Questions inspired by Mikael Passare's mathematics

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1. Introduction

Mikael Passare (1959–2011) was a brilliant mathematician. His PhD thesis from 1984 was a breakthrough in the theory of residues in several complex variables. Later he switched to the theory of amoebas and coamoebas. In discussions with him during the last thirty years many questions have emerged—not all of them were resolved at the time of his premature death. |

I will present some of these unanswered questions, preceded by a discussion leading up to the question. Some of the questions might present challenges to his nine former PhD students, to his many collaborators around the globe—and to anybody interested.

Mikael started his studies at Uppsala University in the fall of 1976 while still a high-school student, merely seventeen and a half. He finished high school in June, 1978, gave his first seminar talk in November 1978, got his Bachelor Degree in 1979, and presented his PhD thesis at Uppsala University on December 15, 1984. |

He was appointed a professor at Stockholm University in 1994, on the chair which was created for Sonja Kovalevsky (1850–1891) and held during seven years, 1957–1964, by his mathematical grandfather Lars Hörmander. |

Mikael Passare died from a sudden cardiac arrest in Oman on September 15, 2011. He is buried not far from Sonja's grave.

I shall first present two questions in complex analysis: the non-associativity of multiplication of principal-value distributions and residue currents, followed by a section on constructions using meromorphic extension (questions from the early 1980s and up to 1988). |

Then I will speak about Mikael's more recent interest: amoebas and tropical geometry (questions from the period 2003–2010). |

Tropical geometry has intriguing connections to digital geometry, mathematical morphology, discrete optimization, and crystallography, including the theory of quasi-crystals. I believe these connections could be further developed—I hope they will.

2. Multiplication of residue currents and principal-value distributions

Let f and g be holomorphic functions of n complex variables. The *principal value* $\mathrm{PV}(f/g)$ of f/g is a distribution defined by the formula

$$\left\langle \mathrm{PV}\left(\frac{f}{g}\right), \varphi \right\rangle = \lim_{\varepsilon\to 0} \int_{|g|>\varepsilon} \frac{f\varphi}{g} = \lim_{\varepsilon\to 0} \int \frac{\chi f \varphi}{g}, \qquad \varphi \in \mathscr{D}(\mathbb{C}^n),$$

where $\chi = \chi(|g|/\varepsilon)$ and χ is a smooth function on the real axis satisfying $0 \leqslant \chi \leqslant 1$ and $\chi(t) = 0$ for $t \leqslant 1$, $\chi(t) = 1$ for $t \geqslant 2$ (in Passare (1985:727) when the $f_j = 1$ and in (1988:39) in general).

The *residue current* is $\bar\partial \mathrm{PV}(f/g)$. Can the products

$$(\mathrm{PV}(f_1/g_1))(\mathrm{PV}(f_2/g_2)), \quad \left(\bar\partial(\mathrm{PV}(f_1/g_1))\right)(\mathrm{PV}(f_2/g_2))$$

and other similar products be defined? |

Schwartz proved (1954) that it is in general impossible to multiply two distributions while keeping the associative law. |He indicated three distributions $u, v, w \in \mathscr{D}'(\mathbb{R})$ where uv, vw, $(uv)w$ and $u(vw)$ all have a good meaning but where $(uv)w \neq u(vw)$. He took $u = \mathrm{PV}(1/x)$, the principal value of $1/x$; v as the identity, i.e., the smooth function $v(x) = x$, which can be multiplied to any distribution; and $w = \delta$, the Dirac measure placed at the origin. |Then we have $uv = 1$, $(uv)w = \delta$, while $vw = 0$, $u(vw) = 0$. Hence there is no associative multiplication in $\mathscr{D}'(\mathbb{R})$. |An easy modification proves the same result for $\mathscr{E}'(\mathbb{R})$, the distributions of compact support.

Mikael's construction of residue currents and principal-value distributions goes as follows. Take $f = (f_1, \dots, f_{p+q})$, $g = (g_1, \dots, g_{p+q})$, two $(p+q)$-tuples of holomorphic functions, and consider the limit

$$\lim_{\varepsilon_j\to 0} \frac{f_1}{g_1} \cdots \frac{f_{p+q}}{g_{p+q}} \bar\partial\chi_1 \wedge \cdots \wedge \bar\partial\chi_p \cdot \chi_{p+1} \cdots \chi_{p+q}, \tag{1}$$

where $\chi_j = \chi(|g_j|/\varepsilon_j)$ and the ε_j tend to zero in some way. |

Coleff and Herrera (1978:35–36) took $q = 0$ or 1 and assumed that ε_j tends to zero much faster than ε_{j+1}, which means that $\varepsilon_j/\varepsilon_{j+1}^m \to 0$ for all $m \in \mathbb{N}$ and $j = 1, \dots, p+q-1$; thus it is almost an iterated limit. |This gives rise to the strange situation that the limit depends in general on the order of the functions (and is not just an alternating product). |However, in the case of complete intersection, the construction is satisfactory, and Mikael's construction then gives the same results as that of Coleff and Herrera, but Mikael's construction (1988) is valid also when we do not have a complete intersection.

Mikael took $\varepsilon_j = \varepsilon^{s_j}$ for fixed $s_1, \dots, s_{p+q}$. The limit, which will be written as $R^p P^q[f/g](s)$, where we now write $[\dots]$ for the principal value, does not exist for arbitrary s_j. But he proved that, if we remove finitely many hyperplanes, then $R^p P^q[f/g](s)$ is locally constant in a finite subdivision of the simplex

$$\Sigma = \left\{ s \in \mathbb{R}^{p+q};\ s_j > 0,\ \sum s_j = 1 \right\},$$

so that the mean value, $R^p P^q \left[\frac{f}{g}\right]$

$$= \fint_{\Sigma} R^p P^q \left[\frac{f}{g}\right](s) = \bar\partial \left[\frac{f_1}{g_1}\right] \wedge \cdots \wedge \bar\partial \left[\frac{f_p}{g_p}\right] \cdot \left[\frac{f_{p+1}}{g_{p+1}}\right] \cdots \left[\frac{f_{p+q}}{g_{p+q}}\right]$$

exists (Definition A in Passare (1987:159)). This is the product of p residue currents and q principal-value distributions.

