Questions inspired by Mikael Passare's mathematics

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Mikael Passare's Day

At Stockholm University

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A presentation in Kiruna 2012-05-11.

Section 10 on lineal convexity is new here.

1. Introduction

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In discussions with Mikael during the last thirty years many questions have emerged—not all of them were resolved at the time of his premature death.

I will present some of these unanswered questions, preceded by a discussion leading up to the question. Some of the questions might present challenges to his nine former PhD students, to his many collaborators around the globe—and to anybody interested.

I shall first present two questions in complex analysis: the non-associativity of multiplication of principal-value distributions and residue currents, followed by a section on constructions using meromorphic extension (questions from the early 1980s and up to 1988).

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I shall first present two questions in complex analysis: the non-associativity of multiplication of principal-value distributions and residue currents, followed by a section on constructions using meromorphic extension (questions from the early 1980s and up to 1988).

Then I will speak about Mikael's more recent interest: amoebas and tropical geometry (questions from the period 2003–2010).

Tropical geometry has intriguing connections to digital geometry, mathematical morphology, discrete optimization, and crystallography, including the theory of quasi-crystals. I believe these connections could be further developed—I hope they will.

2. Multiplication of residue currents and principal-value distributions

Let f and g be holomorphic functions of n complex variables. The *principal value* $\mathrm{PV}(f/g)$ of f/g is a distribution defined by the formula

$$\left\langle \mathrm{PV}\left(\frac{f}{g}\right), \varphi \right\rangle = \lim_{\varepsilon\to 0} \int_{|g|>\varepsilon} \frac{f\varphi}{g} = \lim_{\varepsilon\to 0} \int \frac{\chi f \varphi}{g}, \qquad \varphi \in \mathscr{D}(\mathbb{C}^n),$$

where $\chi = \chi(|g|/\varepsilon)$ and χ is a smooth function on the real axis satisfying $0 \leqslant \chi \leqslant 1$ and $\chi(t) = 0$ for $t \leqslant 1$, $\chi(t) = 1$ for $t \geqslant 2$ (in Passare (1985:727) when $f = 1$ and in (1988:39) in general).

The *residue current* is $\bar\partial \mathrm{PV}(f/g)$. Can the products

$$(\mathrm{PV}(f_1/g_1))(\mathrm{PV}(f_2/g_2)), \quad \left(\bar\partial(\mathrm{PV}(f_1/g_1))\right)(\mathrm{PV}(f_2/g_2))$$

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Mikael's construction of residue currents and principal-value distributions goes as follows. Take $f = (f_1, \ldots, f_{p+q})$, $g = (g_1, \ldots, g_{p+q})$, two $(p+q)$-tuples of holomorphic functions, and consider the limit

$$\lim_{\varepsilon_j \to 0} \frac{f_1}{g_1} \cdots \frac{f_{p+q}}{g_{p+q}} \bar{\partial}\chi_1 \wedge \cdots \wedge \bar{\partial}\chi_p \cdot \chi_{p+1} \cdots \chi_{p+q}, \qquad (1)$$

where $\chi_j = \chi(|g_j|/\varepsilon_j)$ and the ε_j tend to zero in some way.

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Mikael took $\varepsilon_j = \varepsilon^{s_j}$ for fixed $s_1, \dots, s_{p+q}$. The limit, which will be written as $R^p P^q[f/g](s)$, where we now write $[\dots]$ for the principal value, does not exist for arbitrary s_j. But he proved that, if we remove finitely many hyperplanes, then $R^p P^q[f/g](s)$ is locally constant in a finite subdivision of the simplex

$$\Sigma = \left\{ s \in \mathbb{R}^{p+q};\ s_j > 0,\ \sum s_j = 1 \right\},$$

so that the mean value, $R^p P^q \left[\dfrac{f}{g} \right]$

$$= \fint_{\Sigma} R^p P^q \left[\frac{f}{g} \right](s) = \bar\partial \left[\frac{f_1}{g_1} \right] \wedge \cdots \wedge \bar\partial \left[\frac{f_p}{g_p} \right] \cdot \left[\frac{f_{p+1}}{g_{p+1}} \right] \cdots \left[\frac{f_{p+q}}{g_{p+q}} \right]$$

exists (Definition A in Passare (1987:159)). This is the product of p residue currents and q principal-value distributions.

In the little paper (1993), based on his talk when accepting the Lilly and Sven Thuréus Prize in 1991, he discusses the possibility of defining the product $\mathrm{PV}(1/x)\delta$ on the real axis, and finds that it should be $-\frac{1}{2}\delta'$, which is the mean value of $-\delta'$ and zero. This is an analogue in real analysis to the mean value over Σ which he considered in the complex case.

Leibniz' rule for the derivative of a product and some other rules of calculus hold; for example we have (1988:43):

$$\left[\frac{1}{z_1}\right]\left[\frac{z_1}{z_2}\right]=\left[\frac{1}{z_2}\right],$$

which yields

$$\left(\bar{\partial}\left[\frac{1}{z_1}\right]\right)\left\{\left[\frac{1}{z_1}\right]\left[\frac{z_1}{z_2}\right]\right\}=\left(\bar{\partial}\left[\frac{1}{z_1}\right]\right)\left[\frac{1}{z_2}\right]\neq 0,$$

while

$$\left\{\left(\bar{\partial}\left[\frac{1}{z_1}\right]\right)\left[\frac{1}{z_1}\right]\right\}\left[\frac{z_1}{z_2}\right]=\frac{1}{2}z_1\left(\bar{\partial}\left[\frac{1}{z_1^2}\right]\right)\left[\frac{1}{z_2}\right]=\frac{1}{2}\left(\bar{\partial}\left[\frac{1}{z_1}\right]\right)\left[\frac{1}{z_2}\right].$$

Thus the associative law does not hold. It is natural to ask whether these currents are just as bad as the general distributions when it comes to multiplication, or whether there is a subclass of them with nicer properties.

