

WEAK LINEAL CONVEXITY

CHRISTER O. KISELMAN

*Uppsala University, Department of Information Technology
P. O. Box 337, SE-751 05 Uppsala
E-mail: kiselman@it.uu.se; Web site: www.cb.uu.se/~kiselman*

Abstract. A bounded open set with boundary of class C^1 which is locally weakly lineally convex is weakly lineally convex, but, as shown by Yuriĭ Zelinskĭi, this is not true for unbounded domains. The purpose here is to construct explicit examples, Hartogs domains, showing this. Their boundary can have regularity $C^{1,1}$ or C^∞ .

Obstructions to constructing smoothly bounded domains with certain homogeneity properties will be discussed.

1. Introduction. In my paper (1998) I claimed that a differential condition which I called the Behnke–Peschl condition implies that a connected open subset of \mathbf{C}^n with boundary of class C^2 is weakly lineally convex. The proof in the case of bounded domains relied on a result by Yužakov and Krivokolesko (1971a, 1971b), proved also in Hörmander (1994: Proposition 4.6.4), but in the case of unbounded domains, the proof of their result breaks down.

Yuriĭ Zelinskĭi (2002a, 2002b) published a counterexample in the case of an unbounded set. His example is not very explicit. We shall construct here an explicit example—actually a Hartogs domain, which has the advantage of being easily visualized in three real dimensions. We construct domains with boundary of class $C^{1,1}$ and a certain homogeneity property (Example 8.1), and show that this cannot be done with a boundary of class C^2 (Proposition 9.1). However, the boundary can be of class C^∞ if the homogeneity requirement is dropped (Example 8.2).

Notation. The boldface letters \mathbf{N} , \mathbf{Z} , \mathbf{R} , \mathbf{C} have their usual meaning from Bourbaki, thus $\mathbf{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers, etc.

We shall use the l^p -norm $\|z\|_p = (\sum_j |z_j|^p)^{1/p}$, $1 \leq p < +\infty$, and the l^∞ -norm

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$\|z\|_\infty = \sup_j |z_j|$ for $z \in \mathbf{C}^n$. When any norm can serve, we write only $\|z\|$. The *bilinear inner product* is written $\zeta \cdot z = \zeta_1 z_1 + \cdots + \zeta_n z_n$, $(\zeta, z) \in \mathbf{C}^n \times \mathbf{C}^n$.

We shall denote by $B_{<}(c, r)$ and $B_{\leq}(c, r)$ the *open ball* and the *closed ball*, respectively, with center at $c \in \mathbf{C}^n$ and radius $r \in \mathbf{R}$ for the Euclidean norm, thus

$$B_{<}(c, r) = \{z \in \mathbf{C}^n; \|z - c\|_2 < r\} \text{ and } B_{\leq}(c, r) = \{z \in \mathbf{C}^n; \|z - c\|_2 \leq r\}.$$

If $n = 1$, we shall write instead $D_{<}(c, r)$ and $D_{\leq}(c, r)$ for the disks.

The closure and interior of a subset A of a topological space will be denoted by \overline{A} and A° , respectively. Thus $\overline{B_{<}(c, r)} = B_{\leq}(c, r)$ if r is positive, and $B_{\leq}(c, r)^\circ = B_{<}(c, r)$ for all real r .

For derivatives of functions, we shall use the notation

$$f_{x_j} = \frac{\partial f}{\partial x_j}, \quad f_{y_j} = \frac{\partial f}{\partial y_j}, \quad f_{z_j} = \frac{1}{2}(f_{x_j} - if_{y_j}), \quad f_{\bar{z}_j} = \frac{1}{2}(f_{x_j} + if_{y_j}), \quad j = 1, \dots, n.$$

2. Lineal convexity.

DEFINITION 2.1. A subset of \mathbf{C}^n is said to be *lineally convex* if its complement is a union of complex affine hyperplanes. \square

To every set A there exists a smallest lineally convex subset $\mu(A)$ which contains A . Clearly the mapping $\mu: \mathcal{P}(\mathbf{C}^n) \rightarrow \mathcal{P}(\mathbf{C}^n)$, where $\mathcal{P}(\mathbf{C}^n)$ denotes the family of all subsets of \mathbf{C}^n (the power set), is increasing and idempotent, in other words an *ethmorphism* (morphological filter). It is also larger than the identity, so that μ is a *cleistomorphism* (closure operator) in the ordered set $\mathcal{P}(\mathbf{C}^n)$.

This kind of complex convexity was introduced by Heinrich Behnke (1898–1997) and Ernst Ferdinand Peschl (1906–1986). I learnt about it from André Martineau (1930–1972) when I was in Nice during the academic year October 1967 through September 1968. See Martineau (1966, 1967, 1968, 1977).

Are there lineally convex sets which are not convex? This is obvious in one complex variable, and from there we can easily construct, by taking Cartesian products, lineally convex sets in any dimension which are not convex. But these sets do not have smooth boundaries. Hörmander (1994:293, Remark 3) constructs open connected sets in \mathbf{C}^n with boundary of class C^2 as perturbations of a convex set. These sets are lineally convex and close to a convex set in the C^2 topology, and therefore starshaped with respect to some point if the perturbation is small. Also the symmetrized bidisk $\{(z_1 + z_2, z_1 z_2) \in \mathbf{C}^2; |z_1|, |z_2| < 1\}$, studied by Agler & Young (2004) and Pflug & Zwonek (2012), is not convex—not even biholomorphic to a convex domain (Nikolov et al. (2008)—but it is starshaped with respect to the origin (Agler & Young 2004: Theorem 2.3). So we may ask:

QUESTION 2.2. *Does there exist a lineally convex set in \mathbf{C}^n , $n \geq 2$, with smooth boundary which is not starshaped with respect to any point?* \square

We shall return to this question in Section 10.

3. Weak lineal convexity.

DEFINITION 3.1. An open subset Ω of \mathbf{C}^n is said to be *weakly lineally convex* if there passes, through every point on the boundary of Ω , a complex affine hyperplane which does not cut Ω . \square

It is clear that every lineally convex open set is weakly lineally convex. The converse does not hold. This is not difficult to see if we allow sets that are not connected:

EXAMPLE 3.2. Given a number c with $0 < c < 1$, define an open set Ω_c in \mathbf{C}^2 as the union of the set

$$\{z \in \mathbf{C}^2; |y_1| < 1, c < |x_1| < 1, |x_2| < c, |y_2| < c\}$$

with the two sets obtained by permuting x_1, x_2 and y_2 . Thus Ω_c consists of six boxes. It is easy to see that it is weakly lineally convex, but there are many points in its complement such that every complex line passing through that point hits Ω_c .

