Convexity of marginal functions in the discrete case

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Abstract

We define, using difference operators, classes of functions defined on the set of points with integer coordinates which are preserved under the formation of marginal functions. The duality between classes of functions with certain convexity properties and families of second-order difference operators plays an important role and is explained using notions from mathematical morphology.

Keywords. Marginal function, discrete convexity, difference operators, A-lateral convexity, rhomboidal convexity, mathematical morphology, infimal convolution.

Prologue

This paper is dedicated to the memory of Mikael Passare: student, mentor, and friend; a great mathematician and a great human being.

After his brilliant achievements in the theory of several complex variables, in particular residue theory, Mikael turned his energy to amoebas and their spines, which are tropical hyperplanes. Tropical geometry was at the time a rather new research area, and he considered his change of focus as an important one, both mentally and scientifically. As far as we know, he did not work on digital geometry or discrete optimization, but he showed great respect for the problems encountered there, which was evident for instance during the preparation of his manuscript later published as Passare (2009). There are in fact strong analogies between tropicalization and discretization. The operation of taking the marginal function is a special case of infimal convolution, which in turn is a tropicalization of ordinary convolution—we have a link to Mikael's interest in tropical geometry. Euclidean geometry, digital geometry, and tropical geometry are three kinds of geometry with contrasting properties. They can support and enrich each other. Together with mathematical morphology and discrete optimization, they constitute research areas with many applications in technology and the sciences.

1. Introduction

1.1. The marginal function of a function of real variables

A simple everyday observation is that the shadow of a convex body is convex. Mathematically this means that the image under an affine mapping of a convex subset of a vector space is convex. It is convenient to express this in terms of marginal functions:

Definition 1.1. If F is a function defined on $\mathbf{R}^n \times \mathbf{R}^m$ and with values in the set of extended real numbers, which we denote by

$$\mathbf{R}_! = [-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\},$$

then its *marginal function* $H \colon \mathbf{R}^n \to \mathbf{R}_!$ is defined by

$$H(x) = \inf_{y \in \mathbf{R}^m} F(x, y), \qquad x \in \mathbf{R}^n. \qquad \square$$

For completeness we also give the definition of a convex function:

Definition 1.2. A function $F \colon \mathbf{R}^n \to \mathbf{R}_!$ is said to be *convex* if it satisfies Jensen's inequality

$$\text{(1.1)} \quad \begin{cases} \text{For all real numbers } t \text{ with } 0 < t < 1 \text{ and all } x, y \in \mathbf{R}^n \\ \text{such that } F(x), F(y) < +\infty \text{ we have} \\ F((1-t)x + ty) \leqslant (1-t)F(x) + tF(y). \end{cases}$$

We shall denote the set of all convex functions by $CVX(\mathbf{R}^n, \mathbf{R}_!)$ and the subset of functions with finite values by $CVX(\mathbf{R}^n, \mathbf{R})$. $\square$

If F is convex, then so is its marginal function H. The proof of this result is completely elementary—and therefore usually mentioned only in passing in the textbooks. The result has nevertheless manifold uses in the applications of the theory for convex functions of real variables.

1.2. The marginal function of a function of integer variables

It would be of interest to establish a similar result for functions $f\colon \mathbf{Z}^n \times \mathbf{Z}^m \to \mathbf{R}_!$, i.e., functions defined at the points in $\mathbf{R}^n \times \mathbf{R}^m$ with integer coordinates. This is what we shall do here.

Definition 1.3. If $f\colon \mathbf{Z}^n \times \mathbf{Z}^m \to \mathbf{R}_!$, we define its *marginal function* h by

$$h(x) = \inf_{y \in \mathbf{Z}^m} f(x, y), \qquad x \in \mathbf{Z}^n. \qquad \square$$

The question now arises which kind of convexity we shall use. A first, seemingly most natural, definition is the following.

Definition 1.4. A function $f\colon \mathbf{Z}^n \to \mathbf{R}_!$ is said to be *convex extensible*[1] if it is the restriction to $\mathbf{Z}^n$ of a convex function defined in $\mathbf{R}^n$. The set of all convex-extensible functions will be denoted by $CVX(\mathbf{R}^n, \mathbf{R}_!)|_{\mathbf{Z}^n}$; the subset of functions which have a real-valued convex extension by $CVX(\mathbf{R}^n, \mathbf{R})|_{\mathbf{Z}^n}$. $\square$

It should be noted that $CVX(\mathbf{R}^n, \mathbf{R})|_{\mathbf{Z}^n}$ is equal to the set of real-valued functions in $CVX(\mathbf{R}^n, \mathbf{R}_!)|_{\mathbf{Z}^n}$.

For $n = 1$, the convex-extensible functions are precisely those which satisfy the special case of Jensen's inequality (1.1) with $x \in \mathbf{Z}$, $y = x + 2$, $t = \frac{1}{2}$. While there are many different notions of discrete convexity in $\mathbf{Z}^n$, $n \geqslant 2$, there is only one reasonable notion of discrete convexity for $n = 1$: The one just described.

Let us now formulate the problem explicitly.

Problem 1.5. Define, for $n = 1, 2, \dots$, classes $\mathscr{M}_n$ of functions defined in $\mathbf{Z}^n$ such that $\mathscr{M}_1 = CVX(\mathbf{R}, \mathbf{R})|_{\mathbf{Z}}$ and such that the successive marginal functions $h_{n-1}, h_{n-2}, \dots, h_1$ of any function $f \in \mathscr{M}_n$ defined by $h_n = f$,

$$h_k(x) = \inf_{t \in \mathbf{Z}} h_{k+1}(x, t), \qquad x = (x_1, \dots, x_k) \in \mathbf{Z}^k, \quad k = n-1, \dots, 1,$$

belong to $\mathscr{M}_k$ whenever they do not take the value $-\infty$. $\square$

We should also require that the classes are large enough so as to avoid trivial results, e.g., by taking $\mathscr{M}_n$ as the set of all functions $f\colon \mathbf{Z}^n \to \mathbf{R}$ such that $f(x_1, \dots, x_n) = g(x_1)$ for some $g \in CVX(\mathbf{R}, \mathbf{R})|_{\mathbf{Z}}$.

[1] This term has been used in a different, narrower sense by Murota (2003:93); for an example showing this, see Kiselman (2011: Example 3.5).

We shall define in this paper classes of functions defined on the integer points which solve the problem completely; see Theorem 11.1. Since this theorem is very general, we have formulated a corollary (Corollary 11.3), which is perhaps easier to apply. The functions of interest are called *A-laterally convex*, where A is a subset of $\mathbf{Z}^n \times \mathbf{Z}^n$ (see Definition 6.2). This subset A determines a family of second-order difference operators; there is a duality between such families and classes of functions with certain convexity properties, which we explain in Section 7 using basic notions of mathematical morphology.

Moreover, we shall prove that the classes obtained are optimal in a natural sense (see Examples 9.2 and 9.3, and Section 12).

The most obvious attempt at defining a convex function of integer variables, i.e., taking $\mathscr{M}_n = CVX(\mathbf{R}^n, \mathbf{R})|_{\mathbf{Z}^n}$, fails in a very conspicuous way, even in low dimensions, as we shall see now.

Example 1.6. Define $f\colon \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ by

$$f(x, y) = |x - 2my|, \qquad (x, y) \in \mathbf{Z} \times \mathbf{Z},$$

where m is a positive integer. Then its marginal function

$$h(x) = \inf_{y \in \mathbf{Z}} f(x, y), \qquad x \in \mathbf{Z},$$

is a periodic function of period $2m$ which is equal to $|x|$ for $-m \leqslant x \leqslant m$. This means that it is a saw-tooth function with teeth as large as we like. We remark also that if we define f in $\mathbf{R} \times \mathbf{Z}$ by the same expression, then the same phenomenon appears. □

The function f in Example 1.6 is actually convex extensible; indeed, an extension is given by the same expression, while h is not convex extensible (or convex in any reasonable sense). Our conclusion is that the property of being convex extensible is too weak to be of use in this context. In view of this observation, one of us has studied a class of functions defined on $\mathbf{Z} \times \mathbf{Z}$ which is suitable for this and other important properties in convexity theory; see Kiselman (2008; 2010a).

The purpose of the present paper is to extend this study to higher dimensions, i.e., to functions on $\mathbf{Z}^n \times \mathbf{Z}^m$.