In the little paper (1993), based on his talk when accepting the Lilly and Sven Thuréus Prize in 1991, he discusses the possibility of defining the product $\mathrm{PV}(1/x)\delta$ on the real axis, and finds that it should be $-\frac{1}{2}\delta'$, which is the mean value of $-\delta'$ and zero. This is an analogue in real analysis to the mean value over Σ which he considered in the complex case.

Leibniz' rule for the derivative of a product and some other rules of calculus hold; for example we have (1988:43):

$$\left[\frac{1}{z_1}\right]\left[\frac{z_1}{z_2}\right] = \left[\frac{1}{z_2}\right],$$

which yields

$$\left(\bar\partial\left[\frac{1}{z_1}\right]\right)\left\{\left[\frac{1}{z_1}\right]\left[\frac{z_1}{z_2}\right]\right\} = \left(\bar\partial\left[\frac{1}{z_1}\right]\right)\left[\frac{1}{z_2}\right] \neq 0,$$

while

$$\left\{\left(\bar\partial\left[\frac{1}{z_1}\right]\right)\left[\frac{1}{z_1}\right]\right\}\left[\frac{z_1}{z_2}\right] = \frac{1}{2} z_1 \left(\bar\partial\left[\frac{1}{z_1^2}\right]\right)\left[\frac{1}{z_2}\right] = \frac{1}{2}\left(\bar\partial\left[\frac{1}{z_1}\right]\right)\left[\frac{1}{z_2}\right].$$

Thus the associative law does not hold. It is natural to ask whether these currents are just as bad as the general distributions when it comes to multiplication, or whether there is a subclass of them with nicer properties.

Question

2. We saw in Schwartz' example that an associative multiplication is impossible in general; the last example shown here makes us wonder whether it is possible to define an associative multiplication for some residue currents and principal-value distributions.

Is there an associative algebra of residue currents and principal-value distributions?

Is there an interesting non-associative algebra of such currents?

3. Constructions using meromorphic extension

3.1. Extending a given meromorphic function

The Riemann ζ-function is a classical example of meromorphic extension: the series

$$\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s},$$

which converges for $s \in \mathbb{C}$, $\operatorname{Re} s > 1$, is extended to a meromorphic function in the whole plane.

The well-known formula

$$\sum_{0}^{\infty} \alpha^j = \frac{1}{1-\alpha}, \qquad |\alpha| < 1,$$

can be used to define more or less funny results like

$$\sum_{0}^{\infty}(-1)^j = \frac{1}{2}; \ \sum_{0}^{\infty} 2^j = -1; \ \sum_{0}^{\infty}(-2)^j = \frac{1}{3}; \ \sum_{0}^{\infty} 3^j = -\frac{1}{2}; \ \sum_{0}^{\infty}(-3)^j = \frac{1}{4}.$$

This is based on the observation that $1/(1-\alpha)$ is a meromorphic function with a single pole at $\alpha = 1$ but otherwise regular. But why should $\sum \alpha^j$ be meromorphic?

The construction of homogeneous distributions in Hörmander (1990: Section 3.2), in particular of the distributions x_+^a on the real axis, is done by meromorphic extension. |

In the three examples mentioned, we have a given meromorphic function in a nonempty open set of the complex plane, and we know that, if it has a meromorphic extension to the whole plane, then the extension is unique. |

The situation is different when we want to construct an object and have to choose a meromorphic function.

3.2. Finding a meromorphic function

Michael Atiyah proved (1970) that if F is a nonnegative real-analytic function, then $\lambda \mapsto F^\lambda$ can be extended to a meromorphic function in all of $\mathbb{C}$. Bernšteĭn and Gel′fand (1969) proved a similar result for polynomials. Using Atiyah's result, Alain Yger (1987) defined residue currents as meromorphic extensions of $(f\bar{f})^\lambda$ for a holomorphic f, and Mikael compared them (Definition B in (1987:159)) with his own construction (Definition A already mentioned).

In this case the authors construct $(f\bar{f})^\lambda$ for just one value of λ, viz. $\lambda = -1$, which means that they have to choose a parametrized family; the choice is not obvious. |

Another kind of limit of a meromorphic function is

$$\lim_{\varepsilon_j \to 0} \bar{\partial}\frac{f_1}{|f_1|^2+\varepsilon_1} \wedge \cdots \wedge \bar{\partial}\frac{f_p}{|f_p|^2+\varepsilon_p},$$

which is obtained by taking $\chi_j(t) = t/(t+1)$ in (1) (the case $q = 0$). It yields the same current as the former construction for complete intersections (Björk & Samuelsson (2010:35); cf. earlier results by Samuelsson (2006: Corollary 26)).

If we want to evaluate a divergent integral, for instance

$$\int_0^1 x^{-2}dx, \tag{2}$$

one method is to embed the integrand into a family of functions depending on a parameter and define the integral as the value of an extension in the parameter space. In the case mentioned, we can define $f(x,\alpha) = x^\alpha$, and since

$$\Phi(\alpha) = \int_0^1 f(x,\alpha)\,dx = \int_0^1 x^\alpha\,dx = \frac{1}{\alpha+1}$$

is well-defined for $\operatorname{Re}\alpha > -1$ and has a meromorphic extension to the whole complex plane, we can define the integral of x^{-2} as $\Phi(-2) = -1$. The question is now: will we get a different answer if we use a different meromorphic function?

It can be remarked that $\Phi(-2)$ is also the finite part of the integral (2) in view of the formula

$$\int_\varepsilon^1 x^{-2}\,dx = \frac{1}{\varepsilon} - 1, \quad 0 < \varepsilon < 1,$$

where $1/\varepsilon$ is the infinite part (to be thrown away) and -1 is the finite part (to be kept). |

We may conclude that meromorphic extension is an often used method to construct mathematical objects.

3.3. Two-parameter families

While meromorphic functions of one variable can be assigned the value ∞ at a pole, and therefore can be defined as good mappings with values in the Riemann sphere $\mathbb{C} \cup \{\infty\}$, meromorphic functions of two variables may have points of indeterminacy: the function $f(z_1,z_2) = z_2/z_1$ can be assigned the value ∞ at a point $(0,z_2) \neq (0,0)$, but at the origin we cannot do so. This explains the trouble we are in for.