Any complex line intersects the real hyperplane defined by $y_1 = 0$ in the empty set or in a real line or in a real two-dimensional plane, and the three-dimensional set $\{z; y_1 = 0\} \cap \Omega_c$ is easy to visualize. \square

It is less easy to construct a connected set with these properties, but this has been done by Yužakov & Krivokolesko (1971b:325, Example 2). See also an example due to Hörmander in the book by Andersson, Passare & Sigurdsson (2004:20–21, Example 2.1.7).

However, the boundary of the constructed set is not of class C^1 , and this is essential. Indeed, Yužakov & Krivokolesko (1971b:323, Theorem 1) proved that a connected bounded open set with “smooth” boundary is locally weakly lineally convex in the sense of Definition 4.3 below if and only if it is lineally convex. It is then even \mathbf{C} -convex (1971b:324, Assertion). See also Corollary 4.6.9 in Hörmander (1994), which states that a connected bounded open set with boundary of class C^1 is locally weakly lineally convex if and only if it is \mathbf{C} -convex (and every \mathbf{C} -convex open set is lineally convex).

There cannot be any cleistomorphism connected with the notion of weak lineal convexity for the simple reason that the property is defined only for open sets. We might therefore want to define weak lineal convexity for arbitrary sets. We may ask:

QUESTION 3.3. *Is there a reasonable definition of weak lineal convexity for all sets which keeps the definition for open sets and is such that there is a cleistomorphism associating to any $A \subset \mathbf{C}^n$ the smallest set which contains A and is weakly lineally convex?* \square

The operation $L \mapsto L \cap \Omega$ associating to a complex line L its intersection with an open set Ω has continuity properties which seem to be highly relevant for weak lineal convexity. Here the family of complex lines can arguably have only one topology, but for the family of sets $L \cap \Omega$ there is a choice of several topologies, especially if Ω is unbounded.

QUESTION 3.4. *Can an interesting theory be built starting from this remark?*

4. Local weak linear convexity.

DEFINITION 4.1. We shall say that an open set $\Omega \subset \mathbf{C}^n$ is *locally weakly linearly convex* if for every point p there exists a neighborhood V of p such that $\Omega \cap V$ is weakly linearly convex. \square

Obviously, a weakly linearly convex open set has this property, but the converse does not hold, which is obvious for sets which are not connected: Take the union of two open balls whose closures are disjoint. Also for connected sets the converse does not hold:

EXAMPLE 4.2. (Kiselman 1996, Example 3.1.)

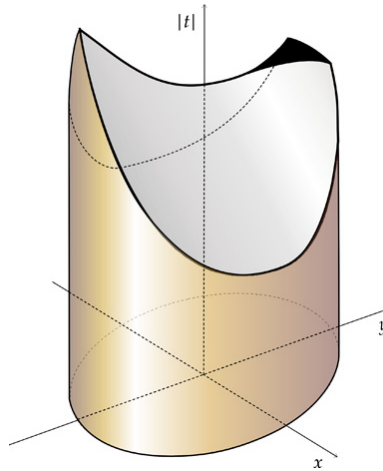


Figure 1. An open connected Hartogs set in \mathbf{C}^2 which is locally weakly linearly convex but not weakly linearly convex. Coordinates $(z, t) \in \mathbf{C}^2$; $(x, y, |t|) \in \mathbf{R}^3$.

Define first

$$\Omega_+ = \{(z, t); |z| < 1 \text{ and } |t| < |z - 2|\};$$

$$\Omega_- = \{(z, t); |z| < 1 \text{ and } |t| < |z + 2|\},$$

and then

$$\Omega_0 = \Omega_+ \cap \Omega_-; \quad \Omega_0^r = \{(z, t) \in \Omega_0; |t| < r\},$$

where r is a constant with $2 < r < \sqrt{5}$. All these sets are linearly convex. The two points $(\pm i, \sqrt{5})$ belong to the boundary of Ω_0 ; in the three-dimensional space of the variables $(\operatorname{Re} z, \operatorname{Im} z, |t|)$, the set representing Ω_0 has two peaks, which have been truncated in Ω_0^r .

We now define Ω^r by glueing together Ω_0 and Ω_0^r : Define Ω^r as the subset of Ω_0 such that $(z, t) \in \Omega_0^r$ if $\operatorname{Im} z > 0$; we truncate only one of the peaks of Ω_0 .

The point $(i - \varepsilon, r)$ for a small positive ε belongs to the boundary of Ω^r and the tangent plane at that point has the equation $t = r$ and so must cut Ω^r at the point $(-i + \varepsilon, r)$. Therefore Ω^r is not linearly convex, but it agrees with the linearly convex sets Ω_0 and Ω_0^r when $\operatorname{Im} z < \delta$ and $\operatorname{Im} z > -\delta$, respectively, for a small positive δ . The set has Lipschitz boundary; in particular it is equal to the interior of its closure. \square

In this example it is essential that the boundary is not smooth.

Zelinskiĭ (1993:118, Example 13.1) constructs an open set which is locally weakly lineally convex but not weakly lineally convex. The set is not equal to the interior of its closure.

DEFINITION 4.3. Let us say that an open set Ω is *locally weakly lineally convex in the sense of Yužakov and Krivokolesko* (1971b:323) if for every boundary point p there exists a complex hyperplane Y passing through p and a neighborhood V of p such that Y does not meet $V \cap \Omega$. \square

Zelinskiĭ (1993:118, Definition 13.1) uses this definition and calls the property *локальная линейная выпуклость*. As we shall see, this property is strictly weaker than the local weak lineal convexity defined above in Definition 4.1.

Hörmander (1994: Proposition 4.6.4) and Andersson et al. (2004: Proposition 2.5.8) use this property only for open sets with boundary of class C^1 . Then the hyperplane Y is unique.

For all open sets, local weak lineal convexity obviously implies local weak lineal convexity in the sense of Yužakov and Krivokolesko. In the other direction, Hörmander's Proposition 4.6.4 shows that for bounded open sets with boundary of class C^1 , local weak lineal convexity in the sense of Yužakov and Krivokolesko implies local weak lineal convexity (even weak lineal convexity if the set is connected).

Nikolov (2012: Proposition 3.7.1) and Nikolov et al. (2013: Proposition 3.3) have a local result in the same direction: If Ω has a boundary of class C^k , $2 \leq k \leq \infty$, and $\Omega \cap B_{<}(p, r)$, where p is a given point, is locally weakly lineally convex in the sense of Yužakov and Krivokolesko at all points near p , then there exists a \mathbf{C} -convex open set ω (hence lineally convex) with boundary of class C^k such that $\omega \cap B_{<}(p, r') = \Omega \cap B_{<}(p, r')$ for some positive r' .