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Question

2.1. We saw in Schwartz' example that an associative multiplication is impossible in general; the last example shown here makes us wonder whether it is possible to define an associative multiplication for some residue currents and principal-value distributions.

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Is there an interesting non-associative algebra of such currents?

3. Constructions using meromorphic extension

3.1. Extending a given meromorphic function

The Riemann ζ-function is a classical example of meromorphic extension: the series

$$\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s},$$

which converges for $s \in \mathbb{C}$, $\operatorname{Re} s > 1$, is extended to a meromorphic function in the whole plane.

The well-known formula

$$\sum_0^\infty \alpha^j = \frac{1}{1-\alpha}, \qquad |\alpha| < 1,$$

can be used to define more or less funny results like

$$\sum_0^\infty (-1)^j = \frac{1}{2}; \;\; \sum_0^\infty 2^j = -1; \;\; \sum_0^\infty (-2)^j = \frac{1}{3}; \;\; \sum_0^\infty 3^j = -\frac{1}{2}; \;\; \sum_0^\infty (-3)^j = \frac{1}{4}.$$

This is based on the observation that $1/(1-\alpha)$ is a meromorphic function with a single pole at $\alpha = 1$ but otherwise regular. But why should $\sum \alpha^j$ be meromorphic?

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The situation is different when we want to construct an object and have to choose a meromorphic function.

3.2. Finding a meromorphic function

Michael Atiyah proved (1970) that if F is a nonnegative real-analytic function, then $\lambda \mapsto F^{\lambda}$ can be extended to a meromorphic function in all of $\mathbb{C}$. Bernšteĭn and Gel′fand (1969) proved a similar result for polynomials. Using Atiyah's result, Alain Yger (1987) defined residue currents as meromorphic extensions of $(f\bar{f})^{\lambda}$ for a holomorphic f, and Mikael compared them (Definition B in (1987:159)) with his own construction (Definition A already mentioned).

In this case the authors construct $(f\bar{f})^{\lambda}$ for just one value of λ, viz. $\lambda = -1$, which means that they have to choose a parametrized family; the choice is not obvious.

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Another kind of limit of a meromorphic function is

$$\lim_{\varepsilon_j \to 0} \bar{\partial} \frac{f_1}{|f_1|^2 + \varepsilon_1} \wedge \cdots \wedge \bar{\partial} \frac{f_p}{|f_p|^2 + \varepsilon_p},$$

which is obtained by taking $\chi_j(t) = t/(t+1)$ in (1) (the case $q = 0$). It yields the same current as the former construction for complete intersections (Björk & Samuelsson (2010:35); cf. earlier results by Samuelsson (2006: Corollary 26), who may have been inspired to consider averaging from a paper on defining residues of a complete intersection by Passare & Tsikh (1996)).

If we want to evaluate a divergent integral, for instance

$$\int_0^1 x^{-2} dx, \tag{2}$$

one method is to embed the integrand into a family of functions depending on a parameter and define the integral as the value of an extension in the parameter space. In the case mentioned, we can define $f(x, \alpha) = x^{\alpha}$, and since

$$\Phi(\alpha) = \int_0^1 f(x, \alpha)\, dx = \int_0^1 x^{\alpha}\, dx = \frac{1}{\alpha + 1}$$

is well-defined for $\mathrm{Re}\, \alpha > -1$ and has a meromorphic extension to the whole complex plane, we can define the integral of x^{-2} as $\Phi(-2) = -1$. The question is now: will we get a different answer if we use a different meromorphic function?

It can be remarked that $\Phi(-2)$ is also the finite part of the integral (2) in view of the formula

$$\int_{\varepsilon}^{1} x^{-2}\, dx = \frac{1}{\varepsilon} - 1, \quad 0 < \varepsilon < 1,$$

where $1/\varepsilon$ is the infinite part (to be thrown away) and -1 is the finite part (to be kept).

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where $1/\varepsilon$ is the infinite part (to be thrown away) and -1 is the finite part (to be kept).

We may conclude that meromorphic extension is an often used method to construct mathematical objects.

3.3. Two-parameter families

While meromorphic functions of one variable can be assigned the value ∞ at a pole, and therefore can be defined as good mappings with values in the Riemann sphere $\mathbb{C} \cup \{\infty\}$, meromorphic functions of two variables may have points of indeterminacy: the function $f(z_1, z_2) = z_2/z_1$ can be assigned the value ∞ at a point $(0, z_2) \neq (0, 0)$, but at the origin we cannot do so. This explains the trouble we are in for.

I will consider a divergent integral where we can get different values for different choices of parametrized families.

We denote by $D(c,r)$ the open disk with center at c and with radius r:

$$D(c,r) = \{z \in \mathbb{C};\ |z-c| < r\}.$$

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A simple idea is to vary the disk. We have

$$\int_{D(c,r)} z^{-2} d\lambda(z) = \pi r^2/c^2 \text{ when } r < |c|,$$

i.e., when the origin is not in the closure of $D(c,r)$. Hence the limit of these values is π as $(c,r) \in \mathbb{C} \times \mathbb{R}$ tends to $(1,1)$ under the restriction $r < |c|$. So this is one possible method.

