Plurisubharmonic functions and potential theory in several complex variables

A contribution to the book project

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Abstract: We survey the development of the theory of plurisubharmonic functions and the potential theory associated with them from their emergence in 1942 to 1997.

1. Introduction

This is a survey of the development of the theory of plurisubharmonic functions since its inception in 1942 up to 1997. We aim at presenting the ideas as they appear during this period. It is our ambition to make the text accessible to all university mathematicians. This has of course some consequences for the choice of topics as well as for the style of presentation.

We shall also take a look at the theory of extremal plurisubharmonic functions, or, what is roughly the same thing, the plurisubharmonic solutions to the complex Monge–Ampère equation. These functions appear in problems analogous to those of classical potential theory, and the field has sometimes been called pluripotential theory. It is a branch of mathematics where crucial properties of plurisubharmonic functions are studied. However, it is by no means the only one, and we do not pretend to cover all areas where plurisubharmonic functions can be put to use.

In the field of pluripotential theory there are several works of survey character available: the survey article by Bedford [1993] and the books by Klimek [1991] and Cegrell [1988], for instance. Also Kołodziej [1998], although it is a research paper, gives a survey of results for the complex Monge–Ampère operator. We refer the interested reader to the surveys mentioned for more details on some of the subjects treated here—as well as for other topics.

Research is going on concerning many of the themes considered here. This means that it is sometimes not so easy to put things into a proper historical perspective. This is especially true concerning sections 12–16.

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2. Setting the stage

For the convenience of the reader we shall give here the main definitions before we go on with the history of the subject.

The plurisubharmonic functions are in many ways analogous to convex functions. Indeed they relate to subharmonic functions of one complex variable as convex functions of several variables do to convex functions of one real variable. In particular, plurisubharmonic functions need for their definition infinitely many inequalites (cf. (2.6) below), just like convex functions of two or more variables, and they are associated to an overdetermined system of differential equations. On the other hand, a plurisubharmonic function is not necessarily continuous or even locally bounded, which makes questions of local regularity much trickier than corresponding questions for a convex function, which is continuous in any open set where it is finite.

From the point of view of differential equations we can describe the theory of several complex variables as being centered around three important equations. First of these is the *Cauchy–Riemann equation*

$$d''u = f, \tag{2.1}$$

where d'' is the component of bidegree (type) $(0,1)$ of the exterior differentiation operator $d = d' + d''$. For functions f, the operators $d' = \partial$ and $d'' = \overline{\partial}$ (the holomorphic and antiholomorphic parts of d) are defined by

$$d'f = \partial f = \sum \frac{\partial f}{\partial z_j} dz_j, \quad d''f = \overline{\partial} f = \sum \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j;$$

for forms of higher degree they are defined inductively by the formulas

$$d'(u \wedge dz_i) = d'u \wedge dz_i, \quad d'(u \wedge d\overline{z}_j) = d'u \wedge d\overline{z}_j$$

and then extended by linearity; similarly for d''. Here we use the customary notation for complex derivatives:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \qquad z_j = x_j + iy_j. \tag{2.2}$$

Already Poincaré [1899:112] calculated with derivatives with respect to the variables $u_k = x_k - iy_k$; Wirtinger [1927:357] used the notation $\partial/\partial\overline{z}_\gamma$.

Thus in the most basic case of functions u and $(0,1)$-forms $f = \sum f_j d\overline{z}_j$, the equation $d''u = f$ takes the form of a system with n equations

$$\frac{\partial u}{\partial \overline{z}_j} = f_j, \qquad j = 1, ..., n.$$

The exterior differentiation operator d satisfies $dd = 0$ (every exact form or current is closed), and similarly we have $d''d'' = 0$. Therefore a necessary condition for solvability of the equation $d''u = f$ is that f be d''-closed, $d''f = 0$, which in terms of the coefficients f_j takes the form

$$\frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\partial \overline{z}_j}, \qquad j, k = 1, ..., n.$$

We shall touch upon the theory of this equation in section 6, but otherwise we consider it as lying outside the scope of this survey.

