is a union of complex hyperplanes passing through the origin. A dual complement A^* is always lineally convex, and we always have $A^{**} \supset A$. The set A^{**} is called the **lineally convex hull** of A . A set A is lineally convex if and only if $A = A^{**}$.

The operation of taking the dual complement is an example of a Galois correspondence, and the operation of taking the lineally convex hull defines a closure operator in the (partially) ordered set of all subsets of $\mathbf{C}^{1+n} \setminus \{0\}$. The general definition of these notions is as follows (see Kuroš [1962:6:11]). Let X and Y be any two ordered sets. A **Galois correspondence** is a pair of decreasing mappings $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g(f(x)) \geq x$ for all $x \in X$ and $f(g(y)) \geq y$ for all $y \in Y$. It follows that $f \circ g \circ f = f$ and $g \circ f \circ g = g$. The operator $g \circ f$ maps X into itself and is a **closure operator**, which means that it is increasing, expanding (larger than the identity), and idempotent. Explicitly, if we write \bar{x} for $g(f(x))$:

$$\begin{aligned} x_1 \leq x_2 &\text{ implies } \bar{x}_1 \leq \bar{x}_2; \\ x &\leq \bar{x}; \text{ and} \\ \overline{\bar{x}} &= \bar{x}. \end{aligned}$$

The elements x such that $\bar{x} = x$ will be called **closed**. Every closure operator comes from some Galois correspondence. Indeed, if a closure operator $x \mapsto \bar{x}$ is given in an ordered set X , define Y as X with the opposite order and $f(x) = \bar{x}$, $g(y) = \bar{y}$. Then $g \circ f(x) = \bar{x}$. The sets which are closed for the closure operator $A \mapsto A^{**}$ are precisely the lineally convex sets.

We shall write $z = (z_0, z') = (z_0, z_1, \dots, z_n)$ for points in $\mathbf{C}^{1+n} \setminus \{0\}$, with $z_0 \in \mathbf{C}^1$ and $z' = (z_1, \dots, z_n) \in \mathbf{C}^n$. Homogeneous sets in $\mathbf{C}^{1+n} \setminus \{0\}$ correspond to subsets of projective n -space \mathbf{P}^n , and we can transfer the notions of dual complement and lineal convexity to \mathbf{P}^n . In the open set where $z_0 \neq 0$ we can use z' as coordinates in \mathbf{P}^n .

We shall write

$$(2.2) \quad Y_\zeta = \{z \in \mathbf{C}^{1+n} \setminus \{0\}; \zeta \cdot z = 0\}$$

for the hyperplane defined by ζ . Then the dual complement can be conveniently defined as

$$(2.3) \quad A^* = \{\zeta; Y_\zeta \cap A = \emptyset\},$$

and its set-theoretic complement in $\mathbf{C}^{1+n} \setminus \{0\}$ is

$$(2.4) \quad \mathfrak{C}A^* = (\mathbf{C}^{1+n} \setminus \{0\}) \setminus A^* = \{\zeta; Y_\zeta \cap A \neq \emptyset\}.$$

The complement of the lineally convex hull A^{**} can be written as

$$\mathfrak{C}A^{**} = \bigcup_{\alpha \in A^*} Y_\alpha.$$

We shall use this idea in the following lemma.

Lemma 2.1. *For any subset Γ of $\mathbf{C}^{1+n} \setminus \{0\}$ we define*

$$A = \mathfrak{C} \bigcup_{\gamma \in \Gamma} Y_\gamma.$$

*Then A is lineally convex. Moreover $A^{**} = A = \Gamma^*$ and $A^* = \Gamma^{**} \supset \Gamma$.*

Proof. Clearly A as the complement of a union of hyperplanes is lineally convex, so $A^{**} = A$. The statement $a \in A$ is equivalent to $\gamma \cdot a \neq 0$ for all $\gamma \in \Gamma$, which by definition means that $a \in \Gamma^*$; thus $A = \Gamma^*$. As a consequence, $A^* = \Gamma^{**}$.

How does the operation of taking the dual complement intertwine with the topological operations of taking the interior and closure? The answer is the following (we write A° for the interior and \overline{A} for the closure of a set A).

Proposition 2.2. *For any homogeneous subset A of $\mathbf{C}^{1+n} \setminus \{0\}$ we have*

$$(2.5) \quad A^{*\circ} = (\overline{A})^* \text{ and}$$

$$(2.6) \quad \overline{A^*} \subset A^{\circ*}.$$

If A is closed, then A^ is open. If A is open, then A^* is closed.*

Proof. To see that A^* is open if A is closed we only have to look at (2.3). Hence $(\overline{A})^*$ is always open, which implies $(\overline{A})^* \subset A^{*\circ}$.

Similarly, (2.4) shows that A^* is closed if A is open. Hence $A^{\circ*}$ is always closed, and we see that $A^{\circ*} \supset \overline{A^*}$.

The inclusion $A^{*\circ} \subset (\overline{A})^*$ remains to be proved. If $\zeta \in A^{*\circ}$, then $Y_\theta \cap A = \emptyset$ for all θ near ζ . The union of these hyperplanes Y_θ is a neighborhood of Y_ζ , so $\zeta \in (\overline{A})^*$. This proves the proposition.

Corollary 2.3. *If a subset A of $\mathbf{C}^{1+n} \setminus \{0\}$ is strongly contained in a set B in the sense that $\overline{A} \subset B^\circ$, then B^* is strongly contained in A^* .*

Proof. Using (2.6) and (2.5) we see that $\overline{A} \subset B^\circ$ implies $\overline{B^*} \subset B^{\circ*} \subset (\overline{A})^* = A^{\circ*}$.

Corollary 2.4. *If a subset A of $\mathbf{C}^{1+n} \setminus \{0\}$ is lineally convex, then its interior A° is also lineally convex.*

Proof. If $A = B^*$, then $A^\circ = B^{\circ*} = (\overline{B})^*$ by (2.5), which shows that A° is lineally convex.

By way of contrast, the closure of a lineally convex set is not necessarily lineally convex if $n \geq 2$. It turns out that the lineal convexity of the closure is connected with the question whether we have equality in (2.6), as shown by the following result.

Corollary 2.5. *Let B be any lineally convex subset of $\mathbf{C}^{1+n} \setminus \{0\}$. Then its closure \overline{B} is lineally convex if and only if its dual complement $A = B^*$ satisfies (2.6) with equality.*

Proof. Using the lineal convexity of B , then (2.6) and (2.5), we get

$$\overline{B} = \overline{A^*} \subset A^{\circ*} = B^{*\circ*} = (\overline{B})^{**}.$$

Thus equality in (2.6) is equivalent to \overline{B} being lineally convex.

The inclusion (2.6) in Proposition 2.2 can be strict simply for dimensionality reasons. This will be clear from the following result, where we use the relative interior instead of the interior with respect to the whole space.

Proposition 2.6. *Let A be a homogeneous set in $\mathbf{C}^{1+n} \setminus \{0\}$ which is contained in a complex subspace F of \mathbf{C}^{1+n} . Let A_F denote the relative interior of A in F . Then $\overline{A^*} \subset (A_F)^* \cup F^\diamond$, where F^\diamond is the set*

$$F^\diamond = \{\zeta \in \mathbf{C}^{1+n} \setminus \{0\}; F \setminus \{0\} \subset Y_\zeta\}.$$

If A is open in F , then $A^ \cup F^\diamond$ is closed.*

Note that when $F = \mathbf{C}^{1+n}$, then F^\diamond is empty and we are reduced to Proposition 2.2.

Proof. Take a point $\zeta \notin (A_F)^* \cup F^\diamond$. Then there is a point $a \in A_F \cap Y_\zeta$ and a non-zero vector $b \in F \setminus Y_\zeta$. If θ is close to ζ , then the hyperplane Y_θ cuts the complex line $\{a + tb; t \in \mathbf{C}\}$ in a unique point $a(t)$ close to a , and since a is in the relative interior of A , $a(t)$ belongs to A as soon as θ is close enough to ζ . Therefore $\theta \notin A^*$ for all these θ , which means that $\zeta \notin \overline{A^*}$.

Finally, if A is open in F , then $\overline{A^* \cup F^\diamond} = \overline{A^*} \cup \overline{F^\diamond} \subset (A_F)^* \cup F^\diamond = A^* \cup F^\diamond$, since $A_F = A$ and F^\diamond is closed.

Example 2.7. It is now obvious that the inclusion (2.6) can be strict. Take a non-empty relatively open set $A \subset F \neq \mathbf{C}^{1+n}$. Then $A^\circ = \emptyset$, and $A^{\circ*} = \mathbf{C}^{1+n} \setminus \{0\}$. But $\overline{A^*} \subset (A_F)^* \cup F^\diamond = A^* \cup F^\diamond \neq \mathbf{C}^{1+n} \setminus \{0\}$.

Example 2.8. However, also the inclusion in Proposition 2.6 can be strict. There are sets A such that $A^\circ = \emptyset$, $(\overline{A})^\circ = B \neq \emptyset$, and $B^* = A^*$. Thus $A^{\circ*} = \mathbf{C}^{1+n} \setminus \{0\}$ and $\overline{A^*} = \overline{B^*} = B^* \neq \mathbf{C}^{1+n} \setminus \{0\}$. Such a set is the set A of all $z \in \mathbf{C}^{1+2}$ with

$|z_1|^2 + |z_2|^2 < |z_0|^2$ and either z_1 is a complex rational or $z_2 = 0$. (Here the only choice for F is the whole space, so that F^\diamond is empty.)

Example 2.9. A simple example of a lineally convex set whose closure is not lineally convex is the following, taken from Andersson, Passare & Sigurdsson [1995:29f]. Define

$$A = \{z \in \mathbf{C}^{1+2} \setminus \{0\}; |z_1| < |z_2|\}; \quad \bar{A} = \{z \in \mathbf{C}^{1+2} \setminus \{0\}; |z_1| \leq |z_2|\}.$$

Then A is lineally convex. Any hyperplane which avoids A must pass through $(z_0, z_1, z_2) = (1, 0, 0)$. But this point belongs to \bar{A} . This shows that \bar{A} is not lineally convex. More generally, let Γ be a lineally convex subset of the Greek copy of $\mathbf{C}^{1+n} \setminus \{0\}$ and define A as in Lemma 2.1. We can easily choose Γ without interior points but still such that

$$\bigcup_{\gamma \in \Gamma} Y_\gamma = \mathfrak{C}A$$

has interior points. Thus $\Gamma^\circ = \emptyset$, $\mathfrak{C}\bar{A} = (\mathfrak{C}A)^\circ \neq \emptyset$. Then $(\bar{A})^{**} = A^{**} = \Gamma^{\circ*} = \mathbf{C}^{1+n} \setminus \{0\}$ (see Lemma 2.1 and (2.5)), but $\bar{A} = \mathfrak{C}((\mathfrak{C}A)^\circ) \neq \mathbf{C}^{1+n} \setminus \{0\}$. This shows that \bar{A} cannot be lineally convex.