Another idea is to remove a small disk around the origin, like in the definition of the principal value:

$$\mathrm{PV}\int_{D(1,1)} z^{-2}\,d\lambda(z) = \lim_{\varepsilon\to 0}\int_{z\in D(1,1),\,|z|>\varepsilon} z^{-2}\,d\lambda(z)$$

$$= 2\int_0^{\pi/2} \cos 2\theta \log\cos\theta\,d\theta = \tfrac{1}{2}\pi.$$

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Are there other ways to approach the divergent integral? Let us look at a two-parameter family.

Lemma

For $(\alpha,\beta)\in\mathbb{C}^2$ *with* $\mathrm{Re}(\alpha+\beta)>-2$ *we have*

$$F(\alpha,\beta)=\int_{D(1,1)} z^{\alpha}\overline{z}^{\beta}\,d\lambda(z)=\frac{\pi\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)}. \tag{3}$$

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Proof.

This follows from Gradštejn & Ryžik (1962:490:3.892.2). □

So the extended values of the integral (3) define a meromorphic function F in all of $\mathbb{C}^2$, with singularities, e.g., on the hyperplane $\alpha + \beta = -2$. But restrictions of F may be free from singularities: the function $\beta \mapsto F(m, \beta)$ is entire (in fact a polynomial of degreee m) for every $m \in \mathbb{N}$. At the point $(\alpha, \beta) = (-1, -1)$ we can assign the value ∞ to F if we like, but the point $(\alpha, \beta) = (-2, 0)$ is a point of indeterminacy.

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We take $(\alpha,\beta)=(-2+\varepsilon,\theta-\varepsilon)$ and consider

$$F(\alpha,\beta)=F(-2+\varepsilon,\theta-\varepsilon)=\frac{\pi\varepsilon}{\theta}\frac{\Gamma(1+\theta)}{\Gamma(1+\varepsilon)\Gamma(2+\theta-\varepsilon)}=\frac{\pi\varepsilon}{\theta}(1+o(1))$$

as $(\theta,\varepsilon)\to(0,0)$ under the restriction $\operatorname{Re}\theta>0$. The quantity $\pi\varepsilon/\theta$ has no limit as $(\varepsilon,\theta)\to(0,0)$, but we may introduce a relation between ε and θ to create a one-parameter family of functions which has a limit, e.g., $\varepsilon=\theta$ or $\varepsilon=0$.

If we take $\varepsilon = \theta$, we obtain

$$G(\theta) = F(-2+\theta, 0) = \int_{D(1,1)} z^{-2+\theta}\, d\lambda = \pi, \quad \mathrm{Re}\,\theta > 0.$$

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More generally, we may take $\varepsilon = c\theta$ and get the limit $c\pi$ for certain values of c, or $\theta = \varepsilon^2$ and get infinity.

Question

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This text was essentially written 1988-10-17 and sent out to some people, among them Mikael Passare and Bo Berndtsson. Bo expressed surprise.

4. The axioms of tropical geometry

4.1. Tropicalization

Roughly speaking, tropical mathematics is the mathematics of a structure with addition and maximum as binary operations. The simplest example is the semiring of real numbers $(\mathbb{R}, +, \vee)$, where $+$ denotes usual addition and $\vee$ is the maximum operation, $x \vee y = \max(x, y)$. Note the distributive law $a + (b \vee c) = (a + b) \vee (a + c)$: addition is distributive over maximum. Sometimes $\mathbb{R}$ is augmented by adding an element $-\infty$, the neutral element for the maximum operation: $x \vee (-\infty) = x$. Another name is *idempotent mathematics*, used because of the idempotency of the maximum operation: $x \vee x = x$.

Tropicalization means that in a semiring with multiplication and addition we replace multiplication by addition, and addition by the maximum operation. This is somewhat reminiscent of taking the logarithm. Start with the semiring $(\mathbb{R}_+, \times, +)$, where $\mathbb{R}_+ = \{x \in \mathbb{R}; x > 0\}$ is the set of positive real numbers, and take the logarithm.

Tropicalization means that in a semiring with multiplication and addition we replace multiplication by addition, and addition by the maximum operation. This is somewhat reminiscent of taking the logarithm. Start with the semiring $(\mathbb{R}_+, \times, +)$, where $\mathbb{R}_+ = \{x \in \mathbb{R}; x > 0\}$ is the set of positive real numbers, and take the logarithm. Then

$$\log(x \times y) = \log x + \log y, \text{ and}$$

$$\log(x) \vee \log(y) \leqslant \log(x + y) \leqslant \log 2 + (\log(x) \vee \log(y)),$$

so that multiplication gives addition of the logarithms, and addition comes close to the maximum of the logarithms—a good approximation if $x \gg 1$ or $y \gg 1$.

If we introduce $s = \log x$ and $t = \log y$, and make a change of scale, we see that

$$\log(e^s \times e^t) = s + t;$$

in the limit,

$$h \log(e^{s/h} \times e^{t/h}) = s + t \to s + t \text{ and } h \log(e^{s/h} + e^{t/h}) \to s \vee t$$

as $h \to 0+$.

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as $h \to 0+$. Thus we may say that tropicalization is a limiting case of logarithmization. Here $h > 0$ is an analogue of Planck's constant—hence the name *dequantization* (Litvinov (2005, 2007); Viro (2001: Section 2.1)).