However, in general the two properties are not equivalent:

EXAMPLE 4.4. There exists a bounded connected open set in \mathbf{C}^2 with Lipschitz boundary which is locally weakly lineally convex in the sense of Yužakov and Krivokolesko but not locally weakly lineally convex. While Ω^r is locally weakly lineally convex for $2 < r < \sqrt{5}$, the set Ω^2 for $r = 2$ is not locally weakly lineally convex: The point $(0, 2)$ does not have a neighborhood with the desired property. But it does satisfy the property of Yužakov and Krivokolesko. \square

5. Approximation by smooth sets. Let $A_j \subset \mathbf{C}^{n_j}$ be two lineally convex sets in \mathbf{C}^{n_j} , $j = 1, 2$. Then it is easy to see that their Cartesian product $A_1 \times A_2 \subset \mathbf{C}^{n_1+n_2}$ is lineally convex. In particular, if $n_1 = n_2 = 1$, then every Cartesian product in \mathbf{C}^2 is lineally convex. However, these sets cannot always be approximated by lineally convex sets with smooth boundaries.

If Ω_j , $j = 1, 2$, are convex open sets, then $\Omega = \Omega_1 \times \Omega_2$ is convex and can be approximated from within by convex open set Ω^ε with C^∞ boundaries, $\Omega^\varepsilon \nearrow \Omega$ as $\varepsilon \searrow 0$.

But if we let Ω_1 be an annulus and Ω_2 a disk, e.g.,

$$\Omega = \Omega_1 \times \Omega_2 = \{z \in \mathbf{C}^2; 1 < |z_1| < 3, |z_2| < 1\},$$

then it cannot be approximated by smooth weakly convex sets from the inside as we shall see in the next proposition and its corollary.

PROPOSITION 5.1. *Let ω be a nonempty bounded open subset of \mathbf{R}^2 with boundary of class C^1 . Suppose that $\inf_{x \in \omega} |x_1| > 0$. Define a Reinhardt domain Ω as*

$$\Omega = \{z \in \mathbf{C}^2; (|z_1|, |z_2|) \in \omega\}.$$

Then Ω is not locally weakly lineally convex.

Proof. Take a point $q = (q_1, q_2) \in \Omega$ with $q_1 < 0, q_2 \geq 0$. Denote by Ω^+ the set of all $x \in \Omega$ such that $x_1 > 0$ and $x_2 > 0$, and by L_α the complex line of equation $z_2 - q_2 = \alpha(z_1 - q_1)$, $\alpha \geq 0$. For $\alpha = 0$, the line cuts Ω^+ in $(-q_1, q_2)$; for large α it does not cut Ω^+ . Now choose the smallest α such that L_α does not cut Ω^+ . Then L_α contains at least one point $p \in \partial\Omega^+$, and L_α is the tangent plane of $\partial\Omega$ at p . Since this line meets Ω in q , Ω is not weakly lineally convex. But we can say more: It is not even locally weakly lineally convex. To see this, first note that $\alpha = (p_2 - q_2)/(p_1 - q_1) > 0$. Then there are points $z \in \Omega$ belonging to the tangent at p arbitrarily close to p . Indeed, since α is positive, a point z satisfying

$$|z_1| > p_1 \text{ and } |z_2| < p_2$$

belongs to Ω if it is close enough to p . In terms of z_1 this means that

$$|z_1| > p_1 \text{ and } |q_2 + \alpha(z_1 - q_1)| < p_2;$$

in other words that $z_1 \notin D_{\leq}(0, p_1)$ and that $z_1 \in D_{<}(c_1, r_1)$, the open disk with center at $c_1 = q_1 - q_2/\alpha$ and radius $r_1 = p_2/\alpha = p_1 - c_1$. Since $r_1 = p_1 - c_1$, there are points $z_1 \in D_{<}(c_1, r_1) \setminus D_{\leq}(0, p_1)$ which are arbitrarily close to p_1 . □

COROLLARY 5.2. *A Reinhardt domain*

$$\Omega = \{z \in \mathbf{C}^2; r_1 < |z_1| < R_1, |z_2| < R_2\}$$

with $r_1 > 0$ is lineally convex but cannot be approximated by lineally convex domains with boundary of class C^1 .

Proof. If a domain Ω^ε approximates Ω from the inside in the sense that

$$\Omega^\varepsilon \subset \Omega \subset \Omega^\varepsilon + B_{<}(0, \varepsilon),$$

then there is also a Reinhardt domain with this property: We may construct such a set by averaging over all rotations.

We can now apply the proposition to Ω^ε . □

6. The Behnke–Peschl and Levi conditions.

DEFINITION 6.1. *The real Hessian of a C^2 function f is*

$$H_f^{\mathbf{R}}(p; s) = \sum f_{x_j x_k}(p) s_j s_k, \quad p \in \mathbf{R}^m, \quad s \in \mathbf{R}^m.$$

The complex Hessian is

$$H_f^{\mathbf{C}}(p; t) = \sum f_{z_j z_k}(p) t_j t_k, \quad p \in \mathbf{C}^n, \quad t \in \mathbf{C}^n.$$

The *Levi form* is

$$L_f(p; t) = \sum f_{z_j \bar{z}_k}(p) t_j \bar{t}_k, \quad p \in \mathbf{C}^n, \quad t \in \mathbf{C}^n. \quad \square$$

If we let the relation between the real s and the complex t be the usual one:

$$t_j = s_{2j-1} + is_{2j}, \quad j = 1, \dots, n, \quad s \in \mathbf{R}^n, \quad t \in \mathbf{C}^n,$$

we get

$$\frac{1}{2} H_f^{\mathbf{R}}(p; s) = \operatorname{Re} H_f^{\mathbf{C}}(p; t) + L_f(p; t), \quad p \in \mathbf{C}^n, \quad s \in \mathbf{R}^{2n}, \quad t \in \mathbf{C}^n.$$

Let now Ω_f be the set of all points where a function $f \in C^2(\mathbf{C}^n)$ is negative. We should assume that $\|\operatorname{grad} f\| + |f| > 0$ everywhere, so that the boundary of Ω_f is of class C^2 .

The complex tangent space $T_{\mathbf{C}}(p)$ at a point $p \in \partial\Omega_f$ is defined by $\sum f_{z_j}(p) t_j = 0$; the real tangent space $T_{\mathbf{R}}(p)$ by $\operatorname{Re} \sum f_{z_j}(p) t_j = 0$.

We recall the following classical definition.

DEFINITION 6.2. An open set Ω with boundary of class C^2 is said to satisfy the *Levi condition* if, for every point $p \in \partial\Omega_f$, we have

$$L_f(p; t) \geq 0 \quad \text{when } t \in T_{\mathbf{C}}(p),$$

i.e., when $\sum f_{z_j}(p) t_j = 0$. We say that it satisfies the *strong Levi condition* if, for $t \in T_{\mathbf{C}}(p) \setminus \{0\}$, we have strict inequality. \square

DEFINITION 6.3. An open set Ω with boundary of class C^2 is said to satisfy the *Behnke–Peschl condition* if, for every point $p \in \partial\Omega_f$, we have

$$\frac{1}{2} H_f^{\mathbf{R}}(p; s) = \operatorname{Re} H_f^{\mathbf{C}}(p; t) + L_f(p; t) \geq 0 \quad \text{when } t \in T_{\mathbf{C}}(p),$$

i.e., when $\sum f_{z_j}(p) t_j = 0$. We say that it satisfies the *strong Behnke–Peschl condition* if, for $t \in T_{\mathbf{C}}(p) \setminus \{0\}$, we have strict inequality. \square

The condition says that the restriction of the real Hessian to the complex tangent space at any boundary point shall be positive semidefinite; in the strong case, positive definite.