The second equation is

$$2id'd''u = f, \tag{2.3}$$

where u is a function and f a differential form or a current of bidegree $(1,1)$. The currents are, roughly speaking, differential forms with distribution coefficients. In this context, one often uses the operator $d^c = i(d'' - d')$, which allows us to write $2id'd'' = dd^c$. The operator $2id'd'' = dd^c$ has real coefficients and maps forms or currents of bidegree (p,q) to currents of bidegree $(p+1, q+1)$. In one complex variable (2.3) can be written

$$2id'd''u = dd^c u = 2i\frac{\partial^2 u}{\partial z \partial \overline{z}} dz \wedge d\overline{z} = (\Delta u) dx \wedge dy = f,$$

where Δ is the Laplace operator, so (2.3) generalizes Poisson's equation. Poincaré [1899:113] studied an overdetermined system of equations

$$\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} = P_{jk}, \qquad j, k = 1, ..., n,$$

(although he did not use the notation $\partial/\partial\overline{z}_k$) with polynomials P_{jk}, which is equivalent to (2.3) with $f = 2i \sum P_{jk} dz_j \wedge d\overline{z}_k$, a differential form with polynomial coefficients. He proved that $n^2(n-1)$ obviously necessary conditions on the P_{jk}, corresponding to the condition that f be closed, are also sufficient for solvability.

The third equation is the *complex Monge–Ampère equation*

$$(dd^c u)^n = g. \tag{2.4}$$

This is a combination of (2.3) and the algebraic equation $f^n = g$, where the n^{th} power of f is calculated in the exterior algebra. If u is a function of class C^2, the equation has an elementary meaning: f^n is a form with a continuous coefficient, which is essentially the determinant of all derivatives $\partial^2 u/\partial z_j \partial \overline{z}_k$. It is therefore a polynomial in these derivatives. In a more general situation, the coefficients of f are measures, and measures cannot always be multiplied. But it is desirable to study equation (2.4) when the right-hand side is in a wider class than the (n,n) forms with continuous coefficients. During the last decades, significant efforts have been devoted to making more general definitions work.

A function f defined in some open subset Ω of the space $\mathbf{C}^n$ of n complex variables is said to be *plurisubharmonic,* $f \in PSH(\Omega)$, if its values are real or $-\infty$; if it is upper semicontinuous, i.e., such that the sublevel sets $\{z \in \Omega;\ f(z) < c\}$ are open for all real c; and, finally, if it satisfies the mean-value inequality

$$f(a) \leqslant \frac{1}{2\pi} \int_0^{2\pi} f(a + e^{i\theta} b) d\theta \tag{2.5}$$

for all $a, b \in \mathbf{C}^n$ such that the disk $\{a + tb;\ |t| \leqslant 1\}$ is contained in Ω. For an upper semicontinuous function the latter property is equivalent to requiring that the pull-back $\varphi^*(f) = f \circ \varphi$ be subharmonic wherever it is defined for all affine mappings $\varphi\colon \mathbf{C} \to \mathbf{C}^n$. It is a remarkable fact that this property implies that the pull-back

$\varphi^*(f)$ is subharmonic also for all holomorphic mappings $\varphi\colon \mathbf{C} \to \mathbf{C}^n$. Thus a definition which a priori depends on the vector-space structure of $\mathbf{C}^n$ in fact does not, and the plurisubharmonic functions form a class which is biholomorphically invariant and can be defined on any complex analytic manifold.

A basic example of a plurisubharmonic function is $c \log |h|$, where c is a positive constant and h is holomorphic. Thus the plurisubharmonic functions generalize the (absolute values of) holomorphic functions. But they are not as rigid as the latter: they are easier to manipulate and glue together—this is why Lelong includes them among "les objets souples de l'analyse complexe" [1985].

For twice continuously differentiable functions the mean-value inequality (2.5) can be replaced by a differential inequality. A function $f \in C^2(\Omega)$ is plurisubharmonic if and only if its *Levi form* $L_f(z; b)$ is positive semidefinite, i.e.,

$$L_f(z;b) = \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k}(z) b_j \overline{b}_k \geqslant 0, \qquad z \in \Omega,\ b \in \mathbf{C}^n. \tag{2.6}$$

This means that $d'd''f$ is a differential form of bidegree $(1,1)$ whose coefficients are continuous functions and which has a certain positivity property, expressed here as positive semidefiniteness of the Hermitian form $L_f(z; b)$.