3. Duality of functions

A function $f: \mathbf{C}^{1+n} \setminus \{0\} \rightarrow [-\infty, +\infty]$ with values in the extended real line will be called **homogeneous** if

$$(3.1) \quad f(tz) = -\log |t| + f(z), \quad z \in \mathbf{C}^{1+n} \setminus \{0\}, \quad t \in \mathbf{C} \setminus \{0\}.$$

For such functions we define the **dual function** or **logarithmic transform** $\mathcal{L}f$:

$$(3.2) \quad \mathcal{L}f(\zeta) = \sup_{z \in \text{dom } f} (-\log |\zeta \cdot z| - f(z)), \quad \zeta \in \mathbf{C}^{1+n} \setminus \{0\}.$$

Here $\text{dom } f$ (the effective domain of f) denotes the set of all points z such that $f(z) < +\infty$, and $\zeta \cdot z = \zeta_0 z_0 + \cdots + \zeta_n z_n$ is the inner product. We define $\log 0 = -\infty$. The difference $-\log |\zeta \cdot z| - f(z)$ is well-defined if $f(z) < +\infty$; another way to formulate the definition is to use lower addition $\dot{+}$:

$$(3.3) \quad \mathcal{L}f(\zeta) = \sup_z ((-\log |\zeta \cdot z|) \dot{+} (-f(z))), \quad \zeta \in \mathbf{C}^{1+n} \setminus \{0\}.$$

Lower addition is an extension of usual addition such that $(+\infty) \dot{+} (-\infty) = -\infty$; the effect is that we disregard points outside $\text{dom } f$. Similarly we shall shortly use upper addition $\dot{+}$ (it satisfies $(+\infty) \dot{+} (-\infty) = +\infty$).

Proposition 3.1. *For any homogeneous function $f: \mathbf{C}^{1+n} \setminus \{0\} \rightarrow [-\infty, +\infty]$ its logarithmic transform $\mathcal{L}f$ is a homogeneous function with*

$$(3.4) \quad \text{dom } \mathcal{L}f \subset (\text{dom } f)^*.$$

Proof. The homogeneity of $\mathcal{L}f$ is obvious from its definition (3.2). To prove (3.4) we note that $\zeta \notin (\text{dom } f)^*$ means by definition that the hyperplane Y_ζ and the effective domain $\text{dom } f$ have a common point z (cf. (2.4)), so that $\mathcal{L}f(\zeta) \geq -\log|\zeta \cdot z| - f(z) = +\infty$, thus $\zeta \notin \text{dom } \mathcal{L}f$. The inclusion (3.4) may be strict as will be shown below: see Example 3.7 and Remark 4.3.

The analogue of Fenchel's inequality holds:

$$(3.5) \quad -\log|\zeta \cdot z| \leq f(z) \dot{+} \mathcal{L}f(\zeta), \quad \zeta, z \in \mathbf{C}^{1+n} \setminus \{0\}.$$

Moreover the usual rules for a Galois correspondence hold: $f \leq g$ implies $\mathcal{L}f \geq \mathcal{L}g$, and we always have $\mathcal{L}\mathcal{L}f \leq f$. As a consequence of these two properties, $\mathcal{L}\mathcal{L}\mathcal{L}f = \mathcal{L}f$. A function f will be called **\mathcal{L} -closed** if $\mathcal{L}\mathcal{L}f = f$ (equivalently, if it belongs to the range of \mathcal{L}). Some simple examples follow.

Examples 3.2. If f assumes the value $-\infty$, then $\mathcal{L}f$ is $+\infty$ identically. The same is true if f never takes the value $+\infty$ ($n \geq 1$). If f is $+\infty$ identically, then $\mathcal{L}f$ is $-\infty$ identically. If $f(z) = -\log|t|$ when $z = ta$ for a fixed $a \in \mathbf{C}^{1+n} \setminus \{0\}$ and $+\infty$ otherwise, then $\mathcal{L}f(\zeta) = -\log|\zeta \cdot a|$. If $f(z) = -\log|\alpha \cdot z|$ for some α , then $\mathcal{L}f(\zeta) = -\log|t|$ when $\zeta = t\alpha$ and $+\infty$ otherwise. All these functions are \mathcal{L} -closed.

As a consequence of (3.3) we note that $\sup_i \mathcal{L}f_i = \mathcal{L}(\inf_i f_i)$ for any indexed family (f_i) of functions. Indeed this follows from the rule $\sup_i (C \dot{+} a_i) = C \dot{+} \sup_i a_i$, which is valid also for an infinite constant C ; cf. Kiselman [1984: Lemma 3.1]. This implies that any supremum of \mathcal{L} -closed functions is \mathcal{L} -closed; in fact, we have

$$(3.6) \quad \sup_i f_i = \sup_i \mathcal{L}\mathcal{L}f_i = \mathcal{L}(\inf_i \mathcal{L}f_i)$$

if the f_i are \mathcal{L} -closed.

Homogeneous functions appear rather naturally in complex analysis. Let μ be an analytic functional in an open subset ω of \mathbf{C}^n , $\mu \in \mathcal{O}'(\omega)$. Its Fantappiè transform is

$$\mathcal{F}\mu(\zeta) = \mu(z \mapsto (\zeta_0 + \zeta_1 z_1 + \cdots + \zeta_n z_n)^{-1}),$$

which is a holomorphic function of $\zeta \in \Omega^*$, where Ω is the set of all $z \in \mathbf{C}^{1+n} \setminus \{0\}$ such that $z_0 \neq 0$ and $(z_1/z_0, \dots, z_n/z_0) \in \omega$. This implies that $\log|\mathcal{F}\mu|$ is plurisubharmonic in Ω^* , and it is moreover homogeneous in the sense of (3.1). (We define it as $+\infty$ outside Ω^* .)

Given f defined in $\mathbf{C}^{1+n} \setminus \{0\}$, we can define a function F in \mathbf{C}^n by putting $F(z') = f(1, z_1, \dots, z_n)$, $z' \in \mathbf{C}^n$. Conversely, if F is defined in \mathbf{C}^n , we can define a homogeneous function f in $\mathbf{C}^{1+n} \setminus \{0\}$ by

$$f(z) = \begin{cases} F(z_1/z_0, \dots, z_n/z_0) - \log|z_0|, & z \in \mathbf{C}^{1+n} \setminus \{0\}, \quad z_0 \neq 0; \\ +\infty & z \in \mathbf{C}^{1+n} \setminus \{0\}, \quad z_0 = 0. \end{cases}$$

The transform (3.2) then takes the form

$$(3.7) \quad \mathcal{L}F(\zeta') = \sup_{F(z') < +\infty} (-\log|1 + \zeta' \cdot z'| - F(z')), \quad \zeta' \in \mathbf{C}^n.$$

In particular, if F is radial (i. e., a function of $|z'| = r$), then the transform becomes

$$(3.8) \quad \mathcal{L}F(\rho) = \sup_{F(r) < +\infty} (-\log(1 - \rho r) - F(r)), \quad \rho = |\zeta'| \geq 0.$$

Example 3.3. Take $F(r) = 0$ when $r \leq R$ and $F(r) = +\infty$ otherwise in (3.8). Then $\mathcal{L}F(\rho) = -\log(1 - R\rho)$, $\rho < 1/R$, and $\mathcal{L}F(\rho) = +\infty$, $\rho \geq 1/R$. The second transform is $\mathcal{L}\mathcal{L}F = F$, so that F is \mathcal{L} -closed.

Example 3.4. The radial function $F(r) = -\frac{1}{2}\log(1 - r^2)$ is selfdual, i. e., $\mathcal{L}F(\rho) = -\frac{1}{2}\log(1 - \rho^2)$. Going back to $\mathbf{C}^{1+n} \setminus \{0\}$, we see that the function

$$f(z) = \begin{cases} -\frac{1}{2}\log(|z_0|^2 - |z'|^2), & z \in \mathbf{C}^{1+n} \setminus \{0\}, \quad |z_0| > |z'|; \\ +\infty, & z \in \mathbf{C}^{1+n} \setminus \{0\}, \quad |z_0| \leq |z'| \end{cases}$$

has this property. This function therefore plays the same role as the convex function $f(x) = \frac{1}{2}|x|^2$ for usual convexity.

Now let A be a homogeneous set in $\mathbf{C}^{1+n} \setminus \{0\}$. We define a function d , the distance to the complement of A relative to $\mathbf{C}^{1+n} \setminus \{0\}$, as

$$(3.9) \quad d(z) = d_A(z) = \inf (|z - w|; w \in (\mathbf{C}^{1+n} \setminus \{0\}) \setminus A), \quad z \in \mathbf{C}^{1+n} \setminus \{0\}.$$

The function $-\log d$ is homogeneous, and it is less than $+\infty$ precisely in the interior of A . Analogously we define a function δ by

$$(3.10) \quad \delta(\zeta) = d_{A^*}(\zeta) = \inf (|\zeta - \theta|; \theta \in (\mathbf{C}^{1+n} \setminus \{0\}) \setminus A^*), \quad \zeta \in \mathbf{C}^{1+n} \setminus \{0\},$$

where A^* is the dual complement of A defined by (2.1). If A is empty, then $d_A = 0$ identically, whereas $d_{A^*} = +\infty$ identically.

Theorem 3.5. *Let $f: \mathbf{C}^{1+n} \setminus \{0\} \rightarrow [-\infty, +\infty]$ be any homogeneous function. Then*

$$(3.11) \quad C - \log |\zeta| \leq \mathcal{L}f(\zeta) \leq C - \log \delta(\zeta), \quad \zeta \in \mathbf{C}^{1+n} \setminus \{0\},$$

where δ is defined by (3.10) taking $A = \text{dom } f$, and $C = -\inf_{|z|=1} f(z) \leq +\infty$. We have $C = -\infty$ if and only if f is $+\infty$ identically; in this case $\mathcal{L}f$ is $-\infty$ identically. We have $C = +\infty$ if and only if f is unbounded from below on the unit sphere S ; then $\mathcal{L}f$ is $+\infty$ identically. If f is bounded from below on S , then $C < +\infty$ and (3.11) shows that $\mathcal{L}f$ has at most logarithmic growth at the boundary of $(\text{dom } f)^*$; moreover

$$(3.12) \quad (\overline{\text{dom } f})^* = (\text{dom } f)^{\circ*} = (\text{dom } \mathcal{L}f)^\circ \subset \text{dom } \mathcal{L}f \subset (\text{dom } f)^*,$$

and

$$(3.13) \quad \overline{\text{dom } \mathcal{L}f} \subset \overline{(\text{dom } f)^*} \subset (\text{dom } f)^{\circ*}.$$

In particular $\text{dom } \mathcal{L}f = (\text{dom } f)^*$ if $\text{dom } f$ is closed.

Lemma 3.6. *For any subset A of $\mathbf{C}^{1+n} \setminus \{0\}$ we have*

$$(3.14) \quad |\zeta \cdot z| \geq \delta(\zeta)|z|, \quad \zeta \in \mathbf{C}^{1+n} \setminus \{0\}, \quad z \in A,$$

and

$$(3.15) \quad |\zeta \cdot z| \geq |\zeta|d(z), \quad \zeta \in A^*, \quad z \in \mathbf{C}^{1+n} \setminus \{0\}.$$

Proof. Given $\zeta \in \mathbf{C}^{1+n} \setminus \{0\}$ and $z \in A$ we define $\alpha = \zeta + t\bar{z}$ where $t = -|z|^{-2}(\zeta \cdot z)$. Then $\alpha \cdot z = 0$, which, if $\alpha \neq 0$, means that $\alpha \in \mathcal{C}A^*$ since $z \in A$. Therefore $\delta(\zeta) \leq |\zeta - \alpha| = |\zeta \cdot z|/|z|$, which proves the first inequality except when $\zeta = |z|^{-2}(\zeta \cdot z)\bar{z}$. Since δ is continuous, this restriction can be removed. If we now interchange the role of ζ and z , we get $|\zeta \cdot z| \geq |\zeta|d_{A^{**}}(z)$. But $A^{**} \supset A$, so $d_{A^{**}}(z) \geq d_A(z) = d(z)$. This proves the lemma. (Interchanging z and ζ once more, we see that (3.14) holds even for all $z \in A^{**}$.)