Because of the different homogeneity of $H^{\mathbf{C}}$ and L , the inequality is equivalent to $L \geq |H^{\mathbf{C}}|$. The inequality $L \geq |H^{\mathbf{C}}| \geq 0$ shows that the Behnke–Peschl condition implies the Levi condition.

In my paper (1998) I proved that a bounded connected open set with boundary of class C^2 is weakly lineally convex if it satisfies the Behnke–Peschl condition.

That this condition is necessary for weak lineal convexity was known since Behnke and Peschl (1935); the sufficiency was unknown.

I stated the result also for unbounded connected open sets with C^2 boundary. The proof relied on Proposition 4.6.4 in Hörmander (1994), which is stated there for bounded connected open sets with boundary of class C^1 .

7. Yužakov and Krivokolesko: Passage from local to global. Let us quote the part of Proposition 4.6.4 in Hörmander (1994) which is important for us:

PROPOSITION 7.1. *Let $\Omega \subset \mathbf{C}^n$ be a bounded connected open set with boundary of class C^1 and assume that Ω is locally weakly linearly convex in the sense of Yužakov and Krivokolesko. Then Ω is weakly linearly convex.*

The result was proved by Yužakov & Krivokolesko (1971a, 1971b) under the condition that the boundary is “smooth.” In my paper (1998) I needed this result for boundaries of class C^2 , and carelessly claimed (1998:4) that it is true also if the set is unbounded.

8. A new example. We shall construct explicit Hartogs domains here with the properties mentioned in Zelinskiĭ’s example. We start with the simplest.

EXAMPLE 8.1. Define a function $\varphi^\diamond: \mathbf{C} \rightarrow \mathbf{R}$ by

$$\varphi^\diamond(z_1) = \begin{cases} -x_1^2 - y_1^2, & x_1 \leq 0 \text{ or } y_1 \leq 0; \\ -x_1^2 + y_1^2, & 0 \leq y_1 \leq x_1; \\ x_1^2 - y_1^2, & 0 \leq x_1 \leq y_1. \end{cases}$$

Then

$$\Omega_{\varphi^\diamond} = \{z \in \mathbf{C}^2; 1 + \varphi^\diamond(z_1) + |z_2|^2 < 0\}$$

has boundary of class $C^{1,1}$ and is locally weakly linearly convex but not weakly linearly convex. □

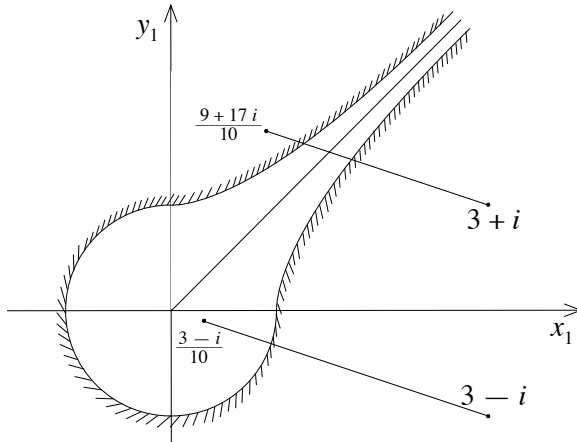


Figure 2. The set $\Omega_{\varphi^\diamond} \cap \{z \in \mathbf{C}^2; z_2 = 0\}$.

The properties of the set in this example will be discussed now and the properties will be seen to hold from Proposition 8.3.

The tangent plane at a boundary point $p = (p_1, p_2)$ with $\text{Re } p_1 > 0$, $\text{Im } p_1 > 0$, and $(\text{Re } p_1)^2 > (\text{Im } p_1)^2 + 1$, has the equation $-p_1(z_1 - p_1) + \bar{p}_2(z_2 - p_2) = 0$ and it passes through the point $q = (p_1 - |p_2|^2/p_1, 0)$. Choosing $p = (3 + i, \sqrt{7})$ we get $q = (\frac{9}{10} + i\frac{17}{10}, 0) \in \Omega_{\varphi^\diamond}$, proving that $\Omega_{\varphi^\diamond}$ is not linearly convex.

We note that the tangent plane at a boundary point p with $\text{Re } p_1 \leq 0$ or $\text{Im } p_1 \leq 0$ is contained in the complement of $\Omega_{\varphi^\diamond}$; in particular, it hits the plane $z_2 = 0$ at the point $q = (p_1/|p_1|^2, 0) \notin \Omega_{\varphi^\diamond}$. We also note that the part of $\Omega_{\varphi^\diamond}$ where $0 < x_1 < y_1$ is

convex, so any tangent plane of this part does not intersect it. Similarly, the part where $0 < y_1 < x_1$ is convex. Therefore Ω_φ^\diamond is the union of two lineally convex sets, taking the subsets where $x_1 < \max(y_1, 0)$, and $y_1 < \max(x_1, 0)$, respectively.

When $x_1 < 0$ or $y_1 < 0$ we get $\varphi_{z_1}^\diamond(z_1) = -\bar{z}_1$, $\varphi_{z_1 z_1}^\diamond(z_1) = 0$, $\varphi_{z_1 \bar{z}_1}^\diamond(z_1) = -1$; when $0 < y_1 \leq x_1$ we have $\varphi_{z_1}^\diamond(z_1) = -z_1$, $\varphi_{z_1 z_1}^\diamond(z_1) = -1$ and $\varphi_{z_1 \bar{z}_1}^\diamond(z_1) = 0$; when $0 < x_1 \leq y_1$ we have $\varphi_{z_1}^\diamond(z_1) = z_1$, $\varphi_{z_1 z_1}^\diamond(z_1) = 1$ and $\varphi_{z_1 \bar{z}_1}^\diamond(z_1) = 0$. In all three cases $|\varphi_{z_1 z_1}^\diamond| - \varphi_{z_1 \bar{z}_1}^\diamond = 1$. An application of Proposition 8.3 below now gives the result, except that it does not give anything at the exceptional points, where the function is not of class C^∞ , i.e., those with $y_1 = 0, x_1 > 0$ or $x_1 = 0, y_1 > 0$. However, we have already seen that at these points, the tangent plane does not cut $\Omega_{\varphi^\diamond}$.