But we can say more about the differential condition (2.6). A link to general plurisubharmonic functions is provided by convolutions with test functions: we form the convolution product $f * \psi_\varepsilon$,

$$(f * \psi_\varepsilon)(z) = \int_{\mathbf{C}^n} f(w)\psi_\varepsilon(z-w)d\lambda(w) = \int_{\mathbf{C}^n} f(z - \varepsilon w)\psi(w)d\lambda(w), \tag{2.7}$$

where $d\lambda$ denotes Lebesgue measure and $\psi \geqslant 0$ is a test function of integral one and which is invariant under multiplication by scalars of modulus one: $\psi(e^{i\theta}z) = \psi(z)$ for real θ, and where $\psi_\varepsilon(z) = \varepsilon^{-2n}\psi(z/\varepsilon)$. The functions $f * \psi_\varepsilon$ are defined in any given relatively compact subdomain of Ω when ε is small, they are plurisubharmonic, of class C^∞, and they tend to f as ε tends to zero.

An upper semicontinuous function is plurisubharmonic if and only if it satisfies (2.6) in the sense of distributions. Moreover any distribution which satisfies (2.6) is the distribution defined by some plurisubharmonic function. Also any such distribution $u \in \mathcal{D}'(\Omega)$ can be locally approximated by smooth plurisubharmonic functions by forming convolutions

$$(u * \psi_\varepsilon)(z) = u(w \mapsto \psi_\varepsilon(z - w)),$$

where ψ is as in (2.7).

Yet another point of view is of interest. All plurisubharmonic functions are subharmonic as functions of the $2n$ real variables $\operatorname{Re} z_1, \operatorname{Im} z_1, ..., \operatorname{Re} z_n, \operatorname{Im} z_n$. Now consider the class of subharmonic functions that remain subharmonic under all real linear changes of the coordinates. This class consists of the convex functions.

Similarly, the class of functions that are subharmonic under all complex linear changes of the coordinates consists precisely of the plurisubharmonic functions, see Klimek [1991: Theorem 2.9.12]. This point of view has been elaborated by Hörmander [1994: Ch. V] to fit various situations where other subgroups of the group of all linear mappings are relevant.

We shall say that a plurisubharmonic function $f \in PSH(\Omega)$ is *maximal* if for any relatively compact subset ω of Ω and any upper semicontinuous function g defined on $\overline{\omega}$, plurisubharmonic in ω, and such that $g \leqslant f$ on the boundary of ω, it is true that $g \leqslant f$ in all of ω; Sadullaev [1981]. In one variable, the maximal plurisubharmonic functions are precisely the harmonic functions and thus characterized as solutions to the Laplace equation $\Delta f = 0$; in more than one variable, the class is much richer, and contains for instance (for degree reasons) all plurisubharmonic functions which are functions of $n-1$ variables. It is known that these functions, if they are locally bounded, are solutions to the homogeneous complex Monge–Ampère equation; see section 12.

3. The emergence of plurisubharmonic functions

To discuss plurisubharmonic functions we must obviously start with subharmonic functions. These functions, or rather the superharmonic[1] functions, were introduced by Friedrich Hartogs in a remarkable paper published in 1906, although he did not give a name to them. He studied holomorphic functions of two complex variables $(x, y) \in \mathbf{C}^2$ and expanded them in a series containing powers of one of the variables:

$$S(x, y) = \sum_{\nu=0}^{\infty} f_\nu(x) y^\nu.$$

He denoted by R'_x the radius of convergence of this power series in y, and he proved that minus the logarithm of R'_x is subharmonic as a function of x in today's terminology; more precisely, he proved [1906:50] that if R'_x is of class C^2, then it satisfies the differential inequality

$$\Delta(\log R'_x) = \frac{\partial^2 \log R'_x}{\partial u^2} + \frac{\partial^2 \log R'_x}{\partial v^2} \leqslant 0,$$

i.e., the now well-known differential inequality for superharmonic functions. (Here $u = \operatorname{Re} x$, $v = \operatorname{Im} x$.) In general he proved that the function $\log R'_x$ majorizes any harmonic function which it majorizes on the boundary of a domain, i.e., that it is a supersolution to the Dirichlet problem. And he went on to prove a converse: given any function U_x of class C^2 which satisfies the differential inequality, then there exists a holomorphic function such that its radius of convergence R'_x is equal to $\exp U_x$ [1906:63]. This amounts to a solution of the Levi problem (to be mentioned later in this section) for domains of the kind which are now known as Hartogs domains and whose boundary is of class C^2.

[1] A function is superharmonic if and only if its negative is subharmonic.