Proof of Theorem 3.5. By Schwartz' inequality and (3.14) applied to $A = \text{dom } f$ we get

$$-\log |\zeta| \leq -\log |\zeta \cdot z| \leq -\log \delta(\zeta), \quad \zeta \in \mathbf{C}^{1+n} \setminus \{0\}, \quad z \in A \cap S.$$

Thus

$$\mathcal{L}f(\zeta) = \sup_{z \in A \cap S} (-\log |\zeta \cdot z| - f(z)) \begin{cases} \leq (-\log \delta(\zeta)) \dot{+} \sup_{A \cap S}(-f); \\ \geq -\log |\zeta| + \sup_{A \cap S}(-f). \end{cases}$$

The cases where $\dot{+}$ and $\dot{-}$ give different results never occur, so we can replace $\dot{+}$ by usual addition. This proves (3.11); note that $\sup_{A \cap S}(-f) = \sup_S(-f) = -\inf_S f$.

We already know that $\text{dom } \mathcal{L}f \subset (\text{dom } f)^*$; see (3.4). If $\zeta \in (\text{dom } f)^{\circ}$ and $C < +\infty$, then $\delta(\zeta) > 0$ and $\mathcal{L}f(\zeta) \leq C - \log \delta(\zeta) < +\infty$, so that $\zeta \in \text{dom } \mathcal{L}f$. This proves that $(\text{dom } f)^{\circ} \subset \text{dom } \mathcal{L}f \subset (\text{dom } f)^*$. Taking the interior of these sets we get (3.12); taking the closure we get (3.13) (cf. Proposition 2.2).

Example 3.7. The effective domain of $\mathcal{L}f$ may fail to be lineally convex, although it is squeezed in between the two lineally convex sets $(\text{dom } f)^{\circ} = \overline{(\text{dom } f)^*}$ and $(\text{dom } f)^*$; see (3.12). Indeed, let $w^k = (k^{-2}, k^{-1}, 1) \in \mathbf{C}^{1+2}$ and define $f(w^k) = \log k$, $k = 1, 2, 3, \dots$, and $f(z) = +\infty$ when $z \notin \mathbf{C}w^k$. Then

$$\mathcal{L}f(\zeta) = \sup_k (-\log |\zeta_0/k + \zeta_1 + k\zeta_2|), \quad \zeta \in \mathbf{C}^{1+2} \setminus \{0\}.$$

Put $\alpha = (1, 0, 0)$ and $\beta = (1, 1, 0)$. Then

$$\mathcal{L}f(\alpha) = \sup_k (-\log |k^{-1}|) = +\infty,$$

so that $\alpha \notin \text{dom } \mathcal{L}f$, whereas

$$\mathcal{L}f(\beta) = \sup_k (-\log |k^{-1} + 1|) = 0,$$

showing that $\beta \in \text{dom } \mathcal{L}f$. The points w^k define hyperplanes

$$Y_{w^k} = \{\zeta; \zeta_0 k^{-2} + \zeta_1 k^{-1} + \zeta_2 = 0\},$$

which converge to a hyperplane $Y_w = \{\zeta; \zeta_2 = 0\}$ with $w = \lim w^k = (0, 0, 1)$. By (3.12),

$$(\overline{\text{dom } f})^* = \mathfrak{C}(Y_w \cup \bigcup Y_{w^k}) \subset \text{dom } \mathcal{L}f \subset (\text{dom } f)^* = \mathfrak{C}(\bigcup Y_{w^k}).$$

Both α and β belong to Y_w , but as $k \rightarrow +\infty$, the hyperplanes Y_{w^k} approach α more rapidly than β (note that $\alpha \cdot w^k = 1/k^2$, while $\beta \cdot w^k = 1/k + 1/k^2$). This explains why $\alpha \notin \text{dom } \mathcal{L}f$ while $\beta \in \text{dom } \mathcal{L}f$. A hyperplane which avoids $\text{dom } \mathcal{L}f$ must be either one of the hyperplanes Y_{w^k} , or (possibly) their limit Y_w . However, the hyperplanes Y_{w^k} do not contain α , and the hyperplane Y_w intersects $\text{dom } \mathcal{L}f$ in β . Therefore there is no hyperplane which passes through α and avoids $\text{dom } \mathcal{L}f$, which shows that $\text{dom } \mathcal{L}f$ is not lineally convex. In particular we must have $(\text{dom } f)^{* \circ} \neq \text{dom } \mathcal{L}f \neq (\text{dom } f)^*$; cf. (3.12).

Example 3.8. A fundamental property of convexity is that the union of an increasing sequence of convex sets is convex. (More generally, this is true for the union of a directed family.) This is not so with lineal convexity. Let A_j be the set of all ζ such that $\mathcal{L}f(\zeta) \leq j$. It is easy to see that this is a lineally convex set; indeed,

$$A_j = \bigcap_{z \in \text{dom } f} \{\zeta \in \mathbf{C}^{1+n} \setminus \{0\}; -\log |\zeta \cdot z| - f(z) \leq j\}.$$

The union of the A_j is $\text{dom } \mathcal{L}f$. If we let f be the function constructed in Example 3.7 we get an example where the A_j are lineally convex but their union is not.

Theorem 3.9. *Let f be a function on $\mathbf{C}^{1+n} \setminus \{0\}$ which is bounded from below on the unit sphere and let $\mathcal{L}f$ be its transform defined by (3.2). Then $\mathcal{L}f$ is plurisubharmonic in the interior of $\text{dom } \mathcal{L}f$, which is a lineally convex set. Moreover $\mathcal{L}f$ is locally Lipschitz continuous in $(\text{dom } \mathcal{L}f)^\circ$; more precisely*

$$\limsup_{t \rightarrow 0+} \frac{\mathcal{L}f(\zeta + t\theta) - \mathcal{L}f(\zeta)}{t} \leq \frac{|\theta|}{\delta(\zeta)}, \quad \zeta \in (\text{dom } \mathcal{L}f)^\circ, \theta \in \mathbf{C}^{1+n},$$

where δ the distance to the complement of $\text{dom } \mathcal{L}f$.

Proof. Consider the function $g(\zeta) = -\log |\zeta \cdot z|$. Its gradient has length $|z|/|\zeta \cdot z|$. At the point $\alpha = \zeta + t\bar{z}$, where $t = -|z|^{-2}(\zeta \cdot z)$, g takes the value $+\infty$, so

$$d_{\text{dom } g}(\zeta) \leq |\alpha - \zeta| \leq \frac{|\zeta \cdot z|}{|z|} = \frac{1}{|\text{grad } g(\zeta)|}.$$

Now $\mathcal{L}f$ is a supremum of functions of the form g plus a constant for various choices of z . All competing functions must satisfy $\text{dom } g \supset \text{dom } \mathcal{L}f$, so that $d_{\text{dom } g} \geq \delta$. Therefore they have a gradient whose length is at most $1/\delta(\zeta)$, which implies that $\mathcal{L}f$ is Lipschitz continuous as indicated. That $\mathcal{L}f$ is plurisubharmonic now follows

from standard properties of such functions: f is a continuous supremum of plurisubharmonic functions.

Finally (3.12) shows that $(\text{dom } \mathcal{L}f)^\circ$ is lineally convex: it is equal to the dual complement of the closure of $\text{dom } f$.

4. Examples of functions in duality

In this section we shall make a detailed study of the functions

$$(4.1) \quad f_c(z) = \begin{cases} -(1-c) \log |z| - c \log d(z), & z \in A; \\ +\infty, & z \in (\mathbf{C}^{1+n} \setminus \{0\}) \setminus A, \end{cases}$$

and

$$(4.2) \quad \varphi_c(\zeta) = \begin{cases} -(1-c) \log |\zeta| - c \log \delta(\zeta), & \zeta \in A^*; \\ +\infty, & \zeta \in (\mathbf{C}^{1+n} \setminus \{0\}) \setminus A^*, \end{cases}$$

where $0 \leq c \leq 1$, A is any homogeneous subset of $\mathbf{C}^{1+n} \setminus \{0\}$, A^* its dual complement, and d and δ are defined by (3.9) and (3.10), respectively.

We shall call $f_0 = I_A$ the **indicator function** of the set A . Its restriction to the unit sphere is the indicator function in the usual sense. And $\mathcal{L}f_0 = \mathcal{L}I_A$ is analogous to the supporting function of A , thus preserving the situation from convex analysis where the supporting function is the Fenchel transform of the indicator function. We shall determine this function explicitly: it is $\varphi_1 = -\log \delta = -\log d_{A^*}$.

More generally, it turns out that the function φ_{1-c} is essentially dual to f_c . It might seem strange to consider functions like f_0 which are not plurisubharmonic. We must have $\mathcal{L}\mathcal{L}f_0 < f_0$ in the interior of A . From this point of view it is more natural to consider

$$(4.3) \quad g_c(z) = \begin{cases} -(1-c) \log |z_0| - c \log d(z), & z \in A; \\ +\infty, & z \in (\mathbf{C}^{1+n} \setminus \{0\}) \setminus A, \end{cases}$$

and

$$(4.4) \quad \psi_c(\zeta) = \begin{cases} -(1-c) \log |\zeta_0| - c \log \delta(\zeta), & \zeta \in A^*; \\ +\infty, & \zeta \in (\mathbf{C}^{1+n} \setminus \{0\}) \setminus A^*. \end{cases}$$

If A is contained in a coordinate patch $z_0 \neq 0$ and if moreover $|z|/|z_0|$ is bounded when $z \in A$, then f_c and g_c are finite in the same set and differ there by a bounded function. If moreover $(1, 0, \dots, 0)$ is an interior point of A , then $\zeta_0 \neq 0$ when $\zeta \in A^*$ and $|\zeta|/|\zeta_0|$ is bounded there, so φ_c and ψ_c are finite in the same set and their difference is bounded there. Therefore our results on f_c and φ_c can easily be translated into inequalities for g_c and ψ_c .

The first result is a simple inequality.

Proposition 4.1. *With f_c and φ_c defined by (4.1) and (4.2) we have $\mathcal{L}f_c \leq \varphi_{1-c}$ for $0 \leq c \leq 1$.*

Proof. If $\zeta \notin A^*$, then $\varphi_{1-c}(\zeta) = +\infty$, so the inequality certainly holds. If on the other hand $\zeta \in A^*$, we can estimate $\mathcal{L}f_c(\zeta)$ using Lemma 3.6:

$$\begin{aligned} \mathcal{L}f_c(\zeta) &= \sup_{z \in A(c)} \left(-\log |\zeta \cdot z| + (1-c) \log |z| + c \log d(z) \right) \\ &= \sup_{z \in A(c)} \left(-(1-c) \log |\zeta \cdot z| + (1-c) \log |z| - c \log |\zeta \cdot z| + c \log d(z) \right) \\ &\leq \sup_{z \in A(c)} \left(-(1-c) \log(\delta(\zeta)|z|) + (1-c) \log |z| - c \log(|\zeta|d(z)) + c \log d(z) \right) \\ &\leq -(1-c) \log \delta(\zeta) - c \log |\zeta| = \varphi_{1-c}(\zeta). \end{aligned}$$

The supremum is over the set $A(c)$ of all z such that $f_c(z) < +\infty$, that is

$$(4.5) \quad A(c) = \begin{cases} A & \text{for } c = 0; \\ A^\circ & \text{for } 0 < c \leq 1. \end{cases}$$

($A(c)$ can of course be empty; in that case $\mathcal{L}f_c$ is $-\infty$ identically.)