The boundary of $\Omega_{\varphi^\diamond}$ is not of class C^2 at the points where $y_1 = 0, x_1 > 0$ or $x_1 = 0, y_1 > 0$. The passage from $-x_1^2 - y_1^2$ for $y_1 \leq 0$ to $-x_1^2 + y_1^2$ for $0 \leq y_1 \leq x_1$ cannot be made analytically.

The function φ^\diamond is not of class $C^{1,1}$ at the points where $x_1 = y_1, x_1 > 0$, but this is of no consequence, since these points do not belong to the closure of the set it defines.

We note that the function φ^\diamond in the example is homogeneous of degree two:

$$\varphi^\diamond(z_1) = \varphi^\diamond(|z_1|e^{it}) = |z_1|^2\psi(t), \quad z_1 \in \mathbf{C}, t \in \mathbf{R}.$$

It is therefore natural to ask if there is a C^∞ homogeneous function φ with the same properties. More precisely, we may ask for functions $\varphi: \mathbf{C} \rightarrow \mathbf{R}$ which yield a locally weakly lineally convex domain which is not weakly lineally convex in four different cases.

- 1.1. *Is there a C^∞ function φ with these properties?*
- 1.2. *Is there a homogeneous C^∞ function φ with these properties?*
- 2.1. *Is there an analytic function φ with these properties?*
- 2.2. *Is there a homogeneous analytic function φ with these properties?*

As we shall see, the answer to the first question is in the affirmative (Example 8.2). But the answer to Question 1.2 is in the negative (Proposition 9.1).

EXAMPLE 8.2. Now define $\varphi^*: \mathbf{C} \rightarrow \mathbf{R}$ by

$$\varphi^*(z_1) = \begin{cases} -x_1^2 + \chi(y_1), & x_1 \geq y_1; \\ -y_1^2 + \chi(x_1), & x_1 \leq y_1, \end{cases}$$

where $\chi \in C^\infty(\mathbf{R})$ is a function of one real variable such that χ' is convex and which satisfies

$$\chi(y_1) = \begin{cases} -y_1^2 + \rho, & y_1 \leq -\frac{1}{2}; \\ y_1^2 + \sigma, & y_1 \geq \frac{1}{2}. \end{cases}$$

The convexity of χ' implies that $2|y_1| \leq |\chi'(y_1)| \leq \max(2|y_1|, 1)$ with equality to the left for $|y_1| \geq \frac{1}{2}$. This implies that we must have $\frac{1}{2} < \chi(\frac{1}{2}) - \chi(-\frac{1}{2}) < 1$, and we can actually choose χ so that $\chi(\frac{1}{2}) - \chi(-\frac{1}{2})$ is any given number in that interval.

For definiteness we now choose $\rho = -\frac{1}{4}, \sigma = 0, \chi(\frac{1}{2}) - \chi(-\frac{1}{2}) = \frac{3}{4}$, which implies that $\varphi^\diamond - \frac{1}{4} \leq \varphi^* \leq \varphi^\diamond$, that Ω_{φ^*} contains $\Omega_{\varphi^\diamond}$, and that the set of points $z \in \Omega_{\varphi^*}$ with

$\operatorname{Re} z_1 \geq \frac{1}{2}$ and $\operatorname{Im} z_1 \geq \frac{1}{2}$ is unchanged compared to Ω_{φ° . We choose χ as a suitable third primitive of

$$\chi'''(y_1) = C \exp(1/(y_1 - c) - 1/(y_1 + c)), \quad -c < y_1 < c,$$

for a number c , $0 < c \leq \frac{1}{2}$, and a positive constant C , taking $\chi'''(y_1)$ equal to zero when $|y_1| \geq c$. This implies that χ' is even and that $\chi(0) = \frac{1}{2}\rho + \frac{1}{2}\sigma = -\frac{1}{8}$. Then

$$\Omega_{\varphi^*} = \{z \in \mathbf{C}^2; 1 + \varphi^*(z_1) + |z_2|^2 < 0\}$$

has boundary of class C^∞ and is locally weakly lineally convex but not lineally convex, since, just as for Ω_{φ° , the tangent plane at the boundary point $p = (3 + i, \sqrt{7})$ passes through $q = (\frac{9}{10} + i\frac{17}{10}, 0) \in \Omega_{\varphi^*}$. □

The properties mentioned in these examples will follow from the next proposition and its corollary.

PROPOSITION 8.3. *Let $\varphi: \mathbf{C} \rightarrow \mathbf{R}$ be a function of class C^k , $k = 2, 3, \dots, \infty, \omega$, and define an open set in \mathbf{C}^2 as*

$$\Omega_\varphi = \{z \in \mathbf{C}^2; 1 + \varphi(z_1) + |z_2|^2 < 0\}.$$

We assume that

$$\varphi_{z_1} \neq 0 \text{ wherever } \varphi = -1, \tag{1}$$

and that

$$(-\varphi - 1)(|\varphi_{z_1 z_1}| - \varphi_{z_1 \bar{z}_1}) \leq |\varphi_{z_1}|^2 \text{ in the set where } -\varphi - 1 \geq 0. \tag{2}$$

Then Ω_φ has boundary of class C^k and satisfies the Behnke–Peschl condition at every boundary point. If the inequality is strict at a certain point, we get the strong Behnke–Peschl condition at that point.

Proof. This result was proved in my paper (1996: Lemma 6.1). There I described the domain by an inequality of the form $|z_2|^2 < h(z_1)$ and found that the Behnke–Peschl condition takes the form $h(h_{z_1 \bar{z}_1} + |h_{z_1 z_1}|) \leq |h_{z_1}|^2$, which, with $h(z_1) = -\varphi(z_1) - 1$ yields (2). □

COROLLARY 8.4. *Let φ have the form $\varphi(z_1) = -x_1^2 + \chi(y_1)$ for $x_1 \geq y_1$ and $\varphi(z_1) = -y_1^2 + \chi(x_1)$ for $y_1 \geq x_1$. We assume that $\chi \in C^k(\mathbf{R})$, $k \geq 2$, with $-2 \leq \chi''$ and such that $\chi(y_1) > -1$ when $\chi'(y_1) = 0$. Then the conclusion of Proposition 8.3 holds under the assumption*

$$\frac{1}{4}\chi'(y_1)^2 + \chi(y_1) + 1 \geq 0, \quad y_1 \in \mathbf{R}. \tag{3}$$

Proof. The condition (1) is satisfied, since the gradient of φ in this case vanishes only when $x_1 = 0$ and $\chi'(y_1) = 0$. Then $1 + \varphi(z_1) + |z_2|^2 = 1 + \chi(y_1) + |z_2|^2 > 0$, so $z = (iy_1, z_2)$ cannot be a boundary point of Ω .