A kind of convexity called integral convexity was introduced by Favati and Tardella (1990) using locally convex functions. A function $f\colon \mathbf{Z}^n \to \mathbf{R}$ is called *integrally convex* if its convex extension over unit cubes is convex in all of $\mathbf{R}^n$. Integrally convex functions are all convex extensible, and their local minima are global. The class has the property of being invariant under simple coordinate transformations: If we put $g(x,y) = f(x,-y)$, $(x,y) \in \mathbf{Z}^2$, then f and g are integrally convex at the same time, and f and g have the same marginal function:

$$\inf_{y\in\mathbf{Z}} f(x,y) = \inf_{y\in\mathbf{Z}} g(x,y), \qquad x \in \mathbf{Z}.$$

Several of the other classes mentioned in Section 2 do not have this property, which implies that they are not suited for the study of marginal functions—and indeed provide poor analogues of convex functions of real variables, which are invariant under such simple coordinate transformations.

In the case of two integer variables, there are several equivalent ways to define integral convexity. In Kiselman (2008) integral convexity was introduced using difference operators. From this characterization it is obvious that the class is closed under addition.

The present paper is an elaborated version of our paper (2010), which was part of the second author's PhD thesis.

1.3. Relations between Minkowski addition, infimal convolution, and the operation of taking the marginal function

The *Minkowski sum* of two sets A and B is defined as

$$A + B = \{a + b;\ a \in A,\ b \in B\}, \qquad A, B \subset \mathbf{R}^n.$$

This very fundamental operation gives rise to *infimal convolution*, which is defined as the operation $(f, g) \mapsto f \sqcap g$, where

$$(f \sqcap g)(x) = \inf_{y\in\mathbf{Z}^n} \big(f(x-y) \dotplus g(y)\big), \qquad x \in \mathbf{R}^n,\ f, g\colon \mathbf{R}^n \to \mathbf{R}_!.$$

Here $x \dotplus y$, $x, y \in \mathbf{R}_!$, is the *upper sum* of x and y, which extends the sum of real numbers and takes the value $+\infty$ if one of x and y equals $+\infty$. As explained in, e.g., Kiselman (2015: §6), this is a tropicalization of the usual bilinear convolution product defined in (5.1) below.

If we choose $g(x)$ to be zero when $x_1 = x_2 = \cdots = x_m = 0$ and $+\infty$ elsewhere, then $f \sqcap g$ is the marginal function h of f defined as

$$h(x_1, \ldots, x_m) = \inf_{x_{m+1},\ldots,x_n} f(x), \qquad (x_1, \ldots, x_m) \in \mathbf{R}^m.$$

Thus marginal functions are a special case of infimal convolution.

In the other direction every infimal convolution is a marginal function, viz. the marginal function of the special function of $2n$ variables $(x, y) \mapsto f(x-y) \dotplus g(y)$ when y varies.

So the two operations are actually equivalent, however at the expense of going up in dimension when viewing infimal convolution as a marginal function.

Infimal convolution in turn is a case of Minkowski addition. Indeed,

$$\mathbf{epi}_{\mathrm{s}}^{\mathrm{F}}(f \sqcap g) = \mathbf{epi}_{\mathrm{s}}^{\mathrm{F}}(f) + \mathbf{epi}_{\mathrm{s}}^{\mathrm{F}}(g),$$

where $\mathbf{epi}_{\mathrm{s}}^{\mathrm{F}}(f)$ is the *strict finite epigraph* of f, defined as

$$\mathbf{epi}_{\mathrm{s}}^{\mathrm{F}}(f) = \{(x,t) \in \mathbf{R}^n \times \mathbf{R};\ t > f(x)\}, \qquad f\colon \mathbf{R}^n \to \mathbf{R}_!.$$

2. Other notions of discrete convexity

Several kinds of discrete convexity have been studied. Miller (1971: 168), introduced discretely convex functions for which local minima are global. These functions are not convex extensible—nor is the class closed under addition; see Murota & Shioura (2001:156, 161).

Two other concepts of convexity were introduced by Murota (1996; 1998). They are called M-convexity and L-convexity, respectively. For functions with either of these two properties, local minima are global. Two other classes of functions are obtained by a special restriction of M- and L-convex functions to a space of one dimension less. These functions are called $\mathrm{M}^\natural$-convex and $\mathrm{L}^\natural$-convex.[2] They were introduced by Murota & Shioura (1999:96) and Fujishige & Murota (2000:135), respectively. The class of $\mathrm{M}^\natural$-convex ($\mathrm{L}^\natural$-convex) functions properly contains the class of M-convex (L-convex) functions. These classes of functions have been studied with respect to some operations such as infimal convolution, addition, and addition by an affine function; see Murota & Shioura (2001). However, these classes are quite small (see Example 9.5).

[2]These expressions should be read, respectively, as “M-natural-convex” (Murota 2003:27, footnote 23), and “L-natural-convex” (Murota 2003:23, footnote 18). Here M stands for *matroid* and L for *lattice* (Murota 2003:xxi).

3. The convex hull and the convex envelope

Definition 3.1. The *convex hull* of a subset A of $\mathbf{R}^n$ is the smallest convex set containing A. It will be denoted by $\mathbf{cvxh}(A)$. □

Definition 3.2. The *convex envelope* of a function $f\colon A \to \mathbf{R}_!$, where A is any subset of $\mathbf{R}^n$, is the largest convex function $G\colon \mathbf{R}^n \to \mathbf{R}_!$ such that $G|_A \leqslant f$. We shall denote it by $\mathbf{cvxe}(f)$. □

The convex envelope is well defined because the supremum of all functions H which are convex and satisfy $H|_A \leqslant f$ has the same properties.

A function f is convex extensible if and only if $\mathbf{cvxe}(f)$ is an extension of f. Indeed, if f admits a convex extension, then also $\mathbf{cvxe}(f)$ is a convex extension. Equivalently, $\mathbf{cvxe}(f)|_A \geqslant f$.

4. The integer neighborhood and the canonical extension

Definition 4.1. We define the *integer neighborhood* of a real number a, denoted by $N(a)$, as the set $\{\lfloor a \rfloor, \lceil a \rceil\} \subset \mathbf{Z}$. We define the *integer neighborhood* of a point $a = (a_1, \dots, a_n) \in \mathbf{R}^n$ as the set

$$N(a) = N(a_1) \times \cdots \times N(a_n) \subset \mathbf{Z}^n. \qquad \square$$

The integer neighborhood has 2^k elements, where k is the number of indices j such that $a_j \in \mathbf{R} \smallsetminus \mathbf{Z}$. Equivalently,

$$N(a) = \big(a + B^\infty_<(0,1)\big) \cap \mathbf{Z}^n, \qquad a \in \mathbf{R}^n,$$

where $B^\infty_<(c,r)$ denotes the strict ball for the l^∞ norm with center at c and of radius r. The mapping

$$\mathbf{R}^n \supset A \mapsto \nu(A) = \bigcup_{a \in A} N(a) \subset \mathbf{Z}^n$$

is one of many digitizations of $\mathbf{R}^n$ and commutes with the formation of arbitrary unions, i.e.,

$$\nu\big(\textstyle\bigcup_{j\in J} A_j\big) = \displaystyle\bigcup_{j \in J} \nu(A_j), \qquad A_j \subset \mathbf{R}^n.$$

In mathematical morphology this is an important concept: a mapping with this property is said to be a *dilation.*

Definition 4.2. The *canonical extension* of a function $f\colon \mathbf{Z}^n \to \mathbf{R}_!$ is defined, for every $a \in \mathbf{R}^n$, as the value at a of the convex envelope of $f|_{N(a)}$, the restriction of f to the integer neighborhood of a. We shall denote it by $\mathbf{can}(f)\colon \mathbf{R}^n \to \mathbf{R}_!$. □

The canonical extension is actually an extension, since $N(a) = \{a\}$ for every $a \in \mathbf{Z}^n$.

Proposition 4.3. *For any function $f\colon \mathbf{Z}^n \to \mathbf{R}_!$, any point $a \in \mathbf{R}^n$, and any $p \in \mathbf{Z}^n$ such that the cube $p + [0,1]^n$ contains a, the value of the canonical extension at a is equal to the value at a of the convex envelope of $f|_{p+\{0,1\}^n}$.*

Proof. For brevity, let us denote by $C(p)$ the cube $p + [0,1]^n$ and by $V(p)$ its set of vertices, $p + \{0,1\}^n$, for any point p with integer coordinates.

If a point a belongs to only one cube $C(p)$, then $N(a) = V(p)$ and there is nothing to prove.

However, a point a may belong to two different cubes $C(p)$ and $C(q)$, $p, q \in \mathbf{Z}^n$, $p \neq q$. Then $N(a)$ is a subset of $V(p) \cap V(q)$. Since $N(a)$ is a subset of $V(p)$ if $a \in V(p)$, we have $\mathbf{cvxe}(f|_{N(a)}) \geqslant \mathbf{cvxe}(f|_{V(p)})$.