We now study inequalities in the other direction. The cases $c = 0$ and $c = 1$ are easy and will be considered first.

Proposition 4.2. *For any homogeneous subset A of $\mathbf{C}^{1+n} \setminus \{0\}$ we have $\mathcal{L}I_A = \mathcal{L}f_0 = \varphi_1 = -\log d_{A^*}$. (The analogue of the supporting function of A .)*

Remark 4.3. Note that here $\text{dom } \mathcal{L}f_0 = A^{*\circ} = (\overline{A})^* = (\text{dom } f_0)^{*\circ}$ is open and lineally convex, whereas $(\text{dom } f_0)^* = A^*$; again we see that the inclusion $\text{dom } \mathcal{L}f \subset (\text{dom } f)^*$ may be strict (cf. (3.4)).

Lemma 4.4. *Assume that A is homogeneous and not empty. For every $\zeta \in A^*$ there is a point $z \in \partial A$, $z \neq 0$, such that $|\zeta \cdot z| \leq \delta(\zeta)|z|$.*

Proof. For every $\zeta \in A^*$ there is a point $\alpha \in \partial A^*$, $\alpha \neq 0$, such that $|\alpha - \zeta| = \delta(\zeta)$. Thus $\alpha \notin A^{*\circ} = (\overline{A})^*$ (cf. (2.5)). Now $\alpha \notin (\overline{A})^*$ means that $Y_\alpha \cap \overline{A} \neq \emptyset$ (see (2.2) and (2.4)). On the other hand $\alpha \in \partial A^* \subset \overline{A^*} \subset A^{\circ*}$ (cf. (2.6)), so that $Y_\alpha \cap A^\circ = \emptyset$. Therefore Y_α meets the boundary of A , and we can choose $z \in S \cap \partial A$ such that $\alpha \cdot z = 0$. Then

$$|\zeta \cdot z| = |(\zeta - \alpha) \cdot z| \leq |\zeta - \alpha||z| = \delta(\zeta)|z|.$$

Proof of Proposition 4.2. If $A = \emptyset$, we have $\mathcal{L}I_A = -\log d_{A^*} = -\infty$. Otherwise the lemma provides us, given any $\zeta \in A^*$, with a point $z \in \partial A \cap S$ such that

$$\mathcal{L}f_0(\zeta) = \sup_{w \in A} \left(-\log |\zeta \cdot w| + \log |w| \right) \geq -\log |\zeta \cdot z| + \log |z| \geq -\log \delta(\zeta) = \varphi_1(\zeta).$$

For $\zeta \in (\mathbf{C}^{1+n} \setminus \{0\}) \setminus A^*$ both $\mathcal{L}f_0$ and φ_1 take the value $+\infty$. Thus $\mathcal{L}f_0 \geq \varphi_1$ everywhere. The inequality $\mathcal{L}f_0 \leq \varphi_1$ was proved already in Proposition 4.1.

Proposition 4.5. *Assume A is open and not empty. Then there is a constant M , which depends on the geometry of A , such that*

$$\varphi_0 = I_{A^*} \geq \mathcal{L}f_1 = \mathcal{L}(-\log d_A) \geq I_{A^*} - M.$$

In fact M can be taken as $\inf_S f_1 = \inf_S (-\log d_A)$.

Here $\text{dom } \mathcal{L}f_1 = A^* = (\text{dom } f_1)^*$ is closed and lineally convex; cf. (3.12).

Lemma 4.6. *Assume A has a nonempty interior and take any point $z \in A^\circ$. Then there is a constant C such that $|\zeta \cdot z| \leq C|\zeta|d(z)$ for all ζ .*

Proof. Given $z \in A^\circ$ define $C = |z|/d(z)$. We have

$$|\zeta \cdot z| \leq |\zeta||z| = C|\zeta|d(z).$$

The best choice is of course a point $z \in S$ such that $d(z) = \sup_S d$, so that $C = 1/\sup_S d$.

Proof of Proposition 4.5. Using the lemma we get for any $\zeta \in A^*$,

$$\begin{aligned} \mathcal{L}f_1(\zeta) &= \sup_{w \in A} (-\log |\zeta \cdot w| + \log d(w)) \geq -\log |\zeta \cdot z| + \log d(z) \geq -\log C - \log |\zeta| \\ &= \varphi_0(\zeta) - M. \end{aligned}$$

When $\zeta \notin A^*$, there is a point $z \in A$ such that $\zeta \cdot z = 0$, and since A is open, $f_1(z) < +\infty$, so that $\mathcal{L}f_1(\zeta) = +\infty$. Thus we have $\mathcal{L}f_1 \geq \varphi_0 - M$ everywhere. The inequality $I_{A^*} \geq \mathcal{L}f_1$ was already proved in Proposition 4.1.

Theorem 4.7. *Let A be an open homogeneous set. Then A is lineally convex if and only if $-\log d_A$ is \mathcal{L} -closed.*

Proof. If $A = B^*$, then $\mathcal{L}I_B = -\log d_{B^*} = -\log d_A$ by Proposition 4.2, so $-\log d_A$ is \mathcal{L} -closed. Conversely, Proposition 4.5 shows that $\mathcal{L}(-\log d_A) \geq I_{A^*} - M$, which implies $\mathcal{L}\mathcal{L}(-\log d_A) \leq -\log d_{A^{**}} + M$. Therefore, if z belongs to the open set A^{**} (cf. Proposition 2.2), then $\mathcal{L}\mathcal{L}(-\log d_A(z))$ is finite. If $-\log d_A$ is \mathcal{L} -closed, this is equivalent to $-\log d_A(z)$ being finite, which implies $z \in A$. Thus $A^{**} \subset A$; this inclusion means that A is lineally convex.

Theorem 4.8. *A closed lineally convex set A can be recovered from $\mathcal{L}I_A$. Indeed, if A is a set with these properties different from $\mathbf{C}^{1+n} \setminus \{0\}$, then $I_A \geq \mathcal{L}\mathcal{L}I_A \geq I_A - M$, so that A is the set where $\mathcal{L}\mathcal{L}I_A$ is finite. If A is equal to $\mathbf{C}^{1+n} \setminus \{0\}$, then $\mathcal{L}\mathcal{L}I_A$ is $-\infty$ identically. If A is a closed and lineally convex set such that $|z'| \leq R|z_0|$ for all $z \in A$, then $\mathcal{L}\mathcal{L}I_A \geq I_A - \log \sqrt{1 + R^2}$.*

This theorem is thus analogous to the result in convexity theory which states that a closed convex set can be recovered from its supporting function. By way of contrast, an open set A can be recovered from $\mathcal{L}I_A$ only under special conditions, since $\mathcal{L}I_A = \mathcal{L}I_{\overline{A}}$. If A is open and equal to the interior of its closure, and if its closure is lineally convex, then A is the interior of the set where $\mathcal{L}\mathcal{L}I_A$ is finite. But an open lineally convex set can of course always be recovered from $\mathcal{L}(-\log d_A)$; see Proposition 4.5.

Proof. If A is closed, lineally convex and not equal to all of $\mathbf{C}^{1+n} \setminus \{0\}$, then A^* is open and non-empty, so we can apply Proposition 4.5 to A^* and obtain

$$I_{A^{**}} \geq \mathcal{L}(-\log d_{A^*}) \geq I_{A^{**}} - M.$$

From Proposition 4.2 we have $\mathcal{L}I_A = -\log d_{A^*}$. Combining this information we deduce

$$I_{A^{**}} \geq \mathcal{L}(-\log d_{A^*}) = \mathcal{L}\mathcal{L}I_A \geq I_{A^{**}} - M.$$

Since A is lineally convex, $A = A^{**}$, and we see that $\mathcal{L}\mathcal{L}I_A$ and I_A are finite in the same set (and differ there at most by a bounded function).

The last statement follows if we keep track of the constant in Lemma 4.6. Alternatively we can compare I_A with the function $F(z) = -\log|z_0|$, $|z'| \leq R|z_0|$, $F(z) = +\infty$ otherwise. This gives $F(z) \geq I_A(z) \geq F + \log|z_0| - \log|z|$ for $z \in A$, so that $I_A \geq F - \log\sqrt{1+R^2}$ everywhere, implying that $\mathcal{L}\mathcal{L}I_A \geq \mathcal{L}\mathcal{L}F - \log\sqrt{1+R^2}$. Since $\mathcal{L}\mathcal{L}F = F$ (cf. Example 3.3), we can conclude that $\mathcal{L}\mathcal{L}I_A \geq F - \log\sqrt{1+R^2} \geq I_A - \log\sqrt{1+R^2}$ in A .

Finally we shall deduce an estimate from below for $\mathcal{L}f_c$ when $0 < c < 1$.

Proposition 4.9. *Assume that A is open, non-empty, and satisfies an interior cone condition in the sense that there exist positive numbers γ and R such that for every $a \in \partial A$ and every $r \leq R$*

$$\sup_z (d(z); |z - a| \leq r|a|) \geq \gamma r|a|.$$

Then there is a constant M such that $\varphi_{1-c} \geq \mathcal{L}f_c \geq \varphi_{1-c} - M$ for every $c \in [0, 1]$.

Here $\text{dom } \mathcal{L}f_c = (\text{dom } f_c)^{\circ}$ for $0 \leq c < 1$, whereas it is closed for $c = 1$ as already noted.

In particular a set with Lipschitz boundary satisfies the interior cone condition. To prove this proposition we shall need a lemma which combines Lemmas 4.4 and 4.6. The requirements concerning the point z are somewhat contradictory, since $z \in \partial A$ in the first and $z \in A^\circ$ in the second. Nevertheless, we can find a compromise:

Lemma 4.10. *With A as in the proposition there exists a constant C such that for every $\zeta \in A^*$ there is a point $z = z_\zeta \in A$ such that*

$$|\zeta \cdot z| \leq C|\zeta|d(z) \text{ and } |\zeta \cdot z| \leq C\delta(\zeta)|z|.$$

Proof. First pick any point $w \in A$. It will serve as the point z_ζ for all ζ such that $\delta(\zeta) \geq R|\zeta|$:

$$|\zeta \cdot w| \leq |\zeta||w| = \frac{|w|}{d(w)}|\zeta|d(w) \leq C|\zeta|d(w)$$

and

$$|\zeta \cdot w| \leq |\zeta||w| = \frac{|\zeta|}{\delta(\zeta)}\delta(\zeta)|w| \leq \frac{1}{R}\delta(\zeta)|w| \leq C\delta(\zeta)|w|$$

for a constant $C \geq \max(R^{-1}, |w|/d(w))$.