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$$\log(e^s \times e^t) = s + t;$$

in the limit,

$$h \log(e^{s/h} \times e^{t/h}) = s + t \to s + t \text{ and } h \log(e^{s/h} + e^{t/h}) \to s \vee t$$

as $h \to 0+$. Thus we may say that tropicalization is a limiting case of logarithmization. Here $h > 0$ is an analogue of Planck's constant—hence the name *dequantization* (Litvinov (2005, 2007); Viro (2001: Section 2.1)).

For basic concepts of tropical geometry, see also Viro (2010, 2011). The book by Itenberg et al. (2007) presents fundamental ideas and key results in tropical algebraic geometry.

4.2. Tropical straight lines

A polynomial function of degree one has the form

$$(x,y) \mapsto f(x,y) = ax + by + c.$$

So if we tropicalize it we get

$$f_{\text{trop}}(x,y) = (a+x) \vee (b+y) \vee c.$$

Thus f_{trop} is a convex function with a simple structure. It is piecewise affine outside the three lines $x+a=c$, $y+b=c$, and $x+a=y+b$, which meet at the point

$$p = (p_1, p_2) = (c-a, c-b).$$

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More precisely, it is affine outside the three rays emanating from that point in the directions $(-1,0)$, $(0,-1)$ and $(1,1)$. We shall call this the *tropical straight line* TSL(p) with *vertex* p.

We see that in general two different tropical lines intersect in a single point. An exception occurs when the lines have their vertices at points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ with $p_1 = q_1$ or $p_2 = q_2$ or $p_2 - p_1 = q_2 - q_1$, i.e., when $q \in \mathrm{TSL}(p)$ or $p \in \mathrm{TSL}(q)$.

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In this way two distinct tropical straight lines always have a unique intersection, just like in spherical geometry. And, similarly, two distinct points always define a tropical straight line, except in certain cases, where again we use stability to impose uniqueness.

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4.1. So we may ask about all of Euclid's axioms! Is it possible to build up an axiomatic theory for tropical geometry? What are the similarities with spherical geometry?

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I asked Mikael these questions a few years ago.

5. Tropical polynomial functions

5.1. Largest tropical minorants

A tropical polynomial function of one real variable is of the form

$$f(x) = \sup_{j \in \mathbb{Z}} (a_j + jx), \qquad x \in \mathbb{R}.$$

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It is a piecewise affine function, where each piece has integer slope.

To any function $G \colon \mathbb{R} \to \mathbb{R}$ we associate its largest tropical minorant f. The example

$$G(x) = 0 \vee (a + \tfrac{1}{2}x) \vee x, \qquad x \in \mathbb{R},$$

shows that the difference $G - f$ can be arbitrarily large even if G is convex: in this case we have $f(x) = 0 \vee x$, so that $G(0) - f(0) = 0 \vee a$.

On the other hand, if

$$G(x) = \sup_{|z|=e^x} \log|h(z)|, \qquad x \in \mathbb{R}, \tag{4}$$

for a holomorphic function h, I proved (1984:168) that

$$f \leqslant G \leqslant f + \log 3.$$

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for a holomorphic function h, I proved (1984:168) that

$$f \leqslant G \leqslant f + \log 3.$$

Question

5.1. Which is the smallest constant c such that $f \leqslant G \leqslant f + c$ if G is of the form (4) for some polynomial h?

We know that $c \leqslant \log 3 \approx 1.09861$. The example $h(z) = 1 + z$ shows that $c \geqslant \log 2 \approx 0.69315$: in this case $G(0) = \log 2$ and $f(0) = 0$.

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I sent this question to Mikael Passare, Jan Boman, David Jacquet, Hans Rullgård, Erik Melin, and Markku Ekonen in a letter of 2003-10-21.

5.2. Approximation of the exponential function

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The *Fenchel transform* $\tilde{f}$ of a function $f\colon \mathbb{R} \to [-\infty, +\infty]$ is defined by

$$\tilde{f}(\xi) = \sup_{x\in\mathbb{R}} \big(\xi x - f(x)\big), \qquad \xi \in \mathbb{R}.$$

We have $\tilde{\tilde{f}} \leqslant f$ with equality if and only if f is convex, lower semicontinuous, and takes the value $-\infty$ only if it is identically $-\infty$.

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The equality $f = \tilde{\tilde{f}}$ means that f is represented as a supremum of affine functions—a tropical integral of the simplest convex functions—just as the Fourier inversion formula $f = \mathscr{F}^{-1}(\mathscr{F}(f))$ represents f as an integral of the simplest oscillations.

Now take $f(x) = e^x$, $x \in \mathbb{R}$. The Fenchel transform of this function is

$$\tilde{f}(\xi) = \begin{cases} +\infty, & \xi < 0, \\ 0, & \xi = 0, \\ \xi \log \xi - \xi, & \xi > 0. \end{cases}$$

Now take $f(x) = e^x$, $x \in \mathbb{R}$. The Fenchel transform of this function is

$$\tilde{f}(\xi) = \begin{cases} +\infty, & \xi < 0, \\ 0, & \xi = 0, \\ \xi \log \xi - \xi, & \xi > 0. \end{cases}$$

Then, for $\xi > 0$,

$$e^{\tilde{f}(\xi)} = \xi^{\xi} e^{-\xi} = \sup_{x \in \mathbb{R}} e^{\xi x - e^x} = \sup_{y > 0} y^{\xi} e^{-y} \approx \int_0^{\infty} y^{\xi} e^{-y} dy = \xi!,$$

for the integral is approximately equal to a tropical integral, i.e., a supremum. This is a crude form of Stirling's formula.