I will consider a divergent integral where we can get different values for different choices of parametrized families.

We denote by $D(c,r)$ the open disk with center at c and with radius r:

$$D(c,r) = \{z \in \mathbb{C};\ |z-c| < r\}.$$

Let us consider the divergent integral

$$\int_{D(1,1)} z^{-2} d\lambda(z).$$

|

A simple idea is to vary the disk. We have

$$\int_{D(c,r)} z^{-2} d\lambda(z) = \pi r^2/c^2 \text{ when } r < |c|,$$

i.e., when the origin is not in the closure of $D(c,r)$. Hence the limit of these values is π as $(c,r) \in \mathbb{C} \times \mathbb{R}$ tends to $(1,1)$ under the restriction $r < |c|$. So this is one possible method.

Another idea is to remove a small disk around the origin, like in the definition of the principal value:

$$\text{PV} \int_{D(1,1)} z^{-2}\, d\lambda(z) = \lim_{\varepsilon \to 0} \int_{z \in D(1,1),\, |z| > \varepsilon} z^{-2}\, d\lambda(z)$$

$$= 2 \int_0^{\pi/2} \cos 2\theta \log \cos \theta \, d\theta = \tfrac{1}{2}\pi.$$

The last integral is evaluated in Gradšteĭn & Ryžik (1962:598:4.384.7). |

Are there other ways to approach the divergent integral? Let us look at a two-parameter family.

Lemma

For $(\alpha,\beta) \in \mathbb{C}^2$ with $\mathrm{Re}(\alpha+\beta) > -2$ we have

$$F(\alpha,\beta) = \int_{D(1,1)} z^{\alpha} \bar{z}^{\beta}\, d\lambda(z) = \frac{\pi\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)}. \qquad (3)$$

|

Proof.

This follows from Gradštejn & Ryžik (1962:490:3.892.2). □

So the extended values of the integral (3) define a meromorphic function F in all of $\mathbb{C}^2$, with singularities, e.g., on the hyperplane $\alpha+\beta = -2$. But restrictions of F may be free from singularities: the function $\beta \mapsto F(m,\beta)$ is entire (in fact a polynomial of degreee m) for every $m \in \mathbb{N}$. At the point $(\alpha,\beta) = (-1,-1)$ we can assign the value ∞ to F if we like, but the point $(\alpha,\beta) = (-2,0)$ is a point of indeterminacy. |

We take $(\alpha,\beta) = (-2+\varepsilon, \theta-\varepsilon)$ and consider

$$F(\alpha,\beta) = F(-2+\varepsilon, \theta-\varepsilon) = \frac{\pi\varepsilon}{\theta} \frac{\Gamma(1+\theta)}{\Gamma(1+\varepsilon)\Gamma(2+\theta-\varepsilon)} = \frac{\pi\varepsilon}{\theta}(1+o(1))$$

as $(\theta,\varepsilon) \to (0,0)$ under the restriction $\mathrm{Re}\,\theta > 0$. The quantity $\pi\varepsilon/\theta$ has no limit as $(\varepsilon,\theta) \to (0,0)$, but we may introduce a relation between ε and θ to create a one-parameter family of functions which has a limit, e.g., $\varepsilon = \theta$ or $\varepsilon = 0$.

If we take $\varepsilon = \theta$, we obtain

$$G(\theta) = F(-2+\theta, 0) = \int_{D(1,1)} z^{-2+\theta}\, d\lambda = \pi, \quad \operatorname{Re}\theta > 0.$$

|

If we take $\varepsilon = 0$, we get

$$H(\theta) = F(-2, \theta) = \int_{D(1,1)} z^{-2}|z|^{\theta}\, d\lambda = 0, \quad \operatorname{Re}\theta > 0.$$

|

More generally, we may take $\varepsilon = c\theta$ and get the limit $c\pi$ for certain values of c, or $\theta = \varepsilon^2$ and get infinity.

Question

3. Meromorphic extension using two parameters easily leads to points of indeterminacy, and so gives rise to infinitely many one-parameter families. Some choice must be made. And what is then the natural choice?

|

This text was essentially written 1988-10-17 and sent out to some people, among them Mikael Passare and Bo Berndtsson. Bo expressed surprise.

4. The axioms of tropical geometry

4.1. Tropicalization

Roughly speaking, tropical mathematics is the mathematics of a structure with addition and maximum as binary operations. The simplest example is the semiring of real numbers $(\mathbb{R}, +, \vee)$, where $+$ denotes usual addition and $\vee$ is the maximum operation, $x \vee y = \max(x, y)$. Note the distributive law $a + (b \vee c) = (a + b) \vee (a + c)$: addition is distributive over maximum. Sometimes $\mathbb{R}$ is augmented by adding an element $-\infty$, the neutral element for the maximum operation: $x \vee (-\infty) = x$. Another name is *idempotent mathematics*, used because of the idempotency of the maximum: $x \vee x = x$.

Tropicalization means that in a semiring with multiplication and addition we replace multiplication by addition, and addition by the maximum operation. This is somewhat reminiscent of taking the logarithm. Start with the semiring $(\mathbb{R}_+, \times, +)$, where $\mathbb{R}_+ = \{x \in \mathbb{R}; x > 0\}$ is the set of positive real numbers, and take the logarithm. |Then

$$\log(x \times y) = \log x + \log y, \text{ and}$$

$$\log(x) \vee \log(y) \leqslant \log(x + y) \leqslant \log 2 + (\log(x) \vee \log(y)),$$

so that multiplication gives addition of the logarithms, and addition comes close to the maximum of the logarithms—a good approximation if $x \gg 1$ or $y \gg 1$.

If we introduce $s = \log x$ and $t = \log y$, and make a change of scale, we see that

$$\log(e^s \times e^t) = s + t;$$

in the limit,

$$h\log(e^{s/h} \times e^{t/h}) = s + t \to s + t \text{ and } h\log(e^{s/h} + e^{t/h}) \to s \vee t$$

as $h \to 0+$. |Thus we may say that tropicalization is a limiting case of logarithmization. Here $h > 0$ is an analogue of Planck's constant—hence the name *dequantization* (Litvinov (2005, 2007); Viro (2001: Section 2.1)). |

For basic concepts of tropical geometry, see also Viro (2010, 2011).