The case $\delta(\zeta) \leq R|\zeta|$ remains to be considered. To a given $\zeta \in A^*$ we choose $\alpha \in \partial A^*$, $\alpha \neq 0$, such that $|\alpha - \zeta| = \delta(\zeta) = r|\zeta|$, $r \leq R$. Since Y_α meets ∂A (cf. the proof of Lemma 4.4), we can choose $a \in \partial A$, $a \neq 0$, such that $\alpha \cdot a = 0$. The interior cone condition now implies the existence of a point $z = z_\zeta \in A$ such that $d(z) \geq \gamma r|a|$ and $|z - a| \leq r|a| \leq R|a|$. Then $|\zeta \cdot a| = |(\zeta - \alpha) \cdot a| \leq |\zeta - \alpha||a| = \delta(\zeta)|a|$ and $|z - a| \leq r|a| = |a|\delta(\zeta)/|\zeta|$, so that

$$|\zeta \cdot z| = |\zeta \cdot a + \zeta(z - a)| \leq |\zeta \cdot a| + |\zeta||z - a| \leq 2\delta(\zeta)|a| \leq \frac{2}{1-R}\delta(\zeta)|z|.$$

Here the last inequality follows from $|z - a| \leq R|a|$; it is no restriction to assume that $R < 1$. On the other hand $d(z) \geq \gamma r|a| = \gamma|a|\delta(\zeta)/|\zeta|$ so that

$$|\zeta \cdot z| \leq 2\delta(\zeta)|a| \leq \frac{2}{\gamma}d(z)|\zeta|.$$

This proves the lemma with the constant

$$C = \max \left[\frac{1}{R}, \frac{|w|}{d(w)}, \frac{2}{1-R}, \frac{2}{\gamma} \right].$$

Proof of Proposition 4.9. Since A is open, $\text{dom } f_c = A$ for all c ; cf. (4.5). Using the lemma we get for any $\zeta \in A^*$, taking $z \in A$ from Lemma 4.10,

$$\begin{aligned} \mathcal{L}f_c(\zeta) &= \sup_{w \in A} \left(-\log |\zeta \cdot w| + (1-c) \log |w| + c \log d(w) \right) \\ &\geq -\log |\zeta \cdot z| + (1-c) \log |z| + c \log d(z) \\ &= (1-c)(\log |z| - \log |\zeta \cdot z|) + c(\log d(z) - \log |\zeta \cdot z|) \\ &\geq (1-c)(\log |z| - \log(C\delta(\zeta)|z|)) + c(\log d(z) - \log(C|\zeta|d(z))) \\ &= -(1-c) \log \delta(\zeta) - c \log |\zeta| - \log C = \varphi_{1-c}(\zeta) - M. \end{aligned}$$

If $\zeta \notin A^*$, then $\mathcal{L}f_c(\zeta) = \varphi_{1-c}(\zeta) = +\infty$.

5. The supporting function of a convex set

Let A_0 be a subset of \mathbf{C}^n . Its supporting function is

$$(5.1) \quad H_{A_0}(\zeta') = \sup_{z' \in A_0} \text{Re}(\zeta' \cdot z'), \quad \zeta' = (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n.$$

If A_0 is closed and convex, then H_{A_0} determines A_0 ; in fact, A_0 is the set of all $z' = (z_1, \dots, z_n)$ such that $\text{Re}(\zeta' \cdot z') \leq H_{A_0}(\zeta')$ for all ζ' . Now let A be the homogeneous set of all $z \in \mathbf{C}^{1+n} \setminus \{0\}$ such that $z_0 \neq 0$ and $z'/z_0 \in A_0$. With the set A we associate the function

$$(5.2) \quad \mathcal{L}I_A(\zeta) = -\log \delta(\zeta) = \sup_{z \in A} (-\log |\zeta \cdot z| + \log |z|), \quad \zeta \in \mathbf{C}^{1+n} \setminus \{0\},$$

the projective analogue of the supporting function. What is the relation between these two supporting functions? To answer this question we first modify I_A a little and define

$$(5.3) \quad h_A(\zeta) = \sup_{z \in A} (-\log |\zeta \cdot z| + \log |z_0|), \quad \zeta \in \mathbf{C}^{1+n} \setminus \{0\}.$$

If A_0 is bounded, then h_A and $\mathcal{L}I_A$ are finite in the same set and differ there by a bounded function.

We shall express h_A in terms of H_{A_0} . We first formulate an auxiliary result, which we shall need for convex sets in the complex plane only, but which is valid in general in \mathbf{R}^n . We shall therefore use the real supporting function

$$(5.4) \quad H_A(\xi) = \sup_{x \in A} \xi \cdot x, \quad \xi \in \mathbf{R}^n.$$

Lemma 5.1. *Let A be a convex set in \mathbf{R}^n . Then*

$$(5.5) \quad - \inf_{x \in A} |x| \leq \inf_{|\xi|=1} H_A(\xi) \leq \inf_{x \notin A} |x|$$

(Euclidean norms) with equality on the left if $0 \notin A^\circ$, and on the right if $0 \in \overline{A}$.

Proof. For any set A we have, writing S for the unit sphere,

$$\inf_S H_A = \inf_{\xi \in S} \sup_{x \in A} \xi \cdot x \geq \sup_{x \in A} \inf_{\xi \in S} \xi \cdot x = \sup_{x \in A} (-|x|) = - \inf_{x \in A} |x|.$$

If A is convex and $x \notin A$, then there is a $\xi \in S$ such that $\xi \cdot x \geq H_A(\xi)$; thus $|x| \geq \xi \cdot x \geq H_A(\xi)$, so that $|x| \geq \inf_S H_A$. This shows that $\inf_{x \notin A} |x| \geq \inf_S H_A$ and proves (5.5) for all convex sets.

Now assume that $0 \notin A^\circ$. Then A must be contained in a half-space $\{x; \xi \cdot x \geq c\}$ for some $\xi \in S$ and $c \geq 0$, which shows that $H_A(-\xi) \leq -c \leq 0$. If A is empty, (5.5) has the form $-\infty \leq -\infty \leq 0$, so the result is true. If A is not empty, then we can choose $c = \inf_{x \in A} |x|$, so that $\inf_S H_A \leq -c = -\inf_{x \in A} |x|$; we have proved equality on the left in (5.5).

On the other hand, if $0 \in \overline{A}$ and $H_A(\xi) < c$ for some $\xi \in S$ and some c , then necessarily $c > 0$ and the vector $c\xi$ cannot belong to A , so that $\inf_{x \notin A} |x| \leq |c\xi| = c$. Thus $\inf_{x \notin A} |x| \leq \inf_S H_A$; we have proved equality on the right in (5.5).

Lemma 5.2. *For any convex set A in \mathbf{R}^n we have*

$$(5.6) \quad \inf_{x \in A} |x| = \sup_{|\xi|=1} H_A(\xi)^-;$$

$$(5.7) \quad \inf_{x \notin A} |x| = \inf_{|\xi|=1} H_A(\xi)^+,$$

where $t^+ = \max(t, 0)$, $t^- = \max(-t, 0)$.

Proof. If $0 \notin A$, then (5.6) is just the first part of (5.5) with equality, and (5.7) reduces to $0 = 0$. If $0 \in A$, then (5.6) reduces to $0 = 0$ while (5.7) is the second part of (5.5) with equality.

Proposition 5.3. *Let A_0 be any convex set in \mathbf{C}^n , and A the homogeneous set of all $z \in \mathbf{C}^{1+n} \setminus \{0\}$ such that $z_0 \neq 0$ and $z'/z_0 \in A_0$. Define H_{A_0} and h_A by (5.1) and (5.3) respectively. Then*

$$(5.8) \quad h_A(\zeta) = \inf_{|t|=1} \log (H_{A_0}(t\zeta') + \operatorname{Re}(t\zeta_0))^- , \quad \zeta \in \mathbf{C}^{1+n} \setminus \{0\}.$$

Proof. If A is empty, (5.8) certainly holds, because both sides are equal to $-\infty$. Fix $\zeta \in \mathbf{C}^{1+n} \setminus \{0\}$ and denote by L the linear mapping $z' \mapsto \zeta' \cdot z'$. If A is not empty, then

$$(5.9) \quad e^{-h_A(\zeta)} = \inf_{z' \in A_0} |\zeta_0 + \zeta' \cdot z'| = \inf_{s \in L(A_0)} |\zeta_0 + s| = |\zeta_0 + a(\zeta_0)|,$$

where $a(\zeta_0)$ denotes the point in the closure of $L(A_0)$ which is closest to $-\zeta_0$. Here the first equality holds because, in view of the homogeneity, it is enough to let z vary with $z_0 = 1$ in the definition of h_A .

We now note that

$$H_{A_0}(t\zeta') = \sup_{z' \in A_0} \operatorname{Re}(t\zeta' \cdot z') = \sup_{z' \in A_0} \operatorname{Re} tL(z') = \sup_{s \in L(A_0)} \operatorname{Re} ts = H_{L(A_0)}(t),$$

which shows that the supporting function of the set $M = L(A_0) + \zeta_0$ is

$$H_M(t) = H_{A_0}(t\zeta') + \operatorname{Re}(t\zeta_0), \quad t \in \mathbf{C}.$$

We can apply (5.6) to the convex set M . This yields

$$e^{-h_A(\zeta)} = |\zeta_0 + a(\zeta_0)| = \sup_{|t|=1} H_M(t)^- = \sup_{|t|=1} (H_{A_0}(t\zeta') + \operatorname{Re}(t\zeta_0))^-,$$

where the first equality is that of (5.9), and thus proves (5.8).

Conversely, we can express H_{A_0} in terms of h_A .

Proposition 5.4. *Let A_0 be a bounded but not necessarily convex set in \mathbf{C}^n , and A the set of all $z \in \mathbf{C}^{1+n} \setminus \{0\}$ such that $z_0 \neq 0$ and $z'/z_0 \in A_0$. Define H_{A_0} and h_A by (5.1) and (5.3) respectively. Then*

$$(5.10) \quad H_{A_0}(\zeta') = \lim_{\substack{\zeta_0 \in \mathbf{R} \\ \zeta_0 \rightarrow -\infty}} (-\zeta_0 - e^{-h_A(\zeta)}), \quad \zeta' \in \mathbf{C}^n.$$

Proof. We can still use (5.9) even though A_0 now is perhaps not convex, if we let $a(\zeta_0)$ denote one of the closest points to $-\zeta_0$ in the closure of $L(A_0)$. Let ζ_0 be real and tend to $-\infty$. Then

$$e^{-h_A(\zeta)} + \zeta_0 = |\zeta_0 + a(\zeta_0)| + \zeta_0 \rightarrow -\operatorname{Re} a(-\infty),$$

where $a(-\infty)$ is an accumulation point of $a(\zeta_0)$ as $\zeta_0 \in \mathbf{R}$ and $\zeta_0 \rightarrow -\infty$. It is a point in the closure of $L(A_0)$ which satisfies

$$H_{A_0}(\zeta') = \sup_{z' \in A_0} \operatorname{Re}(\zeta' \cdot z') = \sup_{s \in L(A_0)} \operatorname{Re} s = \operatorname{Re} a(-\infty).$$

This implies (5.10).