Condition (2) reduces to

$$(x_1^2 - \chi(y_1) - 1) \left(\left| -\frac{1}{2} - \frac{1}{4}\chi''(y_1) \right| + \frac{1}{2} - \frac{1}{4}\chi''(y_1) \right) \leq \left| -x_1 - \frac{1}{2}i\chi'(y_1) \right|^2 = x_1^2 + \frac{1}{4}\chi'(y_1)^2,$$

provided $x_1^2 - \chi(y_1) - 1 \geq 0$. If $-2 \leq \chi''$, we have

$$\left| -\frac{1}{2} - \frac{1}{4}\chi''(y_1) \right| + \frac{1}{2} - \frac{1}{4}\chi''(y_1) = \frac{1}{2} + \frac{1}{4}\chi''(y_1) + \frac{1}{2} - \frac{1}{4}\chi''(y_1) = 1,$$

which gives (3). We then see that in this case the inequality holds also if $x_1^2 - \chi(y_1) - 1 < 0$. \square

In Example 8.2, the defining function $1 - x_1^2 + \chi(y_1) + |z_2|^2$ has nonvanishing gradient everywhere since $\chi' > 0$ everywhere. Smoothness follows.

The function φ^* is not of class C^∞ in the set where $x_1 = y_1$, $x_1 > 0$, but again this is unimportant since these points do not belong to the closure of Ω_{φ^*} . An application of Corollary 8.4 now gives the result. In fact, with the choice of $\rho = -\frac{1}{4}$, $\sigma = 0$, we need only note that $\chi(y_1) \geq -y_1^2 - \frac{1}{4}$ everywhere, and that $\chi'(y_1) \geq 2|y_1|$, so that

$$\frac{1}{4}\chi'(y_1)^2 + \chi(y_1) + 1 \geq \frac{3}{4} > 0, \quad x_1 \geq y_1,$$

thus with strict inequality in (3) and (2); similarly for $x_1 \leq y_1$.

REMARK 8.5. It might be of interest to understand where the proof of Hörmander's Proposition 4.6.4 quoted above breaks down in the unbounded case. An important step in the proof is to see that, if we have a continuous family $(L_t)_{t \in [0,1]}$ of complex lines, the set T of parameter values t such that $L_t \cap \Omega$ is connected is both open and closed. Thus, if $0 \in T$, then also $1 \in T$. We shall see that closedness is no longer true for the sets in Examples 8.1 and 8.2.

Define complex lines

$$L_t = \{z \in \mathbf{C}^2; z_2 = t(z_1 - 1 - i)\}, \quad t \in [0, 1],$$

which all pass through $(1 + i, 0) \notin \Omega_{\varphi^*}$. Then $L_t \cap \Omega_{\varphi^*}$ is connected for $0 \leq t < 1$ while $L_1 \cap \Omega_{\varphi^*}$ is not.

We shall first see that $L_1 \cap \Omega_{\varphi^*}$ is disconnected. If $z \in L_1 \cap \Omega_{\varphi^*}$ and $x_1 \leq 0$ or $y_1 \leq 0$, then

$$f(z) = 1 + \varphi^*(z_1) + |z_2|^2 \geq 1 - |z_1|^2 + \rho + |z_1 - 1 - i|^2 = 3 + \rho - 2(x_1 + y_1).$$

Since we have chosen $\rho = -\frac{1}{4}$, the quantity $f(z)$ can be negative only if $x_1 + y_1 > 0$, which implies that z_1 satisfies either $x_1 > |y_1|$ or $y_1 > |x_1|$. Therefore the real hyperplane of equation $x_1 = y_1$ divides $L_1 \cap \Omega_{\varphi^*}$ into two sets, which are nonempty since $(2, 1 - i)$ and $(2i, i - 1)$ both belong to L_1 , the first with $y_1 < x_1$, the second with $y_1 > x_1$, and that both belong to Ω_{φ^*} in view of the fact that $\chi(0) \leq 0$.

Next we shall see that $L_t \cap \Omega_{\varphi^*}$ is connected when $0 \leq t < 1$. Given t such that $0 \leq t < 1$, we obtain for $z \in L_t \cap \Omega_{\varphi^*}$ with $x_1, y_1 \leq 0$,

$$1 + \varphi^*(z_1) + |z_2|^2 \leq 1 - (1 - t^2)|z_1|^2 - 2t^2(x_1 + y_1) + 2t^2.$$

This yields the estimate

$$1 + \varphi^*(z_1) + |z_2|^2 \leq 3 + 4|z_1| - (1 - t^2)|z_1|^2,$$

which is negative for $|z_1| = R_t$, if R_t large enough. (Obviously R_t tends to $+\infty$ as $t \rightarrow 1$, which explains that $L_1 \cap \Omega_{\varphi^*}$ is disconnected.) Let Γ_t be the arc in L_t with $|z_1| = R_t$ and $x_1 \leq 0$ or $y_1 \leq 0$, thus contained in Ω_{φ^*} .

An arbitrary point $a \in L_t \cap \Omega_{\varphi^*}$ can be joined to a point in Γ_t by a straight line segment contained in Ω_{φ^*} and therefore also contained in $L_t \cap \Omega_{\varphi^*}$. If $\operatorname{Re} a_1 \leq 0$ or $\operatorname{Im} a_1 \leq 0$ this follows from the fact that the set of points in Ω_{φ^*} with argument of

z_1 equal to that of a_1 is convex; otherwise from the fact that the points in Ω_{φ^*} with $0 \leq \text{Im } z_1 < \text{Re } z_1$ is convex, as is the set of points with $0 \leq \text{Re } z_1 < \text{Im } z_1$. \square

9. An impossibility result.

PROPOSITION 9.1. *Let $\Omega_\varphi = \{z \in \mathbf{C}^2; 1 + \varphi(z_1) + |z_2|^2 < 0\}$, where φ is positively homogeneous of degree two and of class C^2 where it is negative. Then either φ is constant and Ω_φ is lineally convex; or φ is not constant and Ω_φ is not connected.*

The set $\Omega_{\varphi^\diamond}$ in Example 8.1 has the properties mentioned here except that its boundary is not of class C^2 : We have a striking contrast between the regularity classes $C^{1,1}$ and C^2 .

Proof. For functions $\varphi: \mathbf{C} \rightarrow \mathbf{R}$ which are positively homogeneous of degree two, i.e., of the form $\varphi(z_1) = |z_1|^2\psi(t)$, $z_1 = |z_1|e^{it} \in \mathbf{C}$, $t \in \mathbf{R}$, condition (2) on φ takes the form

$$(-r^2\psi - 1) \left[-\psi + \sqrt{\left(\frac{1}{4}\psi''\right)^2 + \left(\frac{1}{2}\psi'\right)^2} - \frac{1}{4}\psi'' \right] \leq r^2(\psi^2 + \frac{1}{4}\psi'^2),$$

to hold in the set where $-r^2\psi - 1 \geq 0$; equivalently

$$4\psi + (-r^2\psi - 1) \left[\sqrt{\psi''^2 + 4\psi'^2} - \psi'' \right] \leq r^2\psi'^2.$$

From this we obtain, if we divide by r^2 and let r tend to $+\infty$,

$$(-\psi) \left[\sqrt{\psi''^2 + 4\psi'^2} - \psi'' \right] \leq \psi'^2. \tag{4}$$

But this condition is also sufficient, which follows on multiplication by r^2 and adding the trivial inequality $4\psi - \left[\sqrt{\psi''^2 + 4\psi'^2} - \psi'' \right] \leq 0$.