To prove the converse inequality, we define, given $a \in \mathbf{R}^n$ and $p \in \mathbf{Z}^n$ such that $a \in C(p)$, two sets of indices

$$J_k = \{j \in [1,n]_{\mathbf{Z}};\ a_j = p_j + k\}, \qquad k = 0, 1,$$

and an affine function

$$G(x) = \sum_{j \in J_0} (x_j - p_j) + \sum_{j \in J_1} (p_j + 1 - x_j), \qquad x \in \mathbf{R}^n.$$

If both J_0 and J_1 are empty, then G is identically zero and $C(p)$ is the only cube to which a belongs. We now assume that this is not the case. Then the zero set of G is a hyperplane $Y(a,p)$ in $\mathbf{R}^n$. Obviously G is nonnegative in the cube $C(p)$, and $Y(a,p)$ is a supporting hyperplane of this cube. In general a supporting hyperplane intersects $V(p)$ in a set which contains other vertices than those in $N(a)$, but in view of our construction, the hyperplane has the important property that

$$Y(a,p) \cap V(p) = N(a).$$

This implies that any convex combination of points in $V(p)$ yielding a point in $Y(a,p)$ is already a convex combination of points in $N(a)$.

This proves that the convex envelope of $f|_{V(p)}$ and the convex envelope of $f|_{N(a)}$ have the same value at a. We are done. □

Definition 4.4. We shall say with Favati and Tardella (1990:9), that a function $f\colon \mathbf{Z}^n \to \mathbf{R}_!$ is *integrally convex* if $\mathbf{can}(f)\colon \mathbf{R}^n \to \mathbf{R}_!$ is convex. □

We always have $\mathbf{cvxe}(f) \leqslant \mathbf{can}(f)$ with equality if and only if f is integrally convex. Every integrally convex function is convex extensible, since for such a function, $\mathbf{can}(f)$ is a convex extension.

5. Convolution and convex extensibility

The *convolution product* $f * g$ of two functions $f, g\colon \mathbf{Z}^n \to \mathbf{R}$ is defined by

$$(f * g)(x) = \sum_{y \in \mathbf{Z}^n} f(x-y)g(y), \qquad x \in \mathbf{Z}^n, \tag{5.1}$$

assuming some kind of convergence.

We define for $p = (p^{(1)}, \dots, p^{(k)}) \in (\mathbf{R}^n)^k$ and $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbf{R}^k$ satisfying $\lambda_j \geqslant 0$, $\sum_{j=1}^k \lambda_j = 1$, and $\sum_{j=1}^k \lambda_j p^{(j)} = 0$,

$$\mu_{p,\lambda} = \sum_{j=1}^k \lambda_j \delta_{p^{(j)}}.$$

Here δ_a denotes the *Kronecker delta* placed at a, defined by $\delta_a(a) = 1$ and $\delta_a(x) = 0$ when $x \neq a$. In particular, δ_0 is a neutral element: $f * \delta_0 = f$ for all functions f.

The convex envelope of a function defined on a subset A of $\mathbf{R}^n$ is given by

$$\mathbf{cvxe}(f)(x) = \inf_{p,\lambda} (\mu_{p,\lambda} * f)(x) = \inf_{p,\lambda} \sum_{y \in A} \mu_{p,\lambda}(x-y)f(y), \qquad x \in \mathbf{R}^n.$$

(In view of Carathéodory's theorem it suffices to take $k = n+1$.)

This implies that convex extensibility of a function f defined on a subset A of $\mathbf{R}^n$ can be characterized by means of an infinite family of convolution operators, viz.

$$((\mu_{p,\lambda} - \delta_0) * f)(x) = \sum_{y \in A} \mu_{p,\lambda}(x-y)f(y) - f(x) \geqslant 0, \qquad x \in A,$$

for all p and λ of the kind mentioned.

When $n = 1$, the convex-extensible functions are those which satisfy the inequality $(\mu_{p,\lambda} - \delta_0) * f \geqslant 0$ for $p = (-1, 1)$ and $\lambda = (\frac{1}{2}, \frac{1}{2})$, thus defining the class using a single convolution operator.

6. Lateral convexity: Definition

The following definition extends that for two variables in Kiselman (2008: Definition 2.1); cf. Theorem 2.4 there. See also Kiselman (2011).

Definition 6.1. Given $a \in \mathbf{R}^n$, we define a difference operator $D_a \colon \mathbf{R}^{\mathbf{R}^n} \to \mathbf{R}^{\mathbf{R}^n}$ by

$$(D_aF)(x) = F(x+a)-F(x), \qquad x \in \mathbf{R}^n,\ F \in \mathbf{R}^{\mathbf{R}^n}. \tag{6.1}$$ □

If $a \in \mathbf{Z}^n$, D_a operates from $\mathbf{R}^{\mathbf{Z}^n}$ to $\mathbf{R}^{\mathbf{Z}^n}$ and from $\mathbf{Z}^{\mathbf{Z}^n}$ to $\mathbf{Z}^{\mathbf{Z}^n}$. In particular, $D_{e^{(j)}}$, where $e^{(j)}$ is the vector $(0, 0, \ldots, 1, \ldots, 0)$ with 1 at the j^{th} place, is the difference operator in the j^{th} coordinate.

The operator $f \mapsto D_af$ is a convolution operator: $D_af = \mu_a * f$ with $\mu_a = \delta_{-a} - \delta_0$. The composition of D_a and D_b is the convolution operator given by $D_bD_af = (\mu_b * \mu_a) * f$ with $\mu_b * \mu_a = \delta_{-a-b} - \delta_{-a} - \delta_{-b} + \delta_0$.

The following definition generalizes several definitions used to define discrete convexity. As will be shown, it is highly relevant for problems concerning marginal functions.

Definition 6.2. Given a set $A \subset \mathbf{Z}^n \times \mathbf{Z}^n$, we shall say that a function $f \colon \mathbf{Z}^n \to \mathbf{R}$ is *A-laterally convex* if

$$(D_bD_af)(x) \geqslant 0, \qquad x \in \mathbf{Z}^n, \quad (a, b) \in A. \tag{6.2}$$

We define $\Phi(A)$ as the set of all A-laterally convex functions.

In the other direction, given any subset F of $\mathbf{R}^{\mathbf{Z}^n}$, we define $\Psi(F)$ as the set of all pairs $(a, b) \in \mathbf{Z}^n \times \mathbf{Z}^n$ such that $D_bD_af \geqslant 0$ for all $f \in F$. □

7. Lateral convexity: Morphological aspects

The notions of mathematical morphology are very helpful when it comes to understanding lateral convexity.

The mappings Φ and Ψ are decreasing and $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are larger than the respective identity mappings. One expresses this fact by saying that the pair of mappings (Φ, Ψ) forms a *Galois connection*. This fact can also be expressed using the concept of the lower inverse of a mapping between ordered sets; see Kiselman (2010b: Subsection 4.1).

We define $\widetilde{A} = \Psi(\Phi(A))$ for any subset of $\mathbf{Z}^n \times \mathbf{Z}^n$. It is well known from Galois theory and easy to see that the operation $A \mapsto \widetilde{A}$ is increasing and idempotent, thus an *ethmomorphism* (a morphological filter). It is also larger than the identity, and so it is a *cleistomorphism* (a closure operator).

If a function is A-laterally convex, it is automatically $\widetilde{A}$-laterally convex; any set B satisfying $A \subset B \subset \widetilde{A}$ defines the same class of functions.

From the definition it is obvious that the class of A-laterally convex functions is closed under addition and multiplication by a nonnegative scalar. From the formulas

$$(D_{-a}f)(x) = -(D_a f)(x-a), \qquad (D_{-b}D_{-a}f)(x) = (D_b D_a f)(x-a-b)$$

it follows that

$$-A = \{(-a,-b); (a,b) \in A\}$$

is contained in $\widetilde{A}$. The same is true of

$$A^{\smile} = \{(b,a); (a,b) \in A\}.$$

We define

$$A^{\mathrm{sym}} = A \cup (-A) \cup A^{\smile} \cup (-A)^{\smile},$$

which may have up to four times as many elements as A but still defines the same class, i.e., $\Phi(A^{\mathrm{sym}}) = \Phi(A)$.

The formula

$$D_b D_{-a} f(x) = -D_b D_a f(x-a)$$

shows that f is $\{(-a,b)\}$-laterally convex if and only if $-f$ is $\{(a,b)\}$-laterally convex. So the concepts introduced will enable us to study also A-laterally concave functions and A-laterally affine functions.