6. The dual function expressed as a dual complement

In convexity theory, the Fenchel transform generalizes the supporting function: the supporting function (5.4) is just the Fenchel transform of the indicator function. Conversely, we can express the Fenchel transform \tilde{f} of a function f in terms of the supporting function if we add one dimension: by definition we have $\tilde{f}(\xi) = \sup_x (\xi \cdot x - f(x))$, and we see that $\tilde{f}(\xi) = H_{\text{epi } f}(\xi, -1)$, where $H_{\text{epi } f}$ is the supporting function of the epigraph of f , i.e.,

$$H_{\text{epi } f}(\xi, \eta) = \sup_{(x,y) \in \mathbf{R}^n \times \mathbf{R}} (\xi \cdot x + \eta y; f(x) \leq y), \quad (\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}.$$

We already know that the dual complement A^* of a closed set A can be expressed in terms of the dual function (indeed, A^* is the set where $\mathcal{L}I_A$ is finite; see Proposition 4.2). Conversely, we shall see here that we can express the dual function in terms of a dual complement if we go up one step in dimension (cf. (6.13) below). The functions will then give rise to Hartogs sets, which we proceed to define.

A **Hartogs domain** in $\mathbf{C}^n \times \mathbf{C}$ is a domain which contains, along with a point $(z, t) \in \mathbf{C}^n \times \mathbf{C}$, also every point (z, t') with $|t'| = |t|$. It is said to be a **complete Hartogs domain** if it contains, with (z, t) , also (z, t') for all t' with $|t'| \leq |t|$. We shall generalize this in two ways: first we shall need to study sets which are not necessarily open; second, it is natural to add a hyperplane at infinity and look at subsets of projective space. Thus we consider sets $A \subset (\mathbf{C}^{1+n} \setminus \{0\}) \times \mathbf{C}$ which are homogeneous in the sense of section 2, i.e., such that $(sz, st) \in A$ if $(z, t) \in A$ and $s \in \mathbf{C} \setminus \{0\}$. We shall say that A is a **complete Hartogs set** if (z, t') belongs to A as soon as $(z, t) \in A$ and $|t'| \leq |t|$. Such a set is therefore defined by an inequality $|t| < R(z)$ or $|t| \leq R(z)$ for some function R with $0 \leq R \leq +\infty$. We shall however use $f = -\log R$ to indicate the radius of the disks.

Definition 6.1. *Let $f: \mathbf{C}^{1+n} \setminus \{0\} \rightarrow [-\infty, +\infty]$ be a homogeneous function and X a homogeneous subset of $\mathbf{C}^{1+n} \setminus \{0\}$. We associate to f and X a homogeneous complete Hartogs set $E(X; f)$ in \mathbf{C}^{1+n+1} : it is the set of all $(z, t) \in (\mathbf{C}^{1+n} \setminus \{0\}) \times \mathbf{C}$ such that $|t| \leq e^{-f(z)}$ when $z \in X$ and $|t| < e^{-f(z)}$ when $z \notin X$.*

The fiber of $E(X; f)$ over z is thus the whole t -plane if $f(z) = -\infty$; it is a closed disk of finite positive radius if $z \in (\text{dom } f) \cap X$ and $f(z) > -\infty$; it is an open disk of finite positive radius if $z \in (\text{dom } f) \setminus X$ and $f(z) > -\infty$; it is the origin if $z \in X \setminus \text{dom } f$; finally, the fiber is empty if $z \notin X \cup \text{dom } f$. If $X_1 \subset X_2$ and $f_1 \geq f_2$, we have an obvious inclusion $E(X_1; f_1) \subset E(X_2; f_2)$.

Every complete Hartogs set A is of the form $E(X; f)$ for some X and some f : we can take X as the set of all z such that the fiber is not open and define $f(z)$ as the infimum of all real numbers c such that (z, e^{-c}) belongs to A . A complete Hartogs set defines the set $X \cap \{z; f(z) > -\infty\}$ uniquely: if $f(z) > -\infty$, then $z \in X$ if and only if the fiber over z is closed and non-empty. On the other hand the choice of $X \cap \{z; f(z) = -\infty\}$ is immaterial in the definition of $E(X; f)$.¹

¹To get uniqueness, one could for example require that X always contain $\{z; f(z) = -\infty\}$ or that these two sets be disjoint, or else take the Riemann sphere as the fiber over points in X such that $f(z) = -\infty$, but we shall refrain from doing so.

Theorem 6.2. Consider the dual complement of $E(X; f)$,

$$(6.1) \quad E(X; f)^* = \{(\zeta, \tau) \in \mathbf{C}^{1+n+1} \setminus \{0\}; \zeta \cdot z + \tau t \neq 0 \text{ for all } (z, t) \in E(X; f)\}.$$

If both X and $\text{dom } f$ are empty, then also $E(X; f)$ is empty and $E(X; f)^*$ is equal to $\mathbf{C}^{1+n+1} \setminus \{0\}$. If on the other hand $X \cup \text{dom } f \neq \emptyset$, then $E(X; f)$ is non-empty and its dual complement $E(X; f)^*$ is a subset of $(\mathbf{C}^{1+n} \setminus \{0\}) \times \mathbf{C}$ and a complete Hartogs set, thus

$$E(X; f)^* = E(\Xi; \varphi)$$

for some set Ξ and some function φ . The function φ is uniquely determined:

$$(6.2) \quad \varphi(\zeta) = \begin{cases} \mathcal{L}f(\zeta) & \text{when } \zeta \in (X \cup \text{dom } f)^*, \text{ and} \\ +\infty & \text{when } \zeta \in (\mathbf{C}^{1+n} \setminus \{0\}) \setminus (X \cup \text{dom } f)^*; \end{cases}$$

thus $\varphi = \mathcal{L}f$ as soon as $\text{dom } \mathcal{L}f \subset (X \cup \text{dom } f)^*$, in particular if $X \subset \text{dom } f$. We define the set Ξ as follows. We let $\zeta \in \Xi$ if and only if $\zeta \in (X \cup \text{dom } f)^*$ and either f takes the value $-\infty$ or

$$(6.3) \quad \inf_{z \in \text{dom } f \cap X} |\zeta \cdot z| e^{f(z)} \geq \inf_{w \in \text{dom } f \setminus X} |\zeta \cdot w| e^{f(w)}$$

or else

$$(6.4) \quad \text{for all } z^0 \in X \cap \text{dom } f \text{ we have } |\zeta \cdot z^0| e^{f(z^0)} > \inf_{z \in \text{dom } f \cap X} |\zeta \cdot z| e^{f(z)}.$$

If f is not $+\infty$ identically, then Ξ is uniquely determined, so that this is the only set which satisfies $E(X; f)^* = E(\Xi; \varphi)$. Moreover, we always have

$$(6.5) \quad E(X; f)^* \cap ((\mathbf{C}^{1+n} \setminus \{0\}) \times \{0\}) = (X \cup \text{dom } f)^* \times \{0\},$$

which proves that $\Xi \cup \text{dom } \varphi = (X \cup \text{dom } f)^*$. The particular cases when X is empty or equal to $\text{dom } f$ are of interest. If $\text{dom } f \neq \emptyset$ we have

$$(6.6) \quad E(\emptyset; f)^* = E((\text{dom } f)^*; \mathcal{L}f) \supset E(\text{dom } \mathcal{L}f; \mathcal{L}f).$$

If $\text{dom } f$ is closed and non-empty and f is lower semicontinuous and never takes the value $-\infty$, then

$$(6.7) \quad E(\text{dom } f; f)^* = E(\emptyset; \mathcal{L}f).$$

Remark 6.3. If $X \cup \text{dom } f \neq \emptyset$, then $E(X; f)^*$ contains

$$[E(\emptyset; \mathcal{L}f) \cap ((X \setminus \text{dom } f)^* \times (\mathbf{C} \setminus \{0\}))] \cup ((X \cup \text{dom } f)^* \times \{0\})$$

and is contained in

$$E((\text{dom } \mathcal{L}f) \cup (X \cup \text{dom } f)^*; \mathcal{L}f).$$

If X is a subset of $\text{dom } f$, then $\varphi = \mathcal{L}f$ and these inclusions simplify to:

$$(6.8) \quad E(\emptyset; \mathcal{L}f) \cup ((\text{dom } f)^* \times \{0\}) \subset E(X; f)^* \subset E((\text{dom } f)^*; \mathcal{L}f).$$

We also note the following two special cases. If $f = +\infty$ identically, then

$$E(X; f)^* = X^* \times E = E(\Xi; \varphi)$$

for any $\Xi \subset X^*$. In this case the definition of Ξ in the theorem yields $\Xi = X^*$. If f assumes the value $-\infty$, then

$$(6.9) \quad E(X; f)^* = (X \cup \text{dom } f)^* \times \{0\} = E((X \cup \text{dom } f)^*; \mathcal{L}f).$$

Proof of Theorem 6.2. If $X \cup \text{dom } f$ is empty, then $E(X; f)$ is empty, and its dual complement is the whole space except the origin. If $X \cup \text{dom } f$ is not empty, then the hyperplane $(\mathbf{C}^{1+n} \setminus \{0\}) \times \{0\}$ cuts $E(X; f)$, so that no point $(0, \tau)$ belongs to $E(X; f)^*$, which therefore is contained in $(\mathbf{C}^{1+n} \setminus \{0\}) \times \mathbf{C}$.

We need to find the conditions for (ζ, τ) to belong to $E(X; f)^*$. This happens precisely when $\zeta \cdot z + \tau t$ is non-zero for all $(z, t) \in E(X; f)$. The case $\tau = 0$ is easy: we find that $(\zeta, 0) \in E(X; f)^*$ if and only if $\zeta \cdot z \neq 0$ for all $z \in X \cup \text{dom } f$, thus if and only if $\zeta \in (X \cup \text{dom } f)^*$. This proves (6.5). Now let $\tau \neq 0$. Then we see that $(\zeta, \tau) \in E(X; f)^*$ precisely when the following three conditions hold:

$$(6.10) \quad |\tau| < |\zeta \cdot z| e^{f(z)} \text{ for all } z \in (\text{dom } f) \cap X;$$

$$(6.11) \quad |\tau| \leq |\zeta \cdot w| e^{f(w)} \text{ for all } w \in (\text{dom } f) \setminus X;$$

$$(6.12) \quad |\zeta \cdot z| \neq 0 \text{ for all } z \in X \setminus \text{dom } f.$$

Fix $\zeta \in (X \cup \text{dom } f)^*$. We see that (6.10–12) imply that $|\tau| \leq \exp(-\mathcal{L}f(\zeta))$, and that they are implied by $|\tau| < \exp(-\mathcal{L}f(\zeta))$. This shows that φ is as described in (6.2), and it only remains to be seen when the inequality $|\tau| \leq \exp(-\mathcal{L}f(\zeta))$ is strict. The condition on τ means that it shall belong to all open disks of radius $|\zeta \cdot z| e^{f(z)}$ for $z \in (\text{dom } f) \cap X$, and all closed disks of radius $|\zeta \cdot w| e^{f(w)}$ for $w \in (\text{dom } f) \setminus X$. Now an intersection of a family of closed disks is always closed, and an intersection of non-empty open disks with finite radii is closed exactly when it contains, along with any disk, also a disk of smaller radius. This is what is expressed by conditions (6.3–4). Finally (6.6) and (6.7) follow from an analysis of (6.3–4) in the special cases $X = \emptyset, \text{dom } f$.