To get rid of the square root in (4) we rewrite it as

$$\psi'^2 (\psi'^2 + 2(-\psi)\psi'' - 4\psi^2) \geq 0,$$

where the left-hand side is of degree four.

We now introduce a function g by defining $g(t)$ as the positive square root of $-\psi(t)$ if $\psi(t)$ is negative and as 0 at all other points. Thus g is of class C^2 where it is positive, and $\psi = -g^2$ there. The points where $g = 0$, equivalently $\psi \geq 0$, are not of interest, since for them $1 + |z_1|^2\psi(t) + |z_2|^2 \geq 1 > 0$, implying that z does not belong to the closure of Ω_φ .

We get an inequality of degree eight but which is easy to analyze:

$$g^5 g'^2 (g + g'') \leq 0. \tag{5}$$

Thus, for each t such that $g(t) > 0$, either $g'(t) = 0$ or $g(t) + g''(t) \leq 0$. If g' is zero everywhere, i.e., if g is constant, it is known that Ω_φ is lineally convex, in particular weakly lineally convex. Wherever g is positive and g' is nonzero we get $g + g'' \leq 0$. This implies that any local maximum of g is isolated and that there can only be one point where the maximum is attained.¹ Hence, unless g' vanishes everywhere, $g + g'' \leq 0$

¹Defining a function $g: \mathbf{R} \rightarrow \mathbf{R}$ of period 2π by $g(t) = |\cos 2t|$ when $0 \leq t \leq \pi$ and $g(t) = 1$ when $\pi < t < 2\pi$, we get a function which satisfies inequality (5) in $\mathbf{R} \setminus \pi\mathbf{Z}$, but not the conclusions we have drawn from it. This function is not of class C^2 (but of class $C^{1,1}$).

everywhere. We define $h = g + g'' \leq 0$ and obtain for any $a \in \mathbf{R}$

$$g(t) = g(a) \cos(t - a) + g(a) \int_a^t \sin(t - s)h(s)ds, \quad t \in \mathbf{R}.$$

The function g attains its maximum at some point which we may call a , and the formula then shows that $g(t) \leq g(a) \cos(t - a)$ for all t with $a \leq t \leq a + \pi/2$. In particular, g must have a zero t_0 in the interval $]a, a + \pi/2]$. By symmetry, g has a zero t_1 also in the interval $[a - \pi/2, a[$, hence at least two zeros in a period. This means that Ω is not connected, since the union of the rays $\arg z_1 = t_0$ and $\arg z_1 = t_1$ divides the z_1 -plane. \square

10. A set which is not starshaped. In answer to Question 2.2 we mention a modification of the set $\Omega_{\varphi^\#}$ which is not starshaped.

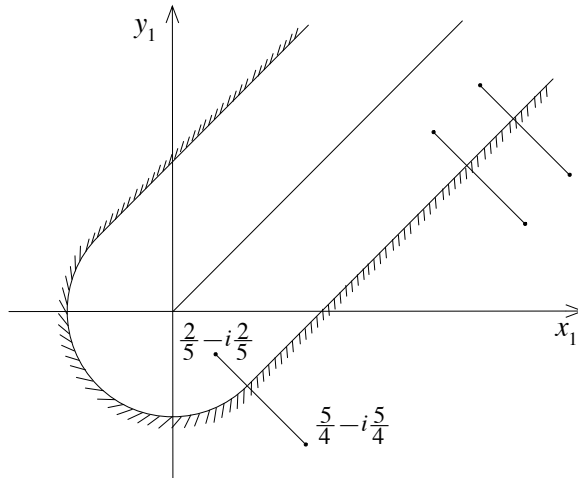


Figure 3. The set $\Omega_{\varphi^\#} \cap \{z \in \mathbf{C}^2; z_2 = 0\}$.

EXAMPLE 10.1. Define $\varphi^\#: \mathbf{C} \rightarrow \mathbf{R}$ by

$$\varphi^\#(z_1) = \begin{cases} -x_1^2 - y_1^2, & x_1 + y_1 \leq 0; \\ -\frac{1}{2}(x_1 - y_1)^2, & x_1 + y_1 \geq 0. \end{cases}$$

Then

$$\Omega_{\varphi^\#} = \{z \in \mathbf{C}^2; 1 + \varphi^\#(z_1) + |z_2|^2 < 0\}$$

has boundary of class $C^{1,1}$ and is lineally convex, but it is not starshaped with respect to any point. \square

This set can conceivably be modified to have a boundary of class C^∞ like in Example 8.2. However, it is unbounded.

QUESTION 10.2. Does there exist a bounded set with boundary of class C^2 which is lineally convex but not starshaped?

QUESTION 10.3. What about a Hartogs domain with these properties?

The set Ω defined in Corollary 5.2 is bounded, lineally convex and not starshaped, but it has only a Lipschitz boundary. It cannot be approximated by a lineally convex set with smooth boundary.

11. A note on terminology. Heinrich Behnke and Ernst Peschl (1935) introduced the notion which is now known as weak lineal convexity. They called it *Planarkonvexität*.

André Martineau used the terms *convexité linéelle* and *linéellement convexe*—see Martineau (1966:73) and (1968:427), reprinted in (1977:228) and (1977:323), respectively. In French there are two adjectives, *linéaire*, corresponding to the English *linear*; and *linéel*, which I rendered as *lineal*. (There is also an adjective *linéal*.) Martineau obviously wanted a distinctive term in order to signal the special meaning of his convexity, not to be misunderstood as ordinary convexity. Diederich & Fornæss (2003) and Diederich & Fischer (2006) write “lineally convex.”

In Russian, the adjective *линейный* is most often used for both French terms *linéel* and *linéaire*, and this is the term used by Aizenberg, Krivokolesko, Yužakov, and others who write in Russian. In the translations into English of these Russian texts, there appears most often *linear convexity* and *linearly convex*.

Also Znamenskii (1979:83; 1990:1037) used *линейный*, as did Znamenskii & Znamenskaya (1996:359).

Later Znamenskii (2001) used *линейчатый* (usually translated as ‘ruled’; a common term is *линейчатая поверхность* ‘ruled surface’). He thus established the distinction between *lineal*, *linéel* and *linear*, *linéaire* in Russian (Yuri Zeliniskii, personal communication 2013 March 26).