The formula

$$(D_b f)(x) + (D_c f)(x+b) = (D_{b+c} f)(x)$$

applied to $D_a f$ yields

$$(D_b D_a f)(x) + (D_c D_a f)(x+b) = (D_{b+c} D_a f)(x), \tag{7.1}$$

which implies that if $D_b D_a f \geqslant 0$ and $D_c D_a f \geqslant 0$, then we also have $D_{b+c} D_a f \geqslant 0$. This means that the set of pairs $\{(a,b) \in \mathbf{Z}^n \times \mathbf{Z}^n\}$ such that the inequality holds is closed under *partial addition:*

$$(a,b) +_2 (a,c) = (a, b+c), \tag{7.2}$$

i.e., if the first elements agree, we may add the second elements. For sets we define

$$B +_2 C = \{(a, b+c); (a,b) \in B, (a,c) \in C\}.$$

Similarly we can define of course

$$(a,b) +_1 (c,b) = (a+c, b) \tag{7.3}$$

and

$$B +_1 C = \{(a+c, b); (a,b) \in B, (c,b) \in C\}$$

when the two second elements are the same.

By repeated use of these formulas we see that $\widetilde{A}$ contains the sets $A^{\mathrm{sym}} +_1 A^{\mathrm{sym}}$, $A^{\mathrm{sym}} +_2 (A^{\mathrm{sym}} +_1 A^{\mathrm{sym}})$ and so on.

We sum up the discussion on $\widetilde{A}$ in the following lemma.

Lemma 7.1. *Let A be any subset of $\mathbf{Z}^n \times \mathbf{Z}^n$ and define $\widetilde{A} = \Psi(\Phi(A))$.*

1. *For any $a \in \mathbf{Z}^n$, $(a, \mathbf{0})$ and $(\mathbf{0}, a)$ belong to $\widetilde{A}$.*
2. *If $(a,b) \in \widetilde{A}$, then $(b,a), (-a,-b), (-b,-a)$ all belong to $\widetilde{A}$.*
3. *If $(a,b), (c,b) \in \widetilde{A}$, then $(a,b) +_1 (c,b) = (a+c, b)$ belongs to $\widetilde{A}$.*
4. *If $(a,b), (a,c) \in \widetilde{A}$, then $(a,b) +_2 (a,c) = (a, b+c)$ belongs to $\widetilde{A}$.*
5. *For any given set F of functions $\mathbf{Z}^n \to \mathbf{R}$, if $\Psi(F)$ contains a set A, it also contains $\widetilde{A}$.* □

When $n = 1$ and $A = \{(1,1)\}$, f is A-laterally convex if and only if it is convex extensible. As already mentioned, this is the only reasonable definition of convexity in one integer variable. We note that it is equivalent to B-lateral convexity for any B such that

$$(1,1) \in B \subset \widetilde{A} \text{ or } (-1,-1) \in B \subset \widetilde{A}.$$

In this case, $\widetilde{A}$ is easy to determine: It is equal to

$$\{(s,t) \in \mathbf{Z} \times \mathbf{Z}; st \geqslant 0\}.$$

More generally, for any n and any $j \in [1, n]_{\mathbf{Z}}$, if $A = \{(e^{(j)}, e^{(j)})\}$, then a function is A-laterally convex if and only if it is convex extensible in the variable x_j when the others are kept fixed. Since this is a convenient property, we shall normally require that

$$(7.4) \qquad (e^{(j)}, e^{(j)}) \in A, \qquad j = 1, \dots, n.$$

If this is so, all A-laterally convex functions are $\{(1,1)\}$-laterally convex in each variable when the others are kept fixed.

8. Lateral convexity: Examples

Example 8.1. If f is the restriction to $\mathbf{Z}^n$ of a polynomial of degree at most two,

$$f(x) = \alpha + \sum_{j=1}^{n} \beta_j x_j + \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{jk} x_j x_k, \qquad x \in \mathbf{Z}^n,$$

with $\gamma_{jk} = \gamma_{kj}$, we see that

$$(D_b D_a f)(x) = 2 \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{jk} a_j b_k,$$

so that f is A-laterally convex if and only if the last expression is nonnegative for all $(a, b) \in A$.

In particular, the restriction to $\mathbf{Z}^n$ of an arbitrary affine function is A-laterally convex.

We also see that the special polynomial $f(x) = x_j^2$ is A-laterally convex if and only if $a_j b_j \geqslant 0$ for all $(a, b) \in A$. Conversely, if $a_j b_j \geqslant 0$ and g is any convex-extensible function of one variable, then the function $x \mapsto g(x_j)$ is $\{(a, b)\}$-laterally convex. $\square$

In view of this example we shall normally require that

$$(8.1) \qquad (a, b) \in A \text{ implies } a_j b_j \geqslant 0, \qquad j = 1, \dots, n.$$

Example 8.2. A special kind of laterally convex functions are the L$^\natural$-convex functions, which are defined by Murota in (2003: 1.33) by the property

$$(8.2) \quad f\left(\left\lfloor \tfrac{1}{2}x + \tfrac{1}{2}y \right\rfloor\right) + f\left(\left\lceil \tfrac{1}{2}x + \tfrac{1}{2}y \right\rceil\right) \leqslant f(x) + f(y), \qquad x, y \in \mathbf{Z}^n.$$

A function $f\colon \mathbf{Z}^n \to \mathbf{R}$ is L$^\natural$-convex if and only if it is Λ-laterally convex with

$$\Lambda = \{(a,b) \in \mathbf{Z}^n \times \mathbf{Z}^n;\ b-a \in \{0,1\}^n \cup \{-1,0\}^n\}.$$

So, in all dimensions, L$^\natural$-convexity is a special case of lateral convexity.

We shall prove first that Λ-lateral convexity implies L$^\natural$-convexity. Let x and y be given and define

$$a = \left\lfloor \tfrac{1}{2}x + \tfrac{1}{2}y \right\rfloor - x \text{ and } b = \left\lceil \tfrac{1}{2}x + \tfrac{1}{2}y \right\rceil - x.$$

Note that $b - a \in \{0,1\}^n \subset \Lambda$. Then $x + a + b = y$, so that, if f is Λ-laterally convex, we obtain $f(\left\lfloor \frac{1}{2}x + \frac{1}{2}y \right\rfloor) + f(\left\lceil \frac{1}{2}x + \frac{1}{2}y \right\rceil) = f(x+a) + f(x+b) \leqslant f(x) + f(x+a+b) = f(x) + f(y)$, proving (8.2).

Next we shall see that L$^\natural$ convexity implies Λ-lateral convexity. If x, a and b are given with $b - a \in \{0,1\}^n \subset \Lambda$, we define $y = x + a + b$. Then $\left\lfloor \frac{1}{2}x + \frac{1}{2}y \right\rfloor = x + a$ and $\left\lceil \frac{1}{2}x + \frac{1}{2}y \right\rceil = x + b$ so that, if f is L$^\natural$-convex, we get $f(x+a) + f(x+b) = f(\left\lfloor \frac{1}{2}x + \frac{1}{2}y \right\rfloor) + f(\left\lceil \frac{1}{2}x + \frac{1}{2}y \right\rceil) \leqslant f(x) + f(y) = f(x) + f(x+a+b)$. If instead $b - a \in \{-1,0\}^n$, we interchange a and b. This shows the implication. □

9. Two variables: rhomboidal convexity

Let us see what Definition 6.2 means for functions of two variables.

Definition 9.1. We shall say that a function $f\colon \mathbf{Z}^2 \to \mathbf{R}$ is *rhomboidally convex* if it is P-laterally convex, where we define $P \subset \mathbf{Z}^2 \times \mathbf{Z}^2$ as

(9.1)
$$P = \{((1,0),(1,t)); t \in [-1,1]_{\mathbf{Z}}\} \cup \{((0,1),(s,1)); s \in [-1,1]_{\mathbf{Z}}\}. \quad \square$$

Given a function f, we consider the set $\Psi(\{f\})$ of all pairs $(a,b) \in \mathbf{Z}^2 \times \mathbf{Z}^2$ such that $D_b D_a f \geqslant 0$. Then we have to take into account several conditions, e.g., the two *one-variable conditions*

$$(e^{(1)}, e^{(1)}),\ (e^{(2)}, e^{(2)}) \in \Psi(\{f\}) \tag{9.2}$$

(which we usually require in order to avoid uninteresting cases—see (7.4)); the two *diagonal conditions*

$$((-1,1),(-1,1)),\ ((1,1),(1,1)) \in \Psi(\{f\}); \tag{9.3}$$

the *left* and *right horizontal lozenge conditions*[3]

$$((-1,0),(-1,1)),\ ((1,0),(1,1)) \in \Psi(\{f\}); \tag{9.4}$$

and finally the *left* and *right vertical lozenge conditions*,

$$((0,1),(-1,1)),\ ((0,1),(1,1)) \in \Psi(\{f\}). \tag{9.5}$$

We note that, by partial addition, $((1,0),(1,1)) +_1 ((0,1),(1,1)) = ((1,1),(1,1))$, which implies that the right horizontal lozenge condition and the right vertical lozenge condition yield the diagonal condition for $((1,1),(1,1))$. Thus we often do not need to consider the diagonal conditions.