Corollary 6.4. *The logarithmic transform $\mathcal{L}f$ of any function f can be obtained from the dual complement of $E(\emptyset; f)$: it is minus the logarithm of a certain distance, viz. the distance from $(\zeta, 0)$ to the complement of $E(\emptyset; f)^*$ in the direction $(0, \dots, 0, 1)$:*

$$(6.13) \quad \mathcal{L}f(\zeta) = -\log \left(\inf_{\tau} (|\tau|; (\zeta, \tau) \in \mathbf{C}^{1+n+1} \setminus E(\emptyset; f)^*) \right).$$

Proof. This follows from (6.6). The result also explains why we cannot expect these functions to be pull-backs of functions on projective space.

7. Lineally convex Hartogs sets

Intuitively, it seems that $E(\emptyset; f)$ and $E(\text{dom } f; f)$ ought to be lineally convex simultaneously. This is not quite true. We shall note three results in the positive direction, Propositions 7.1–3, and one result in the negative direction, Example 7.4. Then we shall establish conditions under which it is true that f is \mathcal{L} -closed if and only if $E(\text{dom } f; f)$ is lineally convex (Corollary 7.6), as well as conditions which guarantee that f is \mathcal{L} -closed if and only if $E(\emptyset; f)$ is lineally convex (Theorem 7.11).

Proposition 7.1. *If $E(X; f)$ is lineally convex, then also $X \cup \text{dom } f$ and $E(X \cup \text{dom } f; f)$ are lineally convex. In particular, if $E(\emptyset; f)$ is lineally convex, then so are $\text{dom } f$ and $E(\text{dom } f; f)$.*

Proof. Suppose that $E(X; f)$ is lineally convex. That $X \cup \text{dom } f$ is lineally convex then follows from the easily proved result that the intersection of a lineally convex set and a complex subspace is lineally convex as a subset of the latter. If $E(X; f)$ is lineally convex, then also $E(X; f + a)$ is lineally convex for any real number a . Any intersection of lineally convex sets has the same property, so we only need to note that $E(X \cup \text{dom } f; f)$ is equal the intersection of all $E(X; f + a)$, $a < 0$.

Proposition 7.2. *If f is upper semicontinuous and there exists a set X such that $E(X; f)$ is lineally convex, then $E(\emptyset; f)$ is lineally convex.*

Proof. We know from Corollary 2.4 that $E(X; f)^\circ$ is lineally convex if $E(X; f)$ is lineally convex. Now $E(X; f)^\circ = E(\emptyset; f)$ if f is upper semicontinuous, hence the result.

However, the semicontinuity of f is not important—it is the fact that the effective domain is open which is relevant as the following result shows.

Proposition 7.3. *If $X \cup \text{dom } f$ is open and $E(X; f)$ is lineally convex, then $E(X \setminus \text{dom } f; f)$ is lineally convex. In particular, $E(\emptyset; f)$ is lineally convex if $\text{dom } f$ is open and $E(X; f)$ is lineally convex for some subset X of $\text{dom } f$.*

Proof. Assume that $E(X; f)$ is lineally convex. Then also $E(X; f + a)$ is lineally convex for any real a , and we shall prove that the union of all $E(X; f + 1/k)$, $k = 1, 2, \dots$, which equals $E(X \setminus \text{dom } f; f)$, is lineally convex.

The hyperplanes in \mathbf{C}^{1+n+1} will be denoted by $Y_{(\zeta, \tau)}$ in analogy with (2.2), thus

$$(7.1) \quad Y_{(\zeta, \tau)} = \{(z, t) \in (\mathbf{C}^{1+n} \times \mathbf{C}) \setminus \{0\}; \zeta \cdot z + \tau t = 0\}.$$

Let $(z, t) \notin E(X \setminus \text{dom } f; f)$ be given with $z \in X \cup \text{dom } f$. For any k there is a hyperplane $Y_{(\zeta^k, \tau^k)}$ which contains the point (z, t) and which does not meet $E(X; f + 1/k)$. We may assume that $|\zeta^k|^2 + |\tau^k|^2 = 1$. Take an accumulation point (ζ, τ) of the sequence (ζ^k, τ^k) . Since $X \cup \text{dom } f$ is open, we can be sure that $\tau \neq 0$. The hyperplane $Y_{(\zeta, \tau)}$ passes through (z, t) and does not meet $E(X \setminus \text{dom } f; f)$ since $\tau \neq 0$. If on the other hand $z \notin X \cup \text{dom } f$, there is a hyperplane $Y_{(\zeta, 0)}$ which passes through (z, t) and does not cut $E(X; f)$.

The openness in Proposition 7.3 cannot be dispensed with as we shall see now.

Example 7.4. There is a function f such that $E(\text{dom } f; f)$ is lineally convex while $E(\emptyset; f)$ is not. Define

$$R(z) = \inf_{k=1,2,\dots} |(k+1)z_1 - z_0 - z_0/k|, \quad z = (z_0, z_1) \in \mathbf{C}^{1+1} \setminus \{0\},$$

and let $f = -\log R$. Then $\text{dom } f$ consists of the complement of the hyperplanes Y_ζ , $\zeta = (1, -k)$, and

$$E(\text{dom } f; f) = \bigcap_{k=1}^{\infty} \{(z, t); z \notin Y_{(1, -k)} \text{ and } |t| \leq |(k+1)z_1 - z_0 - z_0/k|\}$$

is lineally convex. (Note however, that $E(X; f)$ is not lineally convex if X contains $\text{dom } f$ strictly; cf. Example 2.9.) The function f is \mathcal{L} -closed; cf. (3.6) and Theorem 7.5 below. To prove that $E(\emptyset; f)$ is not lineally convex, let us note that $(1, 0, 1) \notin E(\emptyset; f)$, for $R(1, 0) = 1$. Suppose there exists a hyperplane $Y_{(\zeta, \tau)}$ which passes through the point $(1, 0, 1)$ but does not cut $E(\emptyset; f)$. Then $\zeta_0 + \tau = 0$. We must also have $\tau \neq 0$ since $(1, 0, 0) \in E(\emptyset; f)$ as well as $\zeta_1 \neq 0$ since $(1, 2, 1) \in E(\emptyset; f)$. Moreover

$$\frac{|\zeta \cdot z|}{|\tau|} = \frac{|\zeta \cdot z|}{|\zeta_0|} \geq \inf_k |(k+1)z_1 - z_0 - z_0/k| \quad \text{for all } z.$$

Taking $z = (\zeta_1, -\zeta_0)$ we see that there is a number m such that $\zeta_1 = -m\zeta_0$, and we can conclude that, taking $z_0 = 1$,

$$\frac{|\zeta \cdot z|}{|\zeta_0|} = m|z_1 - 1/m| \geq \inf_k |(k+1)z_1 - 1 - 1/k|.$$

However, for z_1 close to $1/m$ we must have

$$\inf_k |(k+1)z_1 - 1 - 1/k| = |(m+1)z_1 - 1 - 1/m|,$$

so that

$$m|z_1 - 1/m| \geq (m+1)|z_1 - 1/m|$$

for all z_1 close to $1/m$. This is impossible, which shows that there is no such hyperplane.

Theorem 7.5. *If $f = \mathcal{L}\mathcal{L}f$ in $(X \cup \text{dom } f)^{**}$, then $E((X \cup \text{dom } f)^{**}; f)$ is lineally convex. In particular, if we assume $\text{dom } f$ to be lineally convex and $X \subset \text{dom } f$, then $\mathcal{L}\mathcal{L}f = f$ in $\text{dom } f$ implies that $E(\text{dom } f; f)$ is lineally convex. Conversely, if $E(X; f)$ is lineally convex, then $f = \mathcal{L}\mathcal{L}f$ in $X \cup \text{dom } f$, which is a lineally convex set. If f is bounded from below on the unit sphere, then $f = \mathcal{L}\mathcal{L}f = +\infty$ outside $(\overline{\text{dom } f})^{**}$. Thus in this case $\mathcal{L}\mathcal{L}f = f$ everywhere if $X \supset (\overline{\text{dom } f})^{**} \setminus \text{dom } f$.*

Proof. Suppose that $f = \mathcal{L}\mathcal{L}f$ in $(X \cup \text{dom } f)^{**}$. Let $(z^0, t^0) \notin E((X \cup \text{dom } f)^{**}; f)$. We shall then prove that there is a hyperplane $Y_{(\zeta, \tau)}$ (see (7.1)) which contains (z^0, t^0)

and does not cut $E((X \cup \text{dom } f)^{**}; f)$. Consider first the case $z^0 \in (X \cup \text{dom } f)^{**}$. We know that $|t^0| > e^{-f(z^0)}$. By the definition of $\mathcal{L}\mathcal{L}f$ and since $\mathcal{L}\mathcal{L}f(z^0) = f(z^0) > -\log |t^0|$, we can choose ζ such that

$$-\log |\zeta \cdot z^0| - \mathcal{L}f(\zeta) > -\log |t^0|.$$

Then we take $\tau = -\zeta \cdot z^0 / t^0$, so that $(z^0, t^0) \in Y_{(\zeta, \tau)}$ and $-\mathcal{L}f(\zeta) > \log |\tau|$. Moreover, for any $(z, t) \in Y_{(\zeta, \tau)}$ we have

$$f(z) \geq \mathcal{L}\mathcal{L}f(z) \geq -\log |\zeta \cdot z| - \mathcal{L}f(\zeta) = -\log |\tau t| - \mathcal{L}f(\zeta) > -\log |t|,$$

which shows that $(z, t) \notin E((X \cup \text{dom } f)^{**}; f)$. The case $z^0 \notin (X \cup \text{dom } f)^{**}$ remains to be considered. In this case there is a hyperplane Y_ζ which contains z^0 and does not meet $(X \cup \text{dom } f)^{**}$, so the hyperplane $Y_{(\zeta, 0)}$ does not cut $E((X \cup \text{dom } f)^{**}; f)$.

Now assume that $E(X; f)$ is linearly convex. We already know that $X \cup \text{dom } f$ is linearly convex (cf. Proposition 7.1). If f assumes the value $-\infty$, then $E(X; f) = (X \cup \text{dom } f) \times \mathbf{C}$, and $f = \mathcal{L}\mathcal{L}f = -\infty$ in $X \cup \text{dom } f$. If $f > -\infty$, let z^0 be any point in $X \cup \text{dom } f$ and take t^0 such that $|t^0| > e^{-f(z^0)}$, thus $(z^0, t^0) \notin E(X; f)$. By hypothesis there is a hyperplane $Y_{(\zeta, \tau)}$ which passes through (z^0, t^0) and does not meet $E(X; f)$. Since $(z^0, 0) \in E(X, f)$, we must have $\tau \neq 0$, so we obtain a minorant of f of the form $-\log |\zeta \cdot z| + \log |\tau| \leq f(z)$, where the left-hand side takes the value $-\log |t^0| < f(z^0)$ at the point z^0 and moreover can be chosen larger than any number less than $f(z^0)$. Thus $\mathcal{L}\mathcal{L}f(z^0) \geq f(z^0)$ and we conclude that $\mathcal{L}\mathcal{L}f = f$ in all of $X \cup \text{dom } f$.