4.2. Tropical straight lines

A polynomial function of degree one has the form

$$(x, y) \mapsto f(x, y) = ax + by + c.$$

So if we tropicalize it we get

$$f_{\text{trop}}(x, y) = (a + x) \vee (b + y) \vee c.$$

Thus f_{trop} is a convex function with a simple structure. It is piecewise affine outside the three lines $x + a = c$, $y + b = c$, and $x + a = y + b$, which meet at the point

$$p = (p_1, p_2) = (c - a, c - b).$$

|

More precisely, it is affine outside the three rays emanating from that point in the directions $(-1, 0)$, $(0, -1)$ and $(1, 1)$. We shall call this the *tropical straight line* TSL(p) with *vertex* p.

We see that in general two different tropical lines intersect in a single point. An exception occurs when the lines have their vertices at points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ with $p_1 = q_1$ or $p_2 = q_2$ or $p_2 - p_1 = q_2 - q_1$, i.e., when $q \in$ TSL(p) or $p \in$ TSL(q). |Then there are infinitely many points in the intersection, but Mikael explained that we should require stability under small perturbations, which means that we should define the intersection as the limit of the unique intersection of the lines TSL(p) and TSL$(q + (\delta, \varepsilon))$ as $(\delta, \varepsilon) \to (0, 0)$ while avoiding the exceptional values (cf. Richter-Gebert et al. (2005: Theorem 4.3) and Tabera (2008: Definition 4)).

In this way two distinct tropical straight lines always have a unique intersection, just like in spherical geometry. And, similarly, two distinct points always define a tropical straight line, except in certain cases, where again we use stability to impose uniqueness. |

Question

4. So we may ask about all of Euclid's axioms! Is it possible to build up an axiomatic theory for tropical geometry? What are the similarities with spherical geometry?

|

I asked Mikael these questions a few years ago.

5. Tropical polynomial functions

5.1. Largest tropical minorants

A tropical polynomial function of one real variable is of the form

$$f(x) = \sup_{j \in \mathbb{Z}} (a_j + jx), \qquad x \in \mathbb{R}.$$

It is a piecewise affine function, where each piece has integer slope.

To any function $G\colon \mathbb{R} \to \mathbb{R}$ we associate its largest tropical minorant f. The example

$$G(x) = 0 \vee (a + \tfrac{1}{2}x) \vee x, \qquad x \in \mathbb{R},$$

shows that the difference $G - f$ can be arbitrarily large even if G is convex: in this case we have $f(x) = 0 \vee x$, so that $G(0) - f(0) = 0 \vee a$.

On the other hand, if

$$G(x) = \sup_{|z|=e^x} \log|h(z)|, \qquad x \in \mathbb{R}, \tag{4}$$

for a holomorphic function h, I proved (1984:168) that

$$f \leqslant G \leqslant f + \log 3.$$

Question

5.1. Which is the smallest constant c such that $f \leqslant G \leqslant f + c$ if G is of the form (4) *for some polynomial h?*

We know that $c \leqslant \log 3 \approx 1.09861$. The example $h(z) = 1 + z$ shows that $c \geqslant \log 2 \approx 0.69315$: in this case $G(0) = \log 2$ and $f(0) = 0$.

I sent this question to Mikael Passare, Jan Boman, David Jacquet, Hans Rullgård, Erik Melin, and Markku Ekonen in a letter of 2003-10-21.

5.2. Approximation of the exponential function

The Fenchel transformation is a tropical analogue of the Fourier transformation or the Laplace transformation.

The *Fenchel transform* $\tilde{f}$ of a function $f\colon \mathbb{R} \to [-\infty, +\infty]$ is defined by

$$\tilde{f}(\xi) = \sup_{x \in \mathbb{R}} \big(\xi x - f(x)\big), \qquad \xi \in \mathbb{R}.$$

We have $\tilde{\tilde{f}} \leqslant f$ with equality if and only if f is convex, lower semicontinuous, and takes the value $-\infty$ only if it is identically $-\infty$.

The equality $f = \tilde{\tilde{f}}$ means that f is represented as a supremum of affine functions—a tropical integral of the simplest convex functions—just as the Fourier inversion formula $f = \mathscr{F}^{-1}(\mathscr{F}(f))$ represents f as an integral of the simplest oscillations.

Now take $f(x) = e^x$, $x \in \mathbb{R}$. The Fenchel transform of this function is

$$\tilde{f}(\xi) = \begin{cases} +\infty, & \xi < 0, \\ 0, & \xi = 0, \\ \xi \log \xi - \xi, & \xi > 0. \end{cases}$$

|

Then, for $\xi > 0$,

$$e^{\tilde{f}(\xi)} = \xi^\xi e^{-\xi} = \sup_{x \in \mathbb{R}} e^{\xi x - e^x} = \sup_{y > 0} y^\xi e^{-y} \approx \int_0^\infty y^\xi e^{-y} dy = \xi!,$$

for the integral is approximately equal to a tropical integral, i.e., a supremum. This is a crude form of Stirling's formula.

Furthermore, since $\tilde{\tilde{f}} = f$,

$$e^{e^x} = e^{\tilde{\tilde{f}}(x)} = \sup_{\xi > 0} e^{\xi x - \tilde{f}(\xi)} = \sup_{\xi > 0} \frac{e^{\xi x}}{e^{\tilde{f}(\xi)}} \approx \sup_{\xi > 0} \frac{e^{\xi x}}{\xi!} \approx \sum_{k \in \mathbb{N}} \frac{e^{kx}}{k!} = e^{e^x},$$

where we have used the former approximation that $e^{\tilde{f}(\xi)} \approx \xi!$ and a new tropical approximation: the sum defining the exponential function is approximately equal to a tropical sum, i.e., to a supremum.

Question

5.2. We have thus showed that e^{e^x} is approximatively equal to e^{e^x}. This is not so remarkable. But the surprising fact is that it is an exact equality. After two approximations we return to the exact value. Is there an explanation? To be precise: is there a more direct explanation why

$$\sup_{\xi > 0} \frac{e^{\xi x}}{e^{\tilde{f}(\xi)}} = \sum_{k=0}^{\infty} \frac{e^{kx}}{k!}, \qquad x \in \mathbb{R}?$$

Are there other, similar examples?

|

I sent this question to Mikael Passare, Jan Boman, David Jacquet, Hans Rullgård, Erik Melin, and Markku Ekonen on 2003-10-21.