Furthermore, since $\tilde{\tilde{f}} = f$,

$$e^{e^x} = e^{\tilde{\tilde{f}}(x)} = \sup_{\xi>0} e^{\xi x - \tilde{f}(\xi)} = \sup_{\xi>0} \frac{e^{\xi x}}{e^{\tilde{f}(\xi)}} \approx \sup_{\xi>0} \frac{e^{\xi x}}{\xi!} \approx \sum_{k\in\mathbb{N}} \frac{e^{kx}}{k!} = e^{e^x},$$

where we have used the former approximation that $e^{\tilde{f}(\xi)} \approx \xi!$ and a new tropical approximation: the sum defining the exponential function is approximately equal to a tropical sum, i.e., to a supremum.

Question

5.2. We have thus showed that e^{e^x} *is approximatively equal to* e^{e^x}*. This is not so remarkable. But the surprising fact is that it is an exact equality. After two approximations we return to the exact value. Is there an explanation? To be precise: is there a more direct explanation why*

$$\sup_{\xi>0} \frac{e^{\xi x}}{e^{\tilde{f}(\xi)}} = \sum_{k=0}^{\infty} \frac{e^{kx}}{k!}, \qquad x \in \mathbb{R}?$$

Are there other, similar examples?

Question

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Are there other, similar examples?

I sent this question to Mikael Passare, Jan Boman, David Jacquet, Hans Rullgård, Erik Melin, and Markku Ekonen on 2003-10-21.

5.3. Representation of tropical polynomial functions by tropical polynomials

Proposition

A tropical polynomial function

$$f(x) = \sup_{\alpha \in \mathbb{Z}^n} (x \cdot \alpha + a_\alpha), \qquad x \in \mathbb{R}^n,$$

with coefficients $a_\alpha \in [-\infty, +\infty]$ *can be represented in several ways by tropical polynomials. The representation*

$$f(x) = \sup_{\alpha \in \mathbb{Z}^n} (x \cdot \alpha + b_\alpha), \qquad x \in \mathbb{R}^n,$$

with $b_\alpha = -\tilde{f}(\alpha)$, $\alpha \in \mathbb{Z}^n$, *is the one with largest coefficients.*

5.4. The exponential of a tropical polynomial function

Let $\varphi \colon \mathbb{Z}^n \to [-\infty, +\infty]$ be a function on the integer points which is $< +\infty$ only at finitely many of them. Equivalently we may take φ defined on $\mathbb{R}^n$ but with the value $+\infty$ at all points in $\mathbb{R}^n \smallsetminus \mathbb{Z}^n$.

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We define $f = \tilde{\varphi}$, the Fenchel transform of φ,

$$f(x) = \tilde{\varphi}(x) = \sup_{\alpha \in \mathbb{Z}^n} \big(x \cdot \alpha - \varphi(\alpha)\big), \qquad x \in \mathbb{R}^n.$$

It is a tropical polynomial function.

Passing to the exponential, we see that

$$e^{f(x)} = e^{\tilde{\varphi}(x)} = \sup_{\alpha \in \mathbb{Z}^n} e^{x \cdot \alpha} e^{-\varphi(\alpha)} \leqslant \sum_{\alpha \in \mathbb{Z}^n} e^{x \cdot \alpha} e^{-\varphi(\alpha)} = g(y),$$

where $y_j = e^{x_j}$ and

$$g(y) = \sum_{\alpha \in \mathbb{Z}^n} e^{-\varphi(\alpha)} y^{\alpha}$$

is a classical Laurent polynomial majorizing $e^{f(x)} = e^{\tilde{\varphi}(x)}$, but actually often rather close to it.

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is a classical Laurent polynomial majorizing $e^{f(x)} = e^{\tilde{\varphi}(x)}$, but actually often rather close to it.

Summing up, we see that φ contains all information, from which $f = \tilde{\varphi}$ and g can be constructed. Also g determines its coefficients $\exp(-\varphi(\alpha))$, thus also φ and f. On the other hand, $f = \tilde{\varphi}$ contains less information, from which φ cannot in general be retrieved.

Question

5.3. Is it possible to pass from $e^f = e^{\tilde{\varphi}}$ *to* g *using some other structure? (Cf. the next section.)*

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Question

5.4. Is it possible to pass to the limit in some way so that the classical polynomials tend to the tropical one?

6. Ghosts in tropical mathematics

In a polynomial function $f(z) = \sum a_j z^j$ all coefficients can be retrieved from the values of f, both in the real case and in the complex case: if $\sum a_j z^j = \sum b_j z^j$ for sufficiently many z, then $a_j = b_j$ for all j.

6. Ghosts in tropical mathematics

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But in a tropical polynomial function $f(x) = \sup_j(a_j + jx)$, a coefficient a_k cannot be retrieved from the values of f if $a_k \leqslant \frac{1}{2}(a_{k-1} + a_{k+1})$: if this is so, we can replace a_k by a smaller value without changing the values of f at any point. In fact, under this hypothesis,

$$a_k + kx \leqslant \big(a_{k-1} + (k-1)x\big) \vee \big(a_{k+1} + (k+1)x\big) \text{ for all } x \in \mathbb{R}.$$

Thus such a coefficient is invisible. We see that many tropical polynomials $P(X) = \bigvee_{j\in\mathbb{Z}}(a_j + jX)$ have the same evaluations on the real axis.