To see which conditions are necessary for the marginal function to be convex extensible, it is instructive to look at the following examples.

Example 9.2. Let f be the function in Example 1.6 with $m = 1$. It does not satisfy $D_{(1,1)}D_{(1,0)}f(0,0) \geqslant 0$, which explains that $\frac{1}{2}h(0) + \frac{1}{2}h(2) = 0$ does not majorize $h(1) = 1$. It does satisfy all other conditions (9.2)–(9.5), i.e., it satisfies seven of the eight conditions, the only exception being the right horizontal lozenge condition $D_{(1,1)}D_{(1,0)}f \geqslant 0$. □

Example 9.3. Let now f be the function defined as

$$f(x,y) = |3x - 2y|, \qquad (x,y) \in \mathbf{Z}^2.$$

It does not satisfy $D_{(1,1)}D_{(0,1)}f(0,0) \geqslant 0$, the right vertical lozenge condition. It does satisfy all other conditions (9.2)–(9.5), i.e., it satisfies seven of the eight conditions, the only exception being the right vertical lozenge condition $D_{(1,1)}D_{(0,1)}f \geqslant 0$. Its marginal function takes the value 0 at even integers and 1 at odd integers, and is thus not convex extensible. □

By forming similar examples we can conclude that for the marginal function to be convex extensible, each of the four lozenge conditions (9.4) and (9.5) is necessary, even in the presence of the other three lozenge conditions, the two one-variable conditions, and the two diagonal conditions. So we conclude that all four lozenge conditions are needed, but that we can then omit the two diagonal conditions: We need six conditions for the marginal function to be convex extensible.

[3] We are aware that *lozenge* and *rhombus* are considered to be synonyms, but we are brave enough to call a set like $\mathbf{cvxh}\{(0,0),(1,0),(1,1),(2,1)\}$ a lozenge, although its sides have Euclidean lengths 1 and $\sqrt{2}$. However, their l^∞ lengths are all equal, so it is actually a rhombus as well as a lozenge for the l^∞ metric.

Example 9.4. When $n = 2$ and $A = \{(e^{(1)}, e^{(2)})\}$, a function is A-laterally convex if and only if it is submodular. Note that (7.4) is not satisfied in this case. (Cf. Murota (2003:26, 206–207).) □

Example 9.5. Since, for $n = 2$, $\Lambda \supset P$ (see Example 8.2), we have $\Phi(\Lambda) \subset \Phi(P)$, i.e., every $\mathrm{L}^\natural$-convex function is rhomboidally convex. In fact, the $\mathrm{L}^\natural$-convex functions form a tiny fraction of the rhomboidally convex functions. To illustrate this fact, let us mention that a function $f(x_1, x_2) = g(x_1 + x_2)$, $(x_1, x_2) \in \mathbf{Z}^2$, is $\mathrm{L}^\natural$-convex if and only if $g\colon \mathbf{Z} \to \mathbf{R}$ is the restriction to $\mathbf{Z}$ of an affine function defined on $\mathbf{R}$, while it is rhomboidally convex if and only if g is convex extensible. (If $f(x_1, x_2) = h(x_1 - x_2)$, the situation is quite different.) □

Proposition 9.6. *Consider the following conditions on a function* $f\colon \mathbf{Z}^2 \to \mathbf{R}$.

(A). *f is rhomboidally convex;*

(B). *f is integrally convex;*

(C). *f is convex extensible;*

(D). *The restriction of f to any digital line $\{c+ta;\ t \in \mathbf{Z}\}$, $c, a \in \mathbf{Z}^2$, is convex extensible.*

Then (A) $\Leftrightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D), *and, in general,* (B) $\not\Leftarrow$ (C) $\not\Leftarrow$ (D).

Proof. (A) $\Leftrightarrow$ (B). See Kiselman (2008: Theorem 2.4).

(B) $\Rightarrow$ (C). See the comment after Definition 4.4.

(C) $\Rightarrow$ (D). If F is a convex extension of f, then $D_a D_a F \geqslant 0$ for all $a \in \mathbf{R}^2$. In particular $D_a D_a f \geqslant 0$ for all $a \in \mathbf{Z}^2$.

(B) $\not\Leftarrow$ (C). Example 1.6 with $m = 1$ shows this. Here $\mathbf{can}(f)(x, \frac{1}{2})$ takes the values $1, \frac{1}{2}, 1, \frac{1}{2}, 1$ for $x = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, respectively, so $\mathbf{can}(f)$ is not convex.

(C) $\not\Leftarrow$ (D). Define

$$G(x, y) = (2y - x - 1)^+ \vee (2x - y - 1)^+ \vee (-x - y - 1)^+, \qquad (x, y) \in \mathbf{R}^2.$$

Here $s \vee t = \max(s, t)$ denotes the maximum of two numbers s and t, and $t^+ = t \vee 0$.

The function G is certainly convex, so its restriction $g = G|_{\mathbf{Z}^2}$ is convex extensible. Now define $f(x, y) = g(x, y)$ for $(x, y) \neq (0, 0)$ and

$f(0,0) = g(0,0) + \frac{1}{2} = \frac{1}{2}$. For f to satisfy $D_a D_a f \geqslant 0$ it is enough to consider $D_a D_a f$ on a digital line

$$L = \{ta;\ t \in \mathbf{Z}\} = \{t(p,q);\ t \in \mathbf{Z}\}$$

which passes through the origin, since we have changed the value of g only at the origin. It is sufficient to prove that $\frac{1}{2}f(p,q)+\frac{1}{2}f(-p,-q) \geqslant f(0,0) = \frac{1}{2}$ for two relatively prime integers p, q, since the points (p,q) and $(-p,-q)$ are the integer points closest to the origin on L. We see that

$$f(p,q) \vee f(-p,-q) \leqslant f(p,q) + f(-p,-q) < 1$$

only if

$$|2p-q| < 2,\ |2q-p| < 2,\ |p+q| < 2.$$

This happens only if $(p,q) = (0,0)$. So the restriction of f to L is convex extensible, but f is not convex extensible. Indeed, the origin is the barycenter of the three points $(1,1)$, $(-1,0)$, $(0,-1)$:

$$(0,0) = \tfrac{1}{3}(1,1) + \tfrac{1}{3}(-1,0) + \tfrac{1}{3}(0,-1),$$

but

$$f(0,0) = \tfrac{1}{2} > \tfrac{1}{3}f(1,1) + \tfrac{1}{3}f(-1,0) + \tfrac{1}{3}f(0,-1) = 0,$$

so Jensen's inequality is not satisfied. □

10. The set where the infimum is attained

We shall first study the relation between A-lateral convexity and the interval (possibly empty) where the infimum defining the marginal function is attained.

Theorem 10.1. *Let us define, for any function $f\colon \mathbf{Z}^n \to \mathbf{R}$,*

$$\begin{aligned} M_f(x_1,\dots,x_{n-1}) &= M_f(x') \\ &= \{b \in \mathbf{Z};\ f(x_1,\dots,x_{n-1},b) = \inf_{t\in\mathbf{Z}} f(x_1,\dots,x_{n-1},t)\}, \end{aligned}$$

where $x' = (x_1,\dots,x_{n-1}) \in \mathbf{Z}^{n-1}$. We also define

$$f_\beta(x) = f(x) - \beta x_n, \qquad x = (x_1,\dots,x_n) \in \mathbf{Z}^n, \quad \beta \in \mathbf{R}.$$

Now fix an element $a = (a', a_n)$ of $\mathbf{Z}^n$, where $a' = (a_1,\dots,a_{n-1})$ and $a_n \geqslant 0$, and define

$$A = \{(e^{(n)}, e^{(n)}), ((a', a_n), e^{(n)}), ((-a', a_n), e^{(n)})\},$$

a subset of $(\mathbf{R}^n)^2$ *with three elements. Then* f *is* A*-laterally convex if and only if* $t \mapsto f(x', t)$ *is convex extensible for every* x' *and a certain Lipschitz property holds:*

$$M_{f_\beta}(x'+a') \subset M_{f_\beta}(x')+[-a_n, a_n]_{\mathbf{Z}}, \qquad x' \in \mathbf{Z}^{n-1}, \quad \beta \in \mathbf{R}. \tag{10.1}$$

Proof. Assume first that f is A-laterally convex. Since A contains $(e^{(n)}, e^{(n)})$, $\mathbf{Z} \ni t \mapsto f(x', t)$ is convex extensible for every x'.