5.3. Representation of tropical polynomial functions by tropical polynomials

Proposition

A tropical polynomial function

$$f(x) = \sup_{\alpha \in \mathbb{Z}^n} (x \cdot \alpha + a_\alpha), \qquad x \in \mathbb{R}^n,$$

with coefficients $a_\alpha \in [-\infty, +\infty]$ can be represented in several ways by tropical polynomials. The representation

$$f(x) = \sup_{\alpha \in \mathbb{Z}^n} (x \cdot \alpha + b_\alpha), \qquad x \in \mathbb{R}^n,$$

with $b_\alpha = -\tilde{f}(\alpha)$, $\alpha \in \mathbb{Z}^n$, is the one with largest coefficients.

5.4. The exponential of a tropical polynomial function

Let $\varphi\colon \mathbb{Z}^n \to [-\infty, +\infty]$ be a function on the integer points which is $< +\infty$ only at finitely many of them. Equivalently we may take φ defined on $\mathbb{R}^n$ but with the value $+\infty$ at all points in $\mathbb{R}^n \smallsetminus \mathbb{Z}^n$.

We define $f = \tilde{\varphi}$, the Fenchel transform of φ,

$$f(x) = \tilde{\varphi}(x) = \sup_{\alpha \in \mathbb{Z}^n} \big(x \cdot \alpha - \varphi(\alpha)\big), \qquad x \in \mathbb{R}^n.$$

It is a tropical polynomial function.

Passing to the exponential, we see that

$$e^{f(x)} = e^{\tilde{\varphi}(x)} = \sup_{\alpha \in \mathbb{Z}^n} e^{x\cdot\alpha} e^{-\varphi(\alpha)} \leqslant \sum_{\alpha \in \mathbb{Z}^n} e^{x\cdot\alpha} e^{-\varphi(\alpha)} = g(y),$$

where $y_j = e^{x_j}$ and

$$g(y) = \sum_{\alpha \in \mathbb{Z}^n} e^{-\varphi(\alpha)} y^\alpha$$

is a classical Laurent polynomial majorizing $e^{f(x)} = e^{\tilde{\varphi}(x)}$, but actually often rather close to it.

Summing up, we see that φ contains all information, from which $f = \tilde{\varphi}$ and g can be constructed. Also g determines its coefficients $\exp(-\varphi(\alpha))$, thus also φ and f. On the other hand, $f = \tilde{\varphi}$ contains less information, from which φ cannot in general be retrieved.

Question

5.3. Is it possible to pass from $e^f = e^{\tilde{\varphi}}$ to g using some other structure? (Cf. the next section.)

Question

5.4. Is it possible to pass to the limit in some way so that the classical polynomials tend to the tropical one?

6. Ghosts in tropical mathematics

In a polynomial function $f(z) = \sum a_j z^j$ all coefficients can be retrieved from the values of f, both in the real case and in the complex case: if $\sum a_j z^j = \sum b_j z^j$ for sufficiently many z, then $a_j = b_j$ for all j.

But in a tropical polynomial function $f(x) = \sup_j (a_j + jx)$, a coefficient a_k cannot be retrieved from the values of f if $a_k \leqslant \frac{1}{2}(a_{k-1} + a_{k+1})$: if this is so, we can replace a_k by a smaller value without changing the values of f at any point. In fact, under this hypothesis,

$$a_k + kx \leqslant \big(a_{k-1} + (k-1)x\big) \vee \big(a_{k+1} + (k+1)x\big) \text{ for all } x \in \mathbb{R}.$$

Thus such a coefficient is invisible. We see that many tropical polynomials $P(X) = \bigvee_{j \in \mathbb{Z}} (a_j + jX)$ have the same evaluations on the real axis.

On the other hand, if the function $\mathbb{Z} \ni j \mapsto a_j$ is strictly concave in the sense that $a_j > \frac{1}{2}a_{j-1} + \frac{1}{2}a_{j+1}$ for all $j \in \mathbb{Z}$, then the coefficients can be retrieved from the values of the function; in fact, under this hypothesis, $a_j = -\tilde{f}(j)$, $j \in \mathbb{Z}$. |

I found this slightly disturbing, and asked Mikael the following question in a letter of 2010-03-26. |

Question

6.1. Is there some structure which will allow us to retrieve all coefficients of a tropical polynomial from its point evaluations?

It is clear that we need more information than just the values on the real axis. |

In his answer of 2010-03-29, Mikael directed me to the preprint by Izhakian and Rowen (2009), published as (2010). (Perhaps the paper by Izhakian (2009) is easier to start with.) It seemed to me that Mikael hinted at a solution in that the ghost elements could be used to retrieve the coefficients.

A first lesson is that, just as for classical polynomials, we must distinguish between a polynomial and the function given by a polynomial. A polynomial $P(X) = \sum a_j X^j$ is a formal expression containing an indeterminate X. If we give X a value as a variable in some ring, we get a polynomial function. For instance, if $P(X) = X^3 - 3X^2 + 2X$, and we replace X by a variable in the finite field $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$, then the value is everywhere zero. |

The situation is similar for tropical polynomials: a tropical polynomial is a formal expression $P(X) = \bigvee_{j \in \mathbb{N}}(a_j + jX)$, where information about all the coefficients a_j is preserved. That the coefficients are not retrievable from an evaluation in $\mathbb{R}$ is no more upsetting than the example with evaluation in $\mathbb{Z}_3$.

In the quoted papers by Izhakian and Rowen, the authors construct a space $\mathbb{T} = (\mathbb{R} \times \{0,1\}) \cup \{-\infty\}$, where we have two copies of $\mathbb{R}$; the first copy consists of the usual real numbers, represented as $(x,0)$, the second of the ghost elements, represented as $(x,1)$. We can evaluate a tropical polynomial at the points of $\mathbb{T}$. |

However, Zur Izhakian explains:

In the tropical framework there is no injection of the polynomial semiring into the function semiring. Namely, a function could have several polynomial descriptions. In particular there are monomials (called *inessential*) which can be omitted without changing the function determined by the polynomial.