On the other hand, if the function $\mathbb{Z} \ni j \mapsto a_j$ is strictly concave in the sense that $a_j > \frac{1}{2}a_{j-1} + \frac{1}{2}a_{j+1}$ for all $j \in \mathbb{Z}$, then the coefficients can be retrieved from the values of the function; in fact, under this hypothesis, $a_j = -\tilde{f}(j)$, $j \in \mathbb{Z}$.

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I found this slightly disturbing, and asked Mikael the following question in a letter of 2010-03-26.

On the other hand, if the function $\mathbb{Z} \ni j \mapsto a_j$ is strictly concave in the sense that $a_j > \frac{1}{2}a_{j-1} + \frac{1}{2}a_{j+1}$ for all $j \in \mathbb{Z}$, then the coefficients can be retrieved from the values of the function; in fact, under this hypothesis, $a_j = -\tilde{f}(j)$, $j \in \mathbb{Z}$.

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Question

6.1. Is there some structure which will allow us to retrieve all coefficients of a tropical polynomial from its point evaluations?

It is clear that we need more information than just the values on the real axis.

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In his answer of 2010-03-29, Mikael directed me to the preprint by Izhakian and Rowen (2009), published as (2010). (Perhaps the paper by Izhakian (2009) is easier to start with.) It seemed to me that Mikael hinted at a solution in that the ghost elements could be used to retrieve the coefficients.

A first lesson is that, just as for classical polynomials, we must distinguish between a polynomial and the function given by a polynomial. A polynomial $P(X) = \sum a_j X^j$ is a formal expression containing an indeterminate X. If we give X a value as a variable in some ring, we get a polynomial function. For instance, if $P(X) = X^3 - 3X^2 + 2X$, and we replace X by a variable in the finite field $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$, then the value is everywhere zero.

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The situation is similar for tropical polynomials: a tropical polynomial is a formal expression $P(X) = \bigvee_{j\in\mathbb{N}}(a_j + jX)$, where information about all the coefficients a_j is preserved. That the coefficients are not retrievable from an evaluation in $\mathbb{R}$ is no more upsetting than the example with evaluation in $\mathbb{Z}_3$.

In the quoted papers by Izhakian and Rowen, the authors construct a space $\mathbb{T} = (\mathbb{R} \times \{0, 1\}) \cup \{-\infty\}$, where we have two copies of $\mathbb{R}$; the first copy consists of the usual real numbers, represented as $(x, 0)$, the second of the ghost elements, represented as $(x, 1)$. We can evaluate a tropical polynomial at the points of $\mathbb{T}$.

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However, Zur Izhakian explains:

In the tropical framework there is no injection of the polynomial semiring into the function semiring. Namely, a function could have several polynomial descriptions. In particular there are monomials (called *inessential*) which can be omitted without changing the function determined by the polynomial.

This phenomena is obtained due to convexity considerations involved in this setting, which cause a loss of information. Accordingly, a full recovery of the exact coefficients of a polynomial from the corresponding function is not always possible. (Zur Izhakian, personal communication 2011-10-26)

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So this is just the observation I made concerning evaluation at real numbers in the beginning of this section, but now extended to the larger space $\mathbb{T}$.

This means that my original Question 6.1 remains unanswered.

Question

6.2. It is well known that the coefficients of a holomorphic function can be retrieved from its values:

$$\textit{if } h(z) = \sum_{j \in \mathbb{N}} a_j z^j, \; z \in \mathbb{C}, \quad \textit{then} \quad a_k = \frac{1}{2\pi i} \int_{|z|=r} \frac{h(z)}{z^{k+1}} dz, \quad k \in \mathbb{N}.$$

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What can be said if we know only the values of the growth function

$$g(r) = \sup_{|z|=r}|h(z)|, \qquad r\geqslant 0?$$

If we have two entire functions h_1 and h_2 with growth functions g_1 and g_2, does it follow that $g_1(r) = g_2(r)$ for all r only if the coefficients of h_1 and h_2 have the same absolute values? Is there even a formula that yields the $|a_j|$ from g?

We can at least retrieve the absolute values of the first and second nonzero coefficients from the growth function:

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Proposition

Let h be an entire function with Taylor series

$$h(z) = \sum_{q}^{\infty} a_j z^j, \qquad z \in \mathbb{C},$$

and let g be its growth function. Then

$$g(r)r^{-q} \to |a_q| \;\; \text{and} \;\; \frac{g(r)r^{-q} - |a_q|}{r} \to |a_{q+1}| \text{ as } r \to 0+.$$

Corollary

If h is a polynomial of the form

$$h(z) = \sum_{q}^{q+3} a_j z^j,$$

then the absolute values of all four coefficients $a_q, a_{q+1}, a_{q+2}, a_{q+3}$ *can be determined from the growth function.*

When the coefficients are real, Jean-Pierre Kahane could give an affirmative answer:

Proposition

(Jean-Pierre Kahane, 2011-11-11.) *If* $h(z) = \sum a_j z^j$ *is an entire function with real coefficients* a_j *and if* a_0 *and* a_1 *are positive, then all coefficients can be determined from the Taylor expansion at the origin of the square of the growth function.*

Proof.

From the following lemma we have that $h(r)\overline{h(r)} = g(r)^2$ for small r. It is easy to see that the coefficients a_j of h can be read off from the Taylor expansion of $h\bar{h}$ at the origin. □

Proof.