We note that for a function which is convex extensible in the last variable,

$$b \in M_f(x') \text{ if and only if } D_{e^{(n)}} f(x', b-1) \leqslant 0 \leqslant D_{e^{(n)}} f(x', b). \tag{10.2}$$

Moreover

$$b, b+1 \in M_f(x') \text{ if and only if } D_{e^{(n)}} f(x', b) = 0. \tag{10.3}$$

Let now f satisfy $D_a D_{e^{(n)}} f \geqslant 0$ and consider two points x' and $x' + a'$ in $\mathbf{Z}^{n-1}$. Then for any $b \in M_f(x')$ we have, since also $((-a', a_n), e^{(n)})$ is in A,

$$\begin{aligned} D_{e^{(n)}} f(x'+a', b-a_n-1) \leqslant D_{e^{(n)}} f(x', b-1) \leqslant 0 \\ \leqslant D_{e^{(n)}} f(x', b) \leqslant D_{e^{(n)}} f(x'+a', b+a_n), \end{aligned} \tag{10.4}$$

which implies that there is a point $c \in [b-a_n, b+a_n]_{\mathbf{Z}}$ with

$$D_{e^{(n)}} f(x'+a', c-1) \leqslant 0 \leqslant D_{e^{(n)}} f(x'+a', c).$$

In view of (10.2), this means that $c \in M_f(x'+a')$. We have proved that $b \in c + [-a_n, a_n]_{\mathbf{Z}} \subset M_f(x'+a') + [-a_n, a_n]_{\mathbf{Z}}$, and, since b was any point in $M_f(x')$, that $M_f(x') \subset M_f(x'+a')+[-a_n, a_n]_{\mathbf{Z}}$. We are done, since the whole argument holds also for f_β.

Conversely, suppose that the function f satisfies $D_{e^{(n)}} D_{e^{(n)}} f \geqslant 0$ but is not A-laterally convex. Then it does not satisfy one of the two inequalities $D_a D_{e^{(n)}} f \geqslant 0$ and $D_{(-a', a_n)} D_{e^{(n)}} f \geqslant 0$. It suffices to consider one of these cases. We thus assume that there exist $(x', b) \in \mathbf{Z}^{n-1} \times \mathbf{Z}$ such that $D_{e^{(n)}} f(x'+a', b+a_n) < D_{e^{(n)}} f(x', b)$. We shall reach a contradiction to the Lipschitz property (10.1).

We take a real number β such that

$$D_{e^{(n)}} f(x'+a', b+a_n) < \beta < D_{e^{(n)}} f(x', b).$$

If we rewrite this for the function f_β, for which $D_{e^{(n)}} f_\beta = D_{e^{(n)}} f - \beta$, we obtain

$$D_{e^{(n)}} f_\beta(x' + a', b + a_n) < 0 < D_{e^{(n)}} f_\beta(x', b), \tag{10.5}$$

which implies that

$$M_{f_\beta}(x' + a') \subset [b + a_n + 1, +\infty[_{\mathbf{Z}} \text{ and that } M_{f_\beta}(x') \subset]-\infty, b]_{\mathbf{Z}}\,.$$

Hence

$$M_{f_\beta}(x' + a') + [-a_n, a_n]_{\mathbf{Z}} \subset [b + 1, +\infty[_{\mathbf{Z}}$$

and

$$M_{f_\beta}(x') + [-a_n, a_n]_{\mathbf{Z}} \subset]-\infty, b + a_n]_{\mathbf{Z}}\,.$$

Thus $M_{f_\beta}(x' + a')$ is not contained in $M_{f_\beta}(x') + [-a_n, a_n]_{\mathbf{Z}}$ unless it is empty, and $M_{f_\beta}(x')$ is not contained in $M_{f_\beta}(x' + a') + [-a_n, a_n]_{\mathbf{Z}}$ unless it is empty. As soon as one of them is nonempty, we get a contradiction to the Lipschitz property (10.1).

So the case when both sets are empty remains to be considered—so far, there is no contradiction in this case. Since $M_{f_\beta}(x'+a')$ is now empty by hypothesis, the function $t \mapsto D_{e^{(n)}} f_\beta(x' + a', t)$ can never change sign, and since $D_{e^{(n)}} f_\beta(x'+a', b+a_n)$ is negative, we must have $D_{e^{(n)}} f_\beta(x' + a', t) < 0$ for all $t \in \mathbf{Z}$. Now define $\gamma = D_{e^{(n)}} f(x', b) > \beta$. Then, by (10.3), $M_{f_\gamma}(x')$ is certainly nonempty; it contains b and $b+1$. And since $\gamma > \beta$ we have

$$D_{e^{(n)}} f_\gamma(x'+a', t) = D_{e^{(n)}} f_\beta(x'+a', t) + \beta - \gamma < D_{e^{(n)}} f_\beta(x'+a', t) < 0$$

for all $t \in \mathbf{Z}$, so that (10.2) shows that $M_{f_\gamma}(x' + a')$ is empty. This contradicts the inclusion $M_{f_\gamma}(x') \subset M_{f_\gamma}(x'+a') + [-a_n, a_n]_{\mathbf{Z}}$. We are done. $\square$

By permuting the variables we easily obtain the following corollary.

Corollary 10.2. *Given a function* $f\colon \mathbf{Z}^n \to \mathbf{R}$, *we define, for* $1 \leqslant j \leqslant n$ *and* $x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbf{Z}^{n-1}$,

$$\begin{aligned} M_{j,f}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}) &= M_{j,f}(x') \\ &= \{b \in \mathbf{Z};\, f(x_1, \dots, x_{j-1}, b, x_{j+1}, \dots, x_{n-1}, x_n) = \inf\nolimits_{x_j \in \mathbf{Z}} f(x)\}. \end{aligned}$$

We also define

$$f_{j,\beta}(x) = f(x) - \beta x_j,\ \ x = (x_1, \dots, x_n) \in \mathbf{Z}^n,\ \ j = 1, \dots, n,\ \ \beta \in \mathbf{R}.$$

Fix a set A *which contains* $(a, e^{(j)})$ *and* $(\bar{a}, e^{(j)})$*, where*

$$\bar{a} = 2a_j e^{(j)} - a = (-a_1, \dots, a_j, \dots, -a_n),$$

and satisfies (7.4) *and* (8.1)*. If* f *is* A*-laterally convex, then* f *is convex extensible in each variable separately and we have*

$$M_{j,f_{j,\beta}}(x' + a') \subset M_{j,f_{j,\beta}}(x') + [-a_j, a_j]_{\mathbf{Z}}, \qquad x' \in \mathbf{Z}^{n-1},$$

where now $a' = (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n)$ *and similarly for* x'*.* □

11. Lateral convexity of marginal functions

11.1. Arbitrary dimensions

In Kiselman (2008: Theorem 3.1), it was shown that for integrally convex functions of two integer variables, the marginal function is convex extensible. We shall now study the marginal function of A-laterally convex functions in arbitrary dimension and for more general choices of A.

Theorem 11.1. *Let* $A \subset \mathbf{Z}^{n-1} \times \mathbf{Z}^{n-1}$ *and* $B \subset \mathbf{Z}^n \times \mathbf{Z}^n$ *be given. We assume that* (7.4) *and* (8.1) *hold for both* A *and* B*. Assume also that*

(11.1)

If $(a, b) \in A$ *and* $s \in [-1, 1]_{\mathbf{Z}}$*, then* $((a, s), (b, 0))$ *belongs to* $\widetilde{B}$;

that

(11.2)

If there exists $c \in \mathbf{Z}^{n-1}$ *such that* $(a, c) \in A$*, then* $((a, 1), e^{(n)}) \in \widetilde{B}$;

and finally that

(11.3) *If* $((a, 1), e^{(n)}) \in B$*, then* $((-a, 1), e^{(n)}) \in \widetilde{B}$.