Hörmander (1994:290, Definition 4.6.1), Andersson et al. (2004:16, Definition 2.1.2) and Jacquet (2008:8, Definition 2.1.2) used *linear* and *linearly* and thus did not keep the distinction introduced by Martineau. In my opinion, these authors unnecessarily copied the usage in the translations from Russian and did not pay attention to the pioneering work of Martineau. It should also be noted that the English *lineal* is actually the older of the two words, being attested since the fourteenth century, while *linear* is attested from 1706 (Webster 1983).²

Another term is *hypoconvex*. The first appearance in this context³ that I have found is in Helton & Marshall (1990:182), where it is used for sets with a boundary of class C^2 and has the meaning of ‘strongly lineally convex’ (satisfying the strong Behnke–Peschl condition as I called it in my paper (1998:3); see Definition 6.3 here); later it was weakened to a synonym of *lineally convex* by Whittelsey (2000:678), and used in this sense by Agler & Young (2004:379). The term helpfully reminds us that it signifies a property weaker than convexity.

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²I mentioned this to Mats Andersson, Mikael Passare and Ragnar Sigurdsson in a letter of 1994 January 16.

³Norberto Salinas (1976:144, 1979:327) used the term *hypoconvex* in a different sense.

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References

- Agler, J.; Young, N. J. 2004. The hyperbolic geometry of the symmetrized bidisc. *J. Geom. Anal.* **14**, no. 3, 375–403.
- Andersson, Mats; Passare, Mikael; Sigurdsson, Ragnar. 2004. *Complex Convexity and Analytic Functionals*. Progress in Mathematics, 225. Basel et al.: Birkhäuser.
- Behnke, H[einrich]; Peschl, E[rnst]. 1935. Zur Theorie der Funktionen mehrerer komplexer Veränderlichen. Konvexität in bezug auf analytische Ebenen im kleinen und großen. *Math. Ann.* **111**, 158–177.
- Diederich, Klas; Fischer, Bert. 2006. Hölder estimates on lineally convex domains of finite type. *Michigan Math. J.* **54**, no. 2, 341–352.
- Diederich, Klas; Fornæss, John Erik. 2003. Lineally convex domains of finite type: holomorphic support functions. *Manuscripta Math.* **112**, no. 4, 403–431.
- Helton, J. William; Marshall, Donald E. 1990. Frequency domain design and analytic selections. *Indiana Univ. Math. J.* **39**, no. 1, 157–184.
- Hörmander, Lars. 1994. *Notions of Convexity*. Progress in Mathematics, 127. Boston et al.: Birkhäuser.
- Jacquet, David. 2008. *On complex convexity*. PhD Thesis, defended at Stockholm University on 2008-04-14. Stockholm: Stockholm University. 70 pp.
- Kiselman, Christer O. 1996. Lineally convex Hartogs domains. *Acta Math. Vietnam.* **21**, No. 1, 69–94.
- Kiselman, Christer O. 1998. A differential inequality characterizing weak lineal convexity. *Math. Ann.* **311**, 1–10.
- Martineau, André. 1966. Sur la topologie des espaces de fonctions holomorphes. *Math. Ann.* **163**, 62–88. (Also in Martineau 1977:215–246.)
- Martineau, André. 1967. Sur les équations aux dérivées partielles à coefficients constants avec second membre, dans le champ complexe. *C. R. Acad. Sci. Paris Sér. A–B* **264**, A400–A401. (Also in Martineau 1977:265–267.)
- Martineau, André. 1968. Sur la notion d'ensemble fortement linéellement convexe. *An. Acad. Brasíl. Ci.* **40**, 427–435. (Also in Martineau 1977:323–334.)

- Martineau, André. 1977. *Œuvres de André Martineau*. Paris: Éditions du Centre national de la Recherche scientifique.
- Nikolov, Nikolai [Marinov]. 2012. Invariant functions and metrics in complex analysis. *Dissertationes Math.* **486**, 1–100. (Based on the author's D.Sc. Dissertation, written in Bulgarian and defended 2010-10.)
- Nikolov, Nikolai [Marinov]; Pflug, Peter; Thomas, Pascal J. 2013. On different extremal bases for \mathbf{C} -convex domains. *Proc. Amer. Math. Soc.* **141**, no. 9, 3223–3230.
- Nikolov, Nikolai [Marinov]; Pflug, Peter; Zwonek, Włodzimierz. 2008. An example of a bounded \mathbf{C} -convex domain which is not biholomorphic to a convex domain. *Math. Scand.* **102**, no. 1, 149–155.
- Pflug, Peter; Zwonek, Włodzimierz. 2012. Exhausting domains of the symmetrized bidisc. *Ark. mat.* **50**, 397–402.
- Salinas, Norberto. 1976. Extensions of C^* -algebras and essentially n -normal operators. *Bull. Amer. Math. Soc.* **82**, no. 1, 143–146.
- Salinas, Norberto. 1979. Hypoconvexity and essentially n -normal operators. *Trans. Amer. Math. Soc.* **256**, 325–351.
- Webster's Ninth Collegiate Dictionary*. 1983. Springfield, MA: Merriam-Webster Inc.
- Whittlesey, Marshall A. 2000. Polynomial hulls and H^∞ control for a hypoconvex constraint. *Math. Ann.* **317**, no. 4, 677–701.
- Южаков, А. П.; Кривоколеско, В. П. 1971a. Некоторые свойства линейно выпуклых областей с гладкими границами в \mathbf{C}^n . *Сиб. Мат. Журн.* **12**, No 2, 452–458.
- Yužakov, A. P.; Krivokolesko, V. P. 1971b. Certain properties of linearly convex domains with smooth boundaries. *Siberian Math. J.* **12**, no. 2, 323–327. Translation of Южаков & Кривоколеско (1971a).
- Зелинский, Ю. Б. 1993. *Многозначные отображения в анализе*. Киев: Наукова Думка. 264 pp.
- Зелинский, Ю. Б. 2002a. О локально линейно выпуклых областях. *Укр. мат. журн.* **54**, no. 2, 280–284.
- Zelinskiĭ, Yu. B. 2002b. On locally linearly convex domains. *Ukrainian Math. J.* **54**, no. 2, 345–349. Translation of Зелинский (2002a).
- Знаменский, С. В. 1979. Геометрический критерий сильной линейной выпуклости. *Функциональный анализ и его приложения* **13**(3), 83–84.
- Знаменский, С. В. 1990. Томография в пространствах аналитических функционалов. *Доклады Академии Наук СССР* **312**(5), 1037–1040.
- Знаменский, С. В. 2001. Семь задач о \mathbf{C} -выпуклости. In: Чирка, Е. (Ed.), *Комплексный анализ в современной математике: к 80-летию со дня рождения Б. В. Шабата*, 123–131. Moscow: FAZIS.
- Знаменский, С. В.; Знаменская, Л. Н. 1996. Спиральная связность сечений и проекций \mathbf{C} -выпуклых множеств. *Математические заметки* **59**(3), 359–369.