From the following lemma we have that $h(r)\overline{h(r)} = g(r)^2$ for small r. It is easy to see that the coefficients a_j of h can be read off from the Taylor expansion of $h\bar{h}$ at the origin. □

Lemma

(Jean-Pierre Kahane, 2011-11-11.) *If $h(z) = \sum a_j z^j$ is an entire function with real coefficients a_j and if a_0 and a_1 are positive, then $g(r) = |h(r)|$ for sufficiently small $r \geqslant 0$.*

7. The discrete Prékopa problem

If $F\colon \mathbb{R}^2 \to [-\infty, +\infty]$ is a function of two real variables, its *marginal function* $H\colon \mathbb{R} \to [-\infty, +\infty]$ is defined by

$$H(x) = \inf_{y \in \mathbb{R}} F(x, y), \qquad x \in \mathbb{R}.$$

It is well known and easy to prove that H is convex if F is convex.

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It is well known and easy to prove that H is convex if F is convex.

There is a more general marginal function H_p, called the p-*marginal function*, of $F\colon \mathbb{R}^2 \to [-\infty, +\infty]$. It is defined by

$$e^{-pH_p(x)} = \int_{\mathbb{R}} e^{-pF(x,y)} dy, \qquad x \in \mathbb{R},$$

for any positive real number p. Prékopa's theorem, first presented by András Prékopa in Budapest in 1972 and published in (1973: Theorem 6), says that H_p is convex if F is convex.

In the discrete case, with $f\colon \mathbb{Z}^2 \to [-\infty, +\infty]$, we define the marginal function h by

$$h(x) = \inf_{y \in \mathbb{Z}} f(x, y), \qquad x \in \mathbb{Z},$$

and the p-marginal function h_p by

$$e^{-ph_p(x)} = \sum_{y \in \mathbb{Z}} e^{-pf(x,y)}, \qquad x \in \mathbb{Z}. \tag{5}$$

The classical marginal function $h = h_\infty$ is a limiting case when $p \to +\infty$ and may be defined by

$$e^{-h_\infty(x)} = \sup_{y \in \mathbb{Z}} e^{-f(x,y)}, \qquad x \in \mathbb{Z}. \tag{6}$$

So we may say that (6) is a dequantization of the sums of the exponential functions in (5): replacing the sum by the sup. We have $h_p \leqslant h_\infty$.

Question

7.1. Is it possible to go from $\sum_{y\in\mathbb{Z}} e^{-f(x,y)}$ *to* $\sup_{y\in\mathbb{Z}} e^{-f(x,y)}$ *under some reasonable hypotheses on* f*? (Cf. Question 4.4.)*

Question

7.1. Is it possible to go from $\sum_{y\in\mathbb{Z}} e^{-f(x,y)}$ to $\sup_{y\in\mathbb{Z}} e^{-f(x,y)}$ under some reasonable hypotheses on f? (Cf. Question 4.4.)

In the digital case, it is not enough to assume that f has a convex extension to all of $\mathbb{R}^2$ to conclude that h is convex. But a stronger convexity property, now called rhomboidal convexity, implies that h is convex (Kiselman 2008). I call $f\colon \mathbb{Z}^2 \to \mathbb{R}$ *rhomboidally convex* if its second differences satisfy six conditions:

$$D_b D_a f \geqslant 0 \text{ for all } (a,b) \in \{((1,0),(1,b_2));\ b_2 = -1,0,1\},$$

as well as

$$D_b D_a f \geqslant 0 \text{ for all } (a,b) \in \{((0,1),(b_1,1));\ b_1 = -1,0,1\}.$$

Here D_a is the difference operator $(D_a f)(x) = f(x+a) - f(x)$. It is not known whether the result holds for the p-marginal function. This is the *discrete Prékopa problem:*

Question

7.2. Is it true that the p-marginal function h_p is convex if f is rhomboidally convex?

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7.2. Is it true that the p-marginal function h_p is convex if f is rhomboidally convex?

I asked Mikael Passare and Bo Berndtsson this question in the spring of 2008.

It is enough here to take $p = 1$. Examples show that it is not enough that f admits a convex extension to all of $\mathbb{R}^2$, and that rhomboidal convexity is sufficient in some special classes.

8. A conjecture on coamoebas and Newton polytopes

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8. A conjecture on coamoebas and Newton polytopes

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Mounir Nisse and Jens Forsgård provided the most recent information on this conjecture.

Definition

A *Laurent polynomial* is a polynomial in z_j and z_j^{-1}, $j = 1, \ldots, n$. It thus has the form

$$f(z) = \sum_{\alpha \in A} a_\alpha z^\alpha$$

for some finite subset A of $\mathbb{Z}^n$. The *Newton polytope* of a Laurent polynomial is defined to be the convex hull in $\mathbb{R}^n$ of the set $\{\alpha \in A;\ a_\alpha \neq 0\}$. We will denote this polytope by Δ_f. A Laurent polynomial is said to be *maximally sparse* if the number of its nonzero terms is equal to the number of vertices of its Newton polytope. □

Definition

The *amoeba* of a function f defined in $(\mathbb{C} \setminus \{0\})^n$ is a set in $\mathbb{R}^n$ defined as follows. We define a mapping

$$\mathrm{Log}\colon (\mathbb{C} \setminus \{0\})^n \to \mathbb{R}^n \text{ by } \mathrm{Log}(z) = (\log|z_1|, \log|z_2|, \ldots, \log|z_n|).$$

Then the *amoeba* $\mathscr{A}_f$ of f is the image under Log of its set of zeros. The *coamoeba* $\mathscr{A}'_f$ is defined analogously but with the mapping Log replaced by the mapping

$$\mathrm{Arg}(z) = (\arg z_1, \arg z_2, \ldots, \arg z_n).$$

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$$\mathrm{Arg}(z) = (\arg z_1, \arg z_2, \ldots, \arg z_n).$$

The amoeba of a Laurent polynomial f is said to be *solid* if the number of components of its complement is as small as it can possibly be, that is, if it equals the number of vertices of the Newton polytope Δ_f. □

Mikael wanted to establish formally the duality between amoebas and coamoebas, and he started to write a paper with Mounir Nisse (2012), which Mounir has now finished (Mounir Nisse, personal communication 2011-11-13, 2012-06-24).