If $f\colon \mathbf{Z}^n \to \mathbf{R}$ *is* B*-laterally convex, then its marginal function*

$$h(x) = \inf_{t \in \mathbf{Z}} f(x, t), \qquad x \in \mathbf{Z}^{n-1},$$

is A*-laterally convex, provided that it does not take the value* $-\infty$*.*

Lemma 11.2. *Let* A *and* B *satisfy the hypotheses in Theorem 11.1. Then*

(11.4) *If* $(a, b) \in A$*, then* $((a, -1), (b, -1)), ((a, 1), (b, 1)) \in \widetilde{B}$.

Proof. From the conditions (11.1) and (11.2) we know that

$$\text{both } ((a,1),(b,0)) \text{ and } ((a,1),e^{(n)}) \text{ belong to } \widetilde{B}.$$

By partial addition $+_2$ we conclude that so does $((a,1),(b,1))$.

From the condition (11.2) we know that $((a,1),e^{(n)})$ and, consequently, in view of (11.3), also $((-a,1),e^{(n)})$ belongs to $\widetilde{B}$. So does the opposite pair $-((-a,1),e^{(n)}) = ((a,-1),-e^{(n)})$.

By condition (11.1) we find that $((a,-1),(b,0))$ is in $\widetilde{B}$, and we now only have to form the partial sum

$$((a,-1),-e^{(n)}) +_2 ((a,-1),(b,0)) = ((a,-1),(b,-1))$$

to conclude. □

By this lemma and (11.1) we know that if A and B satisfy the hypotheses of the theorem and if $(a,b) \in A$, then there are pairs of the form $((a,s),(b,t))$ in $\widetilde{B}$ with $-1 \leqslant s,t \leqslant 1$ and the sum $s+t$ taking any of the five values $-2,-1,0,1,2$.

Proof of Theorem 11.1. It is enough to prove the theorem for functions such that the infimum defining h is attained at a unique point. Indeed, if $t \mapsto f(x,t)$ is convex extensible, then for any positive $\varepsilon > 0$, the infimum defining the marginal function h_ε of $f_\varepsilon(x,t) = f(x,t) + \varepsilon t^2$ is attained at a unique integer $t = \varphi_\varepsilon(x)$, and h_ε tends to h as $\varepsilon \to 0$, preserving the A-lateral convexity of h_ε. We observe that f_ε is B-laterally convex with f provided that $(e^{(n)}, e^{(n)}) \in B$, which we assume. We may therefore suppose that $h(x) = f(x,\varphi(x))$ for some function $\varphi\colon \mathbf{Z}^{n-1} \to \mathbf{Z}$. Moreover, we know that φ is Lipschitz in the sense that

$$(11.5) \qquad |\varphi(x+a) - \varphi(x)| \leqslant 1, \qquad x \in \mathbf{Z}^{n-1},$$

for certain values of $a \in \mathbf{Z}^{n-1}$, viz. when $((a,1),e^{(n)})$ and $((-a,1),e^{(n)})$ both belong to $\widetilde{B}$. For this to happen, it is enough that there exists a c such that $(a,c) \in A$.

Similarly, we know that

$$(11.6) \qquad |\varphi(x+b) - \varphi(x)| \leqslant 1 \qquad x \in \mathbf{Z}^{n-1},$$

for certain values of $b \in \mathbf{Z}^{n-1}$, viz. when $((b,1),e^{(n)})$ and $((-b,1),e^{(n)})$ both belong to $\widetilde{B}$. For this it is enough that there exists a d such that $(b,d) \in A$.

In particular, if (a,b) is in A, we can take $c=b$ and $d=a$ above to conclude that the two Lipschitz conditions (11.5) and (11.6) hold.

We have

$$\begin{aligned} D_bD_ah(x) &= f(x+a+b,\varphi(x+a+b)) \\ &\quad -f(x+a,\varphi(x+a)) - f(x+b,\varphi(x+b)) + f(x,\varphi(x)). \end{aligned} \tag{11.7}$$

The formula holds of course for all $x,a,b \in \mathbf{Z}^{n-1}$, but we shall need it only when $(a,b)\in A$. We shall compare (11.7) with

$$\begin{aligned} D_{(b,t)}D_{(a,s)}f(x,\varphi(x)) &= f(x+a+b,\varphi(x)+s+t) \\ &\quad -f(x+a,\varphi(x)+s) - f(x+b,\varphi(x)+t) + f(x,\varphi(x)) \end{aligned} \tag{11.8}$$

for suitable s and t. This expression is nonnegative if $((a,s),(b,t)) \in \widetilde{B}$.

By the definition of φ we have

$$-f(x+a,\varphi(x+a)) \geqslant -f(x+a,s) \text{ and } -f(x+b,\varphi(x+b)) \geqslant -f(x+b,t)$$

for any s and t, so we get $D_bD_ah(x) \geqslant D_{(b,t)}D_{(a,s)}f(x,\varphi(x))$ as soon as $s+t=\varphi(x+a+b)-\varphi(x)$.

In view of (11.5) and (11.6), which, as we have remarked, are applicable,

$$|\varphi(x+a+b)-\varphi(x)| \leqslant |\varphi(x+a+b)-\varphi(x+a)| + |\varphi(x+a)-\varphi(x)| \leqslant 2,$$

and we know from Lemma 11.2 that there are numbers s,t such that

$$s+t=\varphi(x+a+b)-\varphi(x) \text{ and } ((a,s),(b,t)) \in \widetilde{B}.$$

We are done. □

By iteration we easily obtain the following result.

Corollary 11.3. *Let us define $B^{(0)}=\{(0,0)\}$, $B^{(1)}=\{(1,1)\}$, and generally $B^{(n)} \subset \mathbf{Z}^n \times \mathbf{Z}^n$ such that $B^{(n-1)}$ and $B^{(n)}$ satisfy the conditions for A and B in Theorem 11.1 for $n \geqslant 2$. If $f\colon \mathbf{Z}^n \to \mathbf{R}$ is a given $B^{(n)}$-laterally convex function, then the successive marginal functions $h_n = f$,*

$$h_k(x) = \inf_{t\in\mathbf{Z}} h_{k+1}(x,t), \qquad x=(x_1,\dots,x_k)\in\mathbf{Z}^k, \quad k=n-1,\dots,1,$$

are $B^{(k)}$-laterally convex, provided that $h_1 > -\infty$. In particular, the marginal function h_1 of one variable is $\{(1,1)\}$-laterally convex, equivalently convex extensible. □

In condition (11.1) it is often preferable to replace the pair

$$((a,-1),(b,0)) \text{ by its opposite } ((-a,1),(-b,0)),$$

which determines the same condition. This is to be able to continue as in Corollary 11.3, where the last component should be nonnegative—this is needed in Theorem 10.1. We denote the set B so constructed by $\Theta^n(A)$. We can now define $B^{(n)} = \Theta^n(B^{(n-1)})$ and get Corollary 11.3 to work.

Thus taking $\mathscr{M}_n$ as the set of all $B^{(n)}$-laterally convex functions such that the marginal functions h_1 do not take the value $-\infty$ gives a satisfactory solution to Problem 1.5.

11.2. The case of two variables

Let us look in more detail at the construction of $\Theta^2(A)$. Then the corollary is about three functions: $h_2 = f$ defined on $\mathbf{Z}^2$, $h_1(x) = \inf_{y\in\mathbf{Z}} f(x,y)$ defined on $\mathbf{Z}^1$, and the constant $h_0 = \inf_{(x,y)\in\mathbf{Z}^2} f(x,y)$ defined as a function on $\mathbf{Z}^0 = \{0\}$. But here we do not say anything about the marginal function $k_1(y) = \inf_{x\in\mathbf{Z}} f(x,y)$. To do so, we should permute the variables. However, it turns out, perhaps surprisingly, that this is not necessary, for the conditions are symmetric in the two variables.

If we start with $A = \{(1,1)\} \subset \mathbf{Z}^1 \times \mathbf{Z}^1$ in one variable, the construction in Theorem 11.1 yields, in order,

$(e^{(1)}, e^{(1)})$, $(e^{(2)}, e^{(2)})$, applying (7.4);

$((1,-1),(1,0))$, $((1,1),(1,0))$, applying (11.1);

$((1,1), e^{(2)})$, applying (11.2); and

$((-1,1), e^{(2)})$, applying (11.3).

However, as already remarked, we should replace $((1,-1),(1,0))$ by $((-1,1),(-1,0))$. We thus obtain

$$\begin{aligned} B = \{&(e^{(1)}, e^{(1)}),\ (e^{(2)}, e^{(2)}),\ ((-1,1),(-1,0)), \\ &((1,1),(1,0)),\ ((-1,1),(0,1)),\ ((1,1),(0,1))\}, \end{aligned}$$

This means that the two one-dimensional conditions and the four lozenge conditions are all satisfied, while the two diagonal conditions need not be listed since they follow from the others. We see now that if we permute the variables, the conditions remain the same.