Finally, assume that $f \geq -C$ on the unit sphere without any further assumption. Thus, putting $A = \text{dom } f$, we have $f \geq g = I_A - C$, so that $\mathcal{L}f \leq \mathcal{L}g = C - \log d_{A^*}$ by Proposition 4.2. We now note that $d_{A^*} = d_{A^{*\circ}}$ and take the transformation once again, this time using Proposition 4.5. We get $\mathcal{L}\mathcal{L}f \geq \mathcal{L}\mathcal{L}g \geq I_{A^{*\circ}} - M - C$. In particular $\mathcal{L}\mathcal{L}f(z) = +\infty$ if $z \notin A^{*\circ} = (\overline{A})^{**} = (\overline{\text{dom } f})^{**}$ (cf. (2.5)). This finishes the proof.

Corollary 7.6. *Assume that f is bounded from below on the unit sphere and that $\text{dom } f$ is closed and linearly convex. Then f is \mathcal{L} -closed if and only if $E(\text{dom } f; f)$ is linearly convex.*

We now proceed to study the case when $\text{dom } f$ is open and f tends to $+\infty$ at the boundary. Propositions 7.7 and 7.10 below are applicable when f tends rather fast to $+\infty$, and Theorem 7.11 in a more general situation.

Proposition 7.7. *Let f be a homogeneous function on $\mathbf{C}^{1+n} \setminus \{0\}$ which tends to $+\infty$ at the boundary of $\text{dom } f = A$ in the strong sense that $f \geq -C - \log d_A$ for some constant C , where d_A is defined by (3.9). Then f is \mathcal{L} -closed if and only if $E(\emptyset; f)$ is linearly convex.*

Proof. If f is \mathcal{L} -closed, then its effective domain $A = A^\circ$ must be linearly convex by Theorem 3.5, for $A = (\text{dom } \mathcal{L}\mathcal{L}f)^\circ = (\overline{\text{dom } \mathcal{L}f})^*$ in view of (3.12) applied to $\mathcal{L}f$ (this function is bounded from below on the unit sphere unless f is $+\infty$ identically, a trivial case). Theorem 7.5 now shows that $E(A; f)$ is linearly convex and Proposition 7.3 implies that $E(\emptyset; f)$ is linearly convex.

Conversely, assume that $E(\emptyset; f)$ is lineally convex. In view of Theorem 7.5 it only remains to be proved that $\mathcal{L}\mathcal{L}f = f = +\infty$ outside $A = \text{dom } f$. Now if $f \geq -C - \log d_A$, then $\mathcal{L}f \leq C + \mathcal{L}(-\log d_A) \leq C + I_{A^*}$ by Proposition 4.5. We take the transformation again and obtain $\mathcal{L}\mathcal{L}f \geq -C + \mathcal{L}I_{A^*} = -C - \log d_{A^{**}}$, using Proposition 4.2. But $\text{dom } f$ is lineally convex, so $A^{**} = A$. Hence $\mathcal{L}\mathcal{L}f = +\infty$ in the complement of A .

Functions with bounded logarithmic transforms exhibit the behavior studied in Proposition 7.7:

Proposition 7.8. *Let f be a homogeneous function on $\mathbf{C}^{1+n} \setminus \{0\}$ such that $\text{dom } f = A$ equals the interior of its closure and such that $\overline{\text{dom } f}$ is lineally convex. Assume that f is bounded from below on the unit sphere and that $\mathcal{L}f$ is bounded from above in $S \cap \text{dom } \mathcal{L}f$. Then $f \geq -C - \log d_A$, where C is a constant.*

Proof. We have $\mathcal{L}f \leq C + I_B$, where $B = \text{dom } \mathcal{L}f$. Therefore $f \geq \mathcal{L}\mathcal{L}f \geq -C + \mathcal{L}I_B = -C - \log d_{B^*} = -C - \log d_{B^{*\circ}}$. The next lemma shows that $B^{*\circ} = A$.

Lemma 7.9. *Let f be a homogeneous function on $\mathbf{C}^{1+n} \setminus \{0\}$ such that $(\overline{\text{dom } f})^\circ = \text{dom } f$. Assume that f is bounded from below on the unit sphere and that $\overline{\text{dom } f}$ is lineally convex. Then $(\text{dom } \mathcal{L}f)^{*\circ} = \text{dom } f$.*

Proof. From (3.12) we deduce, recalling that f is bounded from below on S , that

$$(\overline{\text{dom } f})^{**} \supset (\text{dom } \mathcal{L}f)^* \supset (\text{dom } f)^{**}.$$

Now, since $\overline{\text{dom } f}$ is lineally convex, so is its interior $(\overline{\text{dom } f})^\circ = \text{dom } f$ (Corollary 2.4). Therefore $\overline{\text{dom } f} \supset (\text{dom } \mathcal{L}f)^* \supset \text{dom } f$. Taking the interior of these sets we get $(\text{dom } \mathcal{L}f)^{*\circ} = \text{dom } f$.

Under a regularity assumption we can let f tend to infinity at a slower pace:

Proposition 7.10. *Let f be a homogeneous function on $\mathbf{C}^{1+n} \setminus \{0\}$ which tends to $+\infty$ at the boundary of $\text{dom } f = A$ in the sense that $f \geq -C - c \log d_A$ on the unit sphere for some constants C and c with $0 < c \leq 1$. Assume that A^* satisfies the interior cone condition of Proposition 4.9. Then f is \mathcal{L} -closed if and only if $E(\emptyset; f)$ is lineally convex.*

Remark. It can be easily proved that if A is lineally convex and its dual complement A^* satisfies the interior cone condition, then so does its set-theoretic complement $\mathbb{C}A$.

Proof. In view of Theorem 7.5 and the proof of Proposition 7.7, it only remains to be proved that $\mathcal{L}\mathcal{L}f = f = +\infty$ outside $\text{dom } f$ if $E(\emptyset; f)$ is lineally convex. Now if $f \geq -C - c \log d_A$ on the unit sphere S , then we obtain $f \geq -C + f_c$ everywhere, introducing the function f_c of (4.1). We take the logarithmic transformation once to obtain $\mathcal{L}f \leq C + \mathcal{L}f_c \leq C + \varphi_{1-c}$ (Proposition 4.1), and then again to get $\mathcal{L}\mathcal{L}f \geq -C + \mathcal{L}\varphi_{1-c} \geq -C - M + f_c$, this time applying Proposition 4.9 to the function φ_{1-c} and using the interior cone condition on A^* . This shows that $\mathcal{L}\mathcal{L}f$ equals $+\infty$ in the complement of A .

We finally come to the general case of a function which tends to infinity at the boundary.

Theorem 7.11. *Let f be a homogeneous function on $\mathbf{C}^{1+n} \setminus \{0\}$. Assume that f is bounded from below on the unit sphere and tends to $+\infty$ at the boundary of $A = \text{dom } f$ in the sense that $A_s = \{z \in S; f(z) < s\}$ is strongly contained in $\text{dom } f$ for all numbers s , i.e., the closure of A_s is contained in the interior of A . (This of course implies that A is open.) Assume moreover that A^* satisfies the interior cone condition of Proposition 4.9. Then $E(\emptyset; f)$ is lineally convex if and only if f is \mathcal{L} -closed.*

Proof. For the proof we shall need the functions $f_{A,r}$, where A is a homogeneous set and r is a positive number, defined as $f_{A,r} = -\log r + I_A$, thus $f_{A,r} = -\log r - \log |z|$ when $z \in A$ and $f_{A,r} = +\infty$ otherwise. We note that $\mathcal{L}f_{A,r} = \log r - \log d_{A^*}$ (Proposition 4.2), so that (ζ, τ) belongs to $E(A^*; \mathcal{L}f_{A,r})$ if and only if $\zeta \in A^*$ and $r|\tau| \leq d_{A^*}(\zeta)$.

What remains to be done, considering Theorem 7.5 and the proof of Proposition 7.7, is the following, assuming $E(\emptyset; f)$ to be lineally convex. Given any $z^0 \notin \text{dom } f = A$ it is required to find a hyperplane $Y_{(\zeta, \tau)}$ with $\tau \neq 0$ which does not cut $E(\emptyset; f)$ and passes through (z^0, t^0) with $|t^0|$ arbitrarily small; the problem is to avoid the vertical hyperplanes, those with $\tau = 0$. Since f is bounded from below on S , there is a number R such that $E(\emptyset; f)$ is contained in $E(\emptyset; f_{A,R})$. On the other hand, given any $\varepsilon > 0$, there is a homogeneous set K which is strongly contained in A and such that $f \geq -\log \varepsilon$ on $S \setminus K$; this means that $E(\emptyset; f)$ is contained in $E(\emptyset; f_{A,\varepsilon}) \cup E(\emptyset; f_{K,R})$. We shall find a hyperplane which does not cut the latter set for a suitable choice of ε . This amounts to finding (ζ, τ) in $E(\emptyset; f_{A,\varepsilon})^* \cap E(\emptyset; f_{K,R})^*$, equivalently in $E(A^*; \mathcal{L}f_{A,\varepsilon}) \cap E(K^*; \mathcal{L}f_{K,R})$; cf. (6.6). The hyperplane shall also contain a point (z^0, t^0) with $|t^0| = \delta$ positive but arbitrarily small.

We shall thus find ζ and τ such that

$$0 \neq |\tau| \leq \frac{1}{\varepsilon} d_{A^*}(\zeta) \text{ and } |\tau| \leq \frac{1}{R} d_{K^*}(\zeta).$$

We take of course $0 \neq \tau = -(\zeta \cdot z^0)/t^0$ to ensure that (z^0, t^0) belongs to $Y_{(\zeta, \tau)}$, and then the problem is reduced to finding ζ such that

$$(7.2) \quad 0 \neq |\zeta \cdot z^0| \leq \frac{\delta}{\varepsilon} d_{A^*}(\zeta) \text{ and } |\zeta \cdot z^0| \leq \frac{\delta}{R} d_{K^*}(\zeta).$$

Since by hypothesis $z^0 \notin A = A^{**}$, there is a point $\zeta^0 \in A^*$ such that $\zeta^0 \cdot z^0 = 0$. If ζ^0 is in the interior of A^* , then finding ζ is easy: we have $d_{K^*}(\zeta^0) \geq d_{A^*}(\zeta^0) > 0 = |\zeta^0 \cdot z^0|$ and can take ζ close to ζ^0 . If on the other hand $\zeta^0 \in \partial A^*$, we argue as follows. By the interior cone condition,

$$\sup_{|\zeta - \zeta^0| \leq s|\zeta^0|} d_{A^*}(\zeta) \geq \gamma s |\zeta^0|$$

for some positive constant γ . On the other hand

$$\sup_{|\zeta - \zeta^0| \leq s|\zeta^0|} |\zeta \cdot z^0| = s |\zeta^0| |z^0|.$$

Given any positive δ , it is thus enough to choose ε such that $\varepsilon|z^0| \leq \delta\gamma$ to satisfy the first inequality in (7.2) for some ζ close to ζ^0 , more precisely satisfying $|\zeta - \zeta^0| \leq s|\zeta^0|$ for any given sufficiently small positive s . The second is then satisfied strictly when $\zeta = \zeta^0$, because A^* is strongly contained in K^* by Corollary 2.3 so that $d_{K^*}(\zeta^0) > d_{A^*}(\zeta^0) = 0$, and it must therefore also be satisfied for all ζ satisfying $|\zeta - \zeta^0| \leq s|\zeta^0|$ for all sufficiently small positive s . This completes the proof.

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