This phenomena is obtained due to convexity considerations involved in this setting, which cause a loss of information. Accordingly, a full recovery of the exact coefficients of a polynomial from the corresponding function is not always possible. (Zur Izhakian, personal communication 2011-10-26)

|So this is just the observation I made concerning evaluation at real numbers in the beginning of this section, but now extended to the larger space $\mathbb{T}$. |

This means that my original Question 6.1 remains unanswered.

Question

6.2. It is well known that the coefficients of a holomorphic function can be retrieved from its values:

$$\text{if } h(z) = \sum_{j\in\mathbb{N}} a_j z^j, \; z \in \mathbb{C}, \quad \text{then } a_k = \frac{1}{2\pi i}\int_{|z|=r} \frac{h(z)}{z^{k+1}}dz, \quad k \in \mathbb{N}.$$

|

What can be said if we know only the values of the growth function

$$g(r) = \sup_{|z|=r} |h(z)|, \qquad r \geqslant 0?$$

If we have two entire functions h_1 and h_2 with growth functions g_1 and g_2, does it follow that $g_1(r) = g_2(r)$ for all r only if the coefficients of h_1 and h_2 have the same absolute values? Is there even a formula that yields the $|a_j|$ from g?

We can at least retrieve the absolute values of the first and second nonzero coefficients from the growth function: |

Proposition

Let h be an entire function with Taylor series

$$h(z) = \sum_{q}^{\infty} a_j z^j, \qquad z \in \mathbb{C},$$

and let g be its growth function. Then

$$g(r)r^{-q} \to |a_q| \text{ and } \frac{g(r)r^{-q} - |a_q|}{r} \to |a_{q+1}| \text{ as } r \to 0+.$$

Corollary

If h is a polynomial of the form

$$h(z) = \sum_{q}^{q+3} a_j z^j,$$

then the absolute values of all four coefficients $a_q, a_{q+1}, a_{q+2}, a_{q+3}$ can be determined from the growth function.

When the coefficients are real, Jean-Pierre Kahane could give an affirmative answer:

Proposition

(Jean-Pierre Kahane, 2011-11-11.) *If $h(z) = \sum a_j z^j$ is an entire function with real coefficients a_j and if a_0 and a_1 are positive, then all coefficients can be determined from the Taylor expansion at the origin of the square of the growth function.*

Proof.

From the following lemma we have that $h(r)\overline{h(r)} = g(r)^2$ for small r. It is easy to see that the coefficients a_j of h can be read off from the Taylor expansion of $h\bar{h}$ at the origin. □

Lemma

(Jean-Pierre Kahane, 2011-11-11.) *If $h(z) = \sum a_j z^j$ is an entire function with real coefficients a_j and if a_0 and a_1 are positive, then $g(r) = |h(r)|$ for sufficiently small $r \geqslant 0$.*

7. The discrete Prékopa problem

If $F\colon \mathbb{R}^2 \to [-\infty, +\infty]$ is a function of two real variables, its *marginal function* $H\colon \mathbb{R} \to [-\infty, +\infty]$ is defined by

$$H(x) = \inf_{y\in\mathbb{R}} F(x,y), \qquad x \in \mathbb{R}.$$

It is well known and easy to prove that H is convex if F is convex.

There is a more general marginal function H_p, called the *p-marginal function*, of $F\colon \mathbb{R}^2 \to [-\infty, +\infty]$. It is defined by

$$e^{-pH_p(x)} = \int_{\mathbb{R}} e^{-pF(x,y)} dy, \qquad x \in \mathbb{R},$$

for any positive real number p. Prékopa's theorem, first presented by András Prékopa in Budapest in 1972 and published in (1973: Theorem 6), says that H_p is convex if F is convex.

In the discrete case, with $f\colon \mathbb{Z}^2 \to [-\infty, +\infty]$, we define the marginal function h by

$$h(x) = \inf_{y\in\mathbb{Z}} f(x,y), \qquad x \in \mathbb{Z},$$

and the p-marginal function h_p by

$$e^{-ph_p(x)} = \sum_{y\in\mathbb{Z}} e^{-pf(x,y)}, \qquad x \in \mathbb{Z}. \tag{5}$$

The classical marginal function $h = h_\infty$ is a limiting case when $p \to +\infty$ and may be defined by

$$e^{-h_\infty(x)} = \sup_{y\in\mathbb{Z}} e^{-f(x,y)}, \qquad x \in \mathbb{Z}. \tag{6}$$

So we may say that (6) is a dequantization of the sums of the exponential functions in (5): replacing the sum by the sup. We have $h_p \leqslant h_\infty$.

Question

7.1. Is it possible to go from $\sum_{y\in\mathbb{Z}} e^{-f(x,y)}$ to $\sup_{y\in\mathbb{Z}} e^{-f(x,y)}$ under some reasonable hypotheses on f? (Cf. Question 4.4.)

|

In the digital case, it is not enough to assume that f has a convex extension to all of $\mathbb{R}^2$ to conclude that h is convex. But a stronger convexity property, now called rhomboidal convexity, implies that h is convex (Kiselman 2008). I call $f\colon \mathbb{Z}^2 \to \mathbb{R}$ *rhomboidally convex* if its second differences satisfy six conditions:

$$D_b D_a f \geqslant 0 \text{ for all } (a,b) \in \{((1,0),(1,b_2));\ b_2 = -1,0,1\},$$

as well as

$$D_b D_a f \geqslant 0 \text{ for all } (a,b) \in \{((0,1),(b_1,1));\ b_1 = -1,0,1\}.$$

Here D_a is the difference operator $(D_a f)(x) = f(x+a) - f(x)$. It is not known whether the result holds for the p-marginal function. This is the *discrete Prékopa problem:*

Question

7.2. Is it true that the p-marginal function h_p is convex if f is rhomboidally convex?

|

I asked Mikael Passare and Bo Berndtsson this question in the spring of 2008. |

It is enough here to take $p = 1$. Examples show that it is not enough that f admits a convex extension to all of $\mathbb{R}^2$, and that rhomboidal convexity is sufficient in some special classes.