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Maximally sparse polynomials enjoy certain minimality properties. For instance, it has been proved by Mounir Nisse that the amoeba of a maximally sparse polynomial is necessarily solid; see Nisse (2008, 2009:33). This was earlier conjectured by Mikael and others.

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The solidness of the amoeba is also one of the characteristic properties of discriminants according to Passare et al. (2005).

Conjecture

(Mikael Passare, 2010-12.) *For $z \in \mathbb{C}^n$, let $f(z)$ be a maximally sparse Laurent polynomial with generic coefficients. Then the number of holes in the compactified coamoeba $\overline{\mathscr{A}_f'}$ is equal to $n!\,\mathrm{Vol}(\Delta_f)$.*

However, the conjecture is false: Jens Forsgård and Petter Johansson found counterexamples in dimensions 2 and 3. In two dimensions their polynomial is of the form

$$f(z, w) = 1 + z^2 + w^3 + azw^3 + bz^2w^2,$$

where a and b are constants. The normalized area of the Newton polytope is 11, while the maximal number of components in the complement of the closed coamoeba is 10. In three dimensions, the Newton polytope is the cube with side length 1. The normalized volume is then 6, while the maximal number of components in the complement is 4. Part of the results is described in Broms (2012). (Jens Forsgård, personal communication 2012-06-26.)

Also Mounir Nisse and Frank Sottile found a counterexample in dimension two. More precisely, they proved that there exists a 2-dimensional polygon Δ such that, for any complex plane curve with Δ as Newton polygon, the number of components in the complement of its coamoeba is strictly less than $2\text{Area}(\Delta)$ (in particular when it is defined by a maximally sparse polynomial). (Mounir Nisse, personal communication 2012-06-24.)

9. The constant term in powers of a Laurent polynomial

Let $P(X) = \sum_{\alpha \in A} a_\alpha X^\alpha$, A a finite subset of $\mathbb{Z}^n$, be a Laurent polynomial and assume that its Newton polytope contains the origin in its interior. We consider powers $P(X)^k$, $k \in \mathbb{N}$, of $P(X)$ and denote by $c_k(P)$ the constant term in $P(X)^k$. The question is whether there are infinitely many k such that $c_k(P)$ is nonzero. This seems plausible, and for $n = 1$ it is not difficult to prove. Mikael lectured on this problem in the Pluricomplex Seminar on 2000-03-14.

Alain Yger points out that Hans Duistermaat and Wilberd van der Kallen (1998) proved that the answer is in the affirmative as well as a more precise result: the radius of convergence of the formal power series $\sum_{k=1}^{\infty} c_k(P)t^k$ is finite (Alain Yger, personal communication 2011-12-01). However, the proof relies on a very heavy machinery when $n \geqslant 2$.

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Question

9.1. Is there a simpler proof of the result of Duistermaat and van der Kallen (1998) that the constant term $c_k(P)$ in the k-th power of a Laurent polynomial is different from zero for infinitely many values of k?

10. Lineal convexity

André Martineau (1930–1972) gave a couple of seminars on lineal convexity (*convexité linéelle*) in Nice during the academic year 1967-68, when I was there. This is a kind of complex convexity which is stronger than pseudoconvexity and weaker than convexity.

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On the one hand, this piece of advice was certainly very good, for he found a lot of results in cooperation with his friends Mats Andersson and Ragnar Sigurðsson (Mikael's mathematical uncle). On the other hand, it was perhaps not such a good suggestion, for the survey just kept growing, and two preprints were circulating starting in 1991—and then they had already been busy writing a long time.

The article became a book, and it did not appear until 2004. Anyway, it is thanks to André Martineau that lineal convexity came to be studied in the Nordic countries—and the book has become a standard reference.

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In the book, the authors study in detail a property which Martineau called strong lineal convexity (*convexité linéelle forte*), and which he did not characterize geometrically. This notion, in the book called $\mathbb{C}$-*convexity*, is not linked to any cleistomorphism (closure operator), since the intersection of two strongly lineally convex sets need not have the property. Therefore it has a different character than lineal convexity and usual convexity.

The latest on lineal convexity

We say that an open set Ω is *weakly lineally convex* if there passes, through every boundary point $p \in \partial\Omega$ a complex hyperplane which does not cut Ω.

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Conjecture

Let Ω be an open connected subset of $\mathbb{C}^n$ with boundary of class C^1 and assume that Ω is locally weakly lineally convex. Then Ω is weakly lineally convex.

When Ω is bounded this is contained in Proposition 4.6.4 in Hörmander (1994). In my paper (1998) I claimed that his proof is valid also if Ω is unbounded. However, this was careless and must be justified.

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I believe the conjecture is true.

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Thank you!