We see that the set $B = \Theta^2(A) \subset \mathbf{Z}^2 \times \mathbf{Z}^2$, which defines rhomboidal convexity and corresponds to the six conditions (9.2), (9.4) and (9.5), consists of 6 pairs, and that $\Theta^3(B)$ consists of $6^2 = 36$ pairs.

11.3. Symmetric and asymmetric conditions

The condition on a function to have a convex-extensible marginal function is asymmetric. Indeed, the function $f(x, y) = (2x - y)^2$, $(x, y) \in \mathbf{Z}^2$, has a convex-extensible marginal function, whereas the marginal function of its reflection $g(x, y) = (2y - x)^2$ does not. Therefore a symmetric condition can never be necessary and sufficient.

For functions f such that $D_{(0,1)}D_{(0,1)}f \geqslant 0$, a known necessary and sufficient condition for all functions $f_\beta(x, y) = f(x, y) - \beta y$, $\beta \in \mathbf{R}$, to have a convex-extensible marginal function is that all conditions

$$D_{(1,p)}D_{(1,p)}f \geqslant 0, \quad D_{(1,p)}D_{(1,p+1)}f \geqslant 0, \qquad p \in \mathbf{Z},$$

shall be satisfied. These are infinitely many conditions as opposed to the six conditions obtained in our construction: $\Theta^2(A)$ has six elements.

We conclude that there is a choice between a sufficient condition which is finite and symmetric but not necessary, and a sufficient and necessary condition which is infinite—and by necessity asymmetric.

12. Necessity of lateral convexity

As can be guessed from Examples 9.2 and 9.3, the convexity property we have defined is essentially best possible. Before showing this, two remarks are in order.

Let $\varphi\colon \mathbf{Z}^2 \to \mathbf{Z}$ be any function such that $\mathbf{Z} \ni y \mapsto \varphi(x, y) \in \mathbf{Z}$ is a surjection for every $x \in \mathbf{Z}$. Then the function $g(x, y) = f(x, \varphi(x, y))$, $(x, y) \in \mathbf{Z}^2$, has the same marginal function as f. In particular, the values on a vertical line can be arbitrarily scrambled. It follows that no reasonable conclusion concerning regularity of f can be drawn from knowledge of its marginal function. But if we consider the marginal functions h_β of the tilted functions $f_\beta(x, y) = f(x, y) - \beta y$, $\beta \in \mathbf{R}$, things are different.

For simplicity we now restrict attention to functions of two variables $(x, y) \in \mathbf{Z}^2$. We define the *partial Fenchel transform* of a function $f\colon \mathbf{Z}\times\mathbf{Z}\to\mathbf{R}_!$ by

$$f^*(x,\eta)=\sup_{y\in\mathbf{Z}}(\eta y-f(x,y)),\qquad (x,\eta)\in\mathbf{Z}\times\mathbf{R},$$

to be compared with the complete Fenchel transform,

$$\tilde{f}(\xi,\eta)=\sup_{(x,y)\in\mathbf{Z}^2}(\xi x+\eta y-f(x,y)),\qquad (\xi,\eta)\in\mathbf{R}\times\mathbf{R}.$$

Thus the marginal function of f is $h(x)=-f^*(x,0)$. Since the third transform f^{***} is equal to the first, the second transform f^{**} has the same marginal function as f. Therefore, again, it is not reasonable to expect that, from knowledge of a marginal function, one can conclude anything about f, only about its minorant f^{**}.

Proposition 12.1. *Let $f\colon \mathbf{Z}\times\mathbf{Z}\to\mathbf{R}$ be such that the marginal function h_β of $f_\beta(x,y)=f(x,y)-\beta y$ is convex extensible for all real numbers β. Then f^{**} satisfies the one-variable conditions* (9.2)*, the diagonal conditions* (9.3)*, and the horizontal lozenge conditions* (9.4)*.*

Proof. For brevity, let us write g instead of f^{**}.

By replacing $g(x,y)$ by $g(x,y)+\varepsilon y^2$, $\varepsilon>0$, we may assume that the infimum of $y\mapsto g_\beta(x,y)=g(x,y)-\beta y$ is always attained at some point. Afterwards we let ε tend to zero; the properties are stable under this operation.

The vertical condition $g(x,y-1)+g(x,y+1)\geqslant 2g(x,y)$ is always satisfied by assumption.

Consider next the horizontal condition $g(x-1,y)+g(x+1,y)\geqslant 2g(x,y)$ for fixed x and y and define $\beta=g(x,y+1)-g(x,y)$. Then $h_\beta(x)=g_\beta(x,y)=g_\beta(x,y+1)$, and if h_β is convex extensible, we get

$$\begin{gathered}g_\beta(x-1,y)+g_\beta(x+1,z)\geqslant h_\beta(x-1)+h_\beta(x+1)\geqslant\\ 2h_\beta(x)=2g_\beta(x,y)=2g_\beta(x,y+1).\end{gathered}$$

Taking $z=y$ we see that the horizontal one-variable condition is satisfied; taking $z=y+1$ we see that the right horizontal lozenge condition is satisfied; and taking $z=y+2$ we see that one of the diagonal conditions is satisfied. For the left horizontal lozenge condition and the other diagonal condition we can argue similarly. $\square$

Theorem 12.2. *Let $f\colon \mathbf{Z}\times\mathbf{Z}\to\mathbf{R}$ satisfy the one-variable conditions* (9.2)*. Define two marginal functions by*

$$h_\beta(x)=\inf_{y\in\mathbf{Z}}(f(x,y)-\beta y),\qquad x\in\mathbf{Z},\ \beta\in\mathbf{R},$$

and

$$k_\alpha(y)=\inf_{x\in\mathbf{Z}}(f(x,y)-\alpha x),\qquad y\in\mathbf{Z},\ \alpha\in\mathbf{R}.$$

Assume that h_β and k_α are convex extensible for all real numbers α and β. Then f is rhomboidally convex.

Proof. We apply Proposition 12.1 to f and to $(x,y)\mapsto f(y,x)$. □

13. Conclusion

We have studied a kind of convexity called *lateral convexity*, which is defined using second-order difference operators (a special kind of convolution operators). We have proved that this notion of convexity is perfectly adapted for proving that the marginal function of a real-valued function defined on the set of points with integer coordinates remains in the same class.

Notions of mathematical morphology proved to be helpful. We believe that the duality between classes of functions with a convexity property and classes of convolution operators studied here will have several applications in the future.

14. Hints for future work

14.1. Discrete convexity of infimal convolutions

Since, as remarked in Subsection 1.3, the operation of taking the marginal function is a special case of infimal convolution, it may be of interest to extend this study of discrete convexity to more general infimal convolutions.

14.2. Discrete convexity of p-marginal functions

Given a positive number p, we may define the *p-marginal function h_p* of a function $f\colon \mathbf{Z}^n\times\mathbf{Z}^m\to\mathbf{R}$ by

$$e^{-ph_p(x)}=\sum_{y\in\mathbf{Z}^m}e^{-pf(x,y)},\qquad x\in\mathbf{Z}^n.$$

As p tends to $+\infty$ we get the usual marginal function. The question of finding suitable classes that are preserved under passage to the p-marginal function is not resolved. For such a class we would have a discrete analogue of Prékopa's theorem. For more details on Prékopa's theorem for real variables and the problem for discrete variables, see Kiselman (2012, 2014).

14.3. Functions with integer values

It may be of interest also to consider functions $f\colon \mathbf{Z}^n \to \mathbf{Z}$ with integer values and their marginal functions. Then convex extensibility of the marginal function is too strong a condition. Instead it is relevant to require that the functions are $(\mathbf{Z}^n \times \mathbf{Z})$-convex, meaning that there exists a convex subset C of $\mathbf{R}^n \times \mathbf{R}$ such that

$$C \cap (\mathbf{Z}^n \times \mathbf{Z}) = \mathbf{epi}^{\mathrm{F}}(f) = \{(x,t) \in \mathbf{Z}^n \times \mathbf{Z};\ t \geqslant f(x)\}.$$

14.4. Duality defined by convolution inequalities

The duality studied in Sections 6 and 7 should be extended to a duality between sets M of functions μ and classes $\Phi(M)$ of functions f satisfying convolution inequalities $\mu * f \geqslant 0$ for all $\mu \in M$. Adama Koné has pursued this idea in his doctoral thesis (2016: Chapter 4).

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