8. A conjecture on coamoebas and Newton polytopes

This part was contributed by Timur Sadykov and is included here with his permission. The conjecture was formulated by Mikael in Stockholm in December, 2010, and written down on a napkin. Timur kept this napkin and reconstructed the conversation. |

Definition

The *Newton polytope* of a Laurent polynomial $f(z) = \sum_{\alpha\in A} a_\alpha z^\alpha$ with a finite set $A \subset \mathbb{Z}^n$ is defined to be the convex hull in $\mathbb{R}^n$ of the set $\{\alpha \in A;\ a_\alpha \neq 0\}$. We will denote this polytope by Δ_f. □

Let f be a Laurent polynomial in n complex variables $z_1, \dots, z_n$. Assume that for any face F of the Newton polytope Δ_f, the restriction $f|_F$ of f to F, defined as

$$f|_F(z) = \sum_{\alpha\in A\cap F} a_\alpha z^\alpha,$$

does not vanish on $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n;\ x_j > 0\}$.

Let us embed the Newton polytope Δ_f into the complex space with coordinates $z_1, \ldots, z_n$ by identifying it with the following intersection of half spaces.

$$\Delta_f = \bigcap_k \{\sigma \in \mathbb{R}^n; \langle \mu_k, \sigma \rangle \geqslant \nu_k\},$$

where the μ_k are the normals to the faces of Δ_f and

$$\sigma = (\sigma_1, \ldots, \sigma_n) = (\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n).$$

Definition

A Laurent polynomial is said to be *maximally sparse* if the number of its monomials is equal to the number of vertices of its Newton polytope. □

In other words, $f(z) = \sum_{\alpha \in A} a_\alpha z^\alpha$ with a finite set $A \subset \mathbb{Z}^n$ is called maximally sparse if the cardinality of A equals the number of vertices of the convex hull of A.

Maximally sparse polynomials enjoy certain minimality properties. For instance, it has been proved by Mounir Nisse that the amoeba of a maximally sparse polynomial is necessarily solid; see Nisse (2008). (An amoeba of a polynomial f is said to be *solid* if the number of components of its complement is as small as it can possibly be, that is, if it equals the number of vertices of the Newton polytope Δ_f.) The solidness of the amoeba is also one of the characteristic properties of discriminants according to Passare et al. (2005).

Conjecture

(Mikael Passare, 2010-12.) *For $z \in \mathbb{C}^n$, let $f(z)$ be a maximally sparse Laurent polynomial with generic coefficients. Then the number of holes in the compactified coamoeba $\overline{\mathscr{A}'_f}$ is equal to* $n! \operatorname{Vol}(\Delta_f)$.

9. The constant term in powers of a Laurent polynomial

Let $P(X) = \sum_{\alpha \in A} a_\alpha X^\alpha$, A a finite subset of $\mathbb{Z}^n$, be a Laurent polynomial and assume that its Newton polytope contains the origin in its interior. We consider powers $P(X)^k$, $k \in \mathbb{N}$, of $P(X)$ and denote by $c_k(P)$ the constant term in $P(X)^k$. The question is whether there are infinitely many k such that $c_k(P)$ is nonzero. This seems plausible, and for $n = 1$ it is not difficult to prove. Mikael lectured on this problem in the Pluricomplex Seminar on 2000-03-14.

Alain Yger points out that Hans Duistermaat and Wilberd van der Kallen (1998) proved that the answer is in the affirmative as well as a more precise result: the radius of convergence of the formal power series $\sum_{k=1}^{\infty} c_k(P) t^k$ is finite (Alain Yger, personal communication 2011-12-01). However, the proof relies on a very heavy machinery when $n \geqslant 2$. |

Alain writes that Mikael was "deeply concerned" about finding a simpler proof of this result. So we may list a new question: |

Question

9. Is there a simpler proof of the result of Duistermaat and van der Kallen (1998) that the constant term $c_k(P)$ in the k-th power of a Laurent polynomial is different from zero for infinitely many values of k?

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Mikael's PhD students

Mikael served as advisor of nine PhD students who successfully completed their degrees. They are registered in the *Mathematics Genealogy Project* and are:

Yang Xing, 1992, Stockholm University: *Zeros and Growth of Entire Functions of Several Variables, the Complex Monge–Ampère Operator and Some Related Topics*.

Mikael Forsberg, 1998, The Royal Institute of Technology: *Amoebas and Laurent Series*.

Lars Filipsson, 1999, The Royal Institute of Technology: *On Polynomial Interpolation and Complex Convexity*.

Timur Sadykov, 2002, Stockholm University: *Hypergeometric Functions in Several Complex Variables*. Although not listed in the Genealogy Project, August Tsikh served as a coadvisor for Timur.

Hans Rullgård, 2003, Stockholm University: *Topics in Geometry, Analysis and Inverse Problems*.

Johan Andersson, 2006, Stockholm University: *Summation Formulae and Zeta Functions*.

Alexey Shchuplev, 2007, Stockholm University: *Toric Varieties and Residues*. August Tsikh was second advisor.

David Jacquet, 2008, Stockholm University: *On Complex Convexity*.

Lisa Nilsson, 2009, Stockholm University: *Amoebas, Discriminants, and Hypergeometric Functions*. August Tsikh was second advisor.

Africa

Mikael was deeply involved in the development of mathematics in Africa: he was a member of the Board of the International Science Programme (ISP), Uppsala, and a member of the Board of the Pan-African Centre for Mathematics (PACM) in Dar es-Salaam, Tanzania. He was a driving force in the creation of this Pan-African Centre, which is a collaborative project between Stockholm University and the University of Dar es-Salaam, and was actively searching for a director of PACM.

Lineal convexity

André Martineau (1930–1972) gave a couple of seminars on lineal convexity (*convexité linéelle*) in Nice during the academic year 1967-68, when I was there. This is a kind of complex convexity which is stronger than pseudoconvexity and weaker than convexity. |Since I was of the opinion that the results for this convexity property were too scattered in the literature and did not always have optimal proofs, I suggested that Mikael write a survey article on the topic.|

On the one hand, this piece of advice was certainly very good, for he found a lot of results in cooperation with his friends Mats Andersson and Ragnar Sigurðsson (Mikael's mathematical uncle). On the other hand, it was perhaps not such a good suggestion, for the survey just kept growing, and two preprints were circulating starting in 1991—and then they had already been busy writing a long time.

The article became a book, and it did not appear until 2004. Anyway, it is thanks to André Martineau that lineal convexity came to be studied in the Nordic countries—and the book has become a standard reference. |

In the book, the authors study in detail a property which Martineau called strong lineal convexity (*convexité linéelle forte*), and which he did not characterize geometrically. This notion, in the book called $\mathbb{C}$-*convexity*, is not linked to any cleistomorphism (closure operator), since the intersection of two strongly lineally convex sets need not have the property. Therefore it has a different character than lineal convexity and usual convexity.

Seminars

Mikael's gave his first seminar talk during the Fall Semester of 1978. He reported on chosen sections of the little book by Lev Isaakovič Ronkin (1931–1998), *The Elements of the Theory of Analytic Functions of Several Variables* (1977), which had been published in Russian in 2,700 copies in Kiev the year before and cost 93 kopecks. The task was a part of the examination for the course *Mathematics D*. |

He gave a total of 29 seminar talks over the period 1978–2010.

The *Nordan* Meetings

Together with Mats Andersson and Peter Ebenfelt, Mikael Passare initiated a series of encounters on complex analysis in the five Nordic countries. Mikael and Peter organized the first conference, which took place in Trosa, Sweden, March 14–16, 1997; Mats the second, in Marstrand, Sweden, April 24–26, 1998. |

Following a voting procedure at the end of the first meeting, these yearly meetings were named *Nordan*. |This is the name in Swedish of a chilly wind from the north, but also reminds us of the original purpose: to promote Nordic Analysis. |

It is a clear reference to *Les Journes complexes du Sud*, which during a long time have taken place in the south of France.

Mikael edited abstracts in Swedish of the lectures—which had all been given in English. These brochures were published with a delay of a few years. Twelve of them have come out; he was preparing the thirteenth, which was to report on *Nordan 13* held in Borgarfjordur in 2009, and asked Ragnar Sigurðsson on September 10, 2011, to write a preface in Icelandic (Ragnar Sigurðsson, personal communication 2011-10-04). |

Lars Filipsson emphasizes (personal communication 2011-10-06) that Mikael wrote these brochures in Swedish to develop Swedish terms in higher mathematics, especially in complex analysis—otherwise the Swedish mathematical terms reach up to the first, possibly the second, university year only.

Nordic meetings like these were something that Mikael and Mats had discussed and planned during many years; both of them wanted to create a forum with a more relaxed atmosphere, where Nordic complex analysts, in particular the young ones, could feel more at home than at international conferences, and which would give those that worked in the Nordic countries occasion to get to know each other better. And the initiative turned out to be a long-lasting success: the fifteenth encounter took place in Röstånga in southern Sweden, May 06–08, 2011; |the sixteenth is going on right now!

An extraordinary curiosity

Andrei Khrennikov writes:

> I would like mention Mikael's extraordinary curiosity, which was extended to a large variety of fields. In particular, he discussed with excitement the possibility of mathematical modeling of cognition, human psychological behavior, and consiousness. I met Mikael and Galina the last time in July 2010, in Stockholm, and during one evening we discussed a large variety of topics: complex and p-adic analysis, mathematical foundations of quantum physics, quantum nonlocality, Bell's inequality and experiments [...] (Andrei Khrennikov, personal communication 2011-11-26)

A passionate traveler

Mikael was a passionate traveler. He visited 152 countries. |

The United Arab Emirates and Oman turned out to be the last ones. Land number 153 should have been Iran: he planned to arrive at Tehran Imam Khomeini International Airport on September 17 at 21:25 (Mikael Passare, electronic letter 2011-09-15 to mathematicians in Tehran). Siamak Yassemi, Head of the School of Mathematics, University of Tehran, was ready to meet him there.

Finally

Mikael's significance goes much beyond his own research. Many persons have testified to his positive view of life, his humor, and to his genuine interest in people he met. He was an unusually stimulating partner in discussions; listening, inspiring, and supportive, in professional situations as well as private ones.

Obituary,

Abridged Obituary,

Questions inspired by Mikael Passare's mathematics,

and other texts are available at the web site

Mikael Passare In Memoriam

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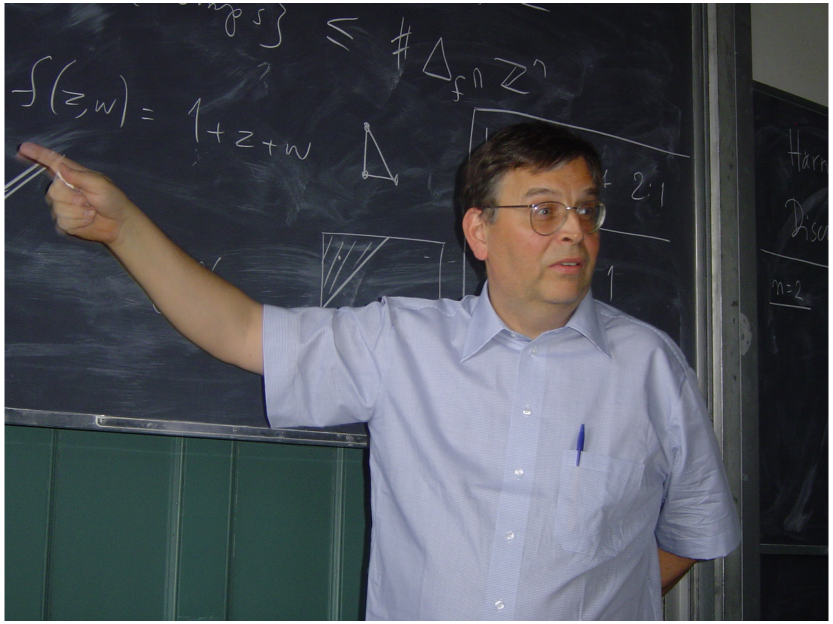
Google the words

Passare Memoriam

and you will find.


30


$f(z,w) = 1+z+w$

Thank you!

Kiitos!

Giitu eatnat!

Tack!