It was F. Riesz who gave a name to this class of functions in a talk in Stockholm on September 15, 1924; see Riesz [1925, 1926:329]. There he discussed subharmonic functions and their relation to potential theory. He called a continuous function of two real variables *subharmonique*[2] if it satisfies an inequality

$$u(x_0, y_0) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\varphi, y_0 + r\sin\varphi) d\varphi$$

for all sufficiently small values of the radius r. This is the mean value inequality (2.5) but he required it only for small r. He showed that this property is equivalent to the fact that the function is a subsolution to the Dirichlet problem, and then, a few pages later, extended his definition to the case of an upper semicontinuous function, using now the property of being a subsolution [1926:333]; in other words, he followed Hartogs [1906].

A little more than a decade after Riesz, a rather complete, now classical, monograph by Radó appeared—published in English in Berlin in the year 1937.

It is somewhat amusing now to note how subharmonic functions were approximated by smooth functions in those days. One first formed the areal mean

$$A_r(x, y; u) = \frac{1}{r^2\pi} \iint\limits_{\xi^2+\eta^2<r^2} u(x+\xi, y+\eta) d\xi d\eta,$$

which is a continuous function. To get a function of class C^2 one had to repeat this procedure; the third areal mean $A_r^{(3)}(x, y; u)$ is of class C^2, and the sequence $u_k^{(3)}(x, y) = A_{1/k}^{(3)}(x, y; u)$ tends to u in any given relatively compact subdomain; see Riesz [1930:342ff] and Radó [1937:11]. Expressed in terms of convolutions, one thus formed $u * \chi_\varepsilon * \chi_\varepsilon * \chi_\varepsilon$, where χ denotes the characteristic function of the unit ball. Nowadays one forms a convolution $u * \psi_\varepsilon$ with one conveniently chosen smooth function ψ; cf. (2.7).

The plurisubharmonic functions were introduced by Oka [1942] and Lelong [1942a], working independently in Japan and France, respectively. In fact Oka did his research work already in 1935[3] at Hiroshima University, where he was assistant professor during the years 1932–1938, while Lelong worked in Paris.[4] The two mathematicians never met.[4] Oka's paper was received by the *Tôhoku Mathematical Journal* on October 25, 1941, and published in May, 1942. Lelong's definition appeared in a note in the *Comptes Rendus* presented on November 3, 1942.

[2]This French term was not well chosen and soon afterwards the functions were called *sousharmoniques* (Szpilrajn [1933] and Frostman [1935:10]). Oka [1942:40] kept Riesz's term, however.

[3]Toshio Nishino, personal communication, October 3, 1997.

[4]Pierre Lelong, personal communication, September 24, 1997.

Oka called his new functions *pseudoconvex,* just like the domains in which they live most naturally.[5] Lelong, on the other hand, used the subharmonic functions as his point of departure; the functions are subharmonic on any complex line, and therefore subharmonic "in many ways." In his note [1942a], the term *plurisousharmonique* appeared for the first time. Earlier Poincaré [1899:112] had used the term *biharmonique* for functions of n variables that we now call pluriharmonic, i.e., functions f such that both f and $-f$ are plurisubharmonic. Thorin [1948:18] followed Poincaré concerning this usage and extended it in a natural way: he called the plurisubharmonic functions *bisubharmonic.* It is Lelong's choice that has survived.

Oka writes [1942:40]:

> "Nous appellerons fonction pseudoconvexe par rapport à x, y dans $\mathfrak{D}$, toute fonction réelle $\varphi(x, y)$ bien définie et satisfaisant aux conditions suivantes: 1°. La fonction $e^{\varphi(x,y)}$ est finie et semi-continue supérieurement par rapport à x, y dans $\mathfrak{D}$. 2°. Sur tout plan caractéristique[6] L passant par un point de $\mathfrak{D}$, $\varphi(x, y)$ est une fonction subharmonique de x ou de y sur la portion de L dans $\mathfrak{D}$.

We note that his definition, albeit in two variables, agrees with the one in use ever since. The easy generalization to n variables appeared in [1953]. He called the function real, but it is clear from his wording that the function may assume the value $-\infty$; this is why he imposes the condition of semicontinuity not on φ but on e^{φ}, which is zero at the points where φ is minus infinity. The constant $-\infty$ is admitted, as is apparent from this definition and also explicitly stated [1942:39]. He noted that his class enjoys the same properties as the class of subharmonic functions with respect to sums, maxima, and passage to the limit.

The reason for introducing this "nouvelle classe de fonctions réelles" is quite clear in Oka's case: the author proved in the quoted article that a pseudoconvex open set in the space of two complex variables is a domain of holomorphy.[7] This had been an open problem since the beginning of the century, called the *inverse Hartogs problem* or the *Levi problem.*[8] As a preliminary to that great theorem, almost *en passant,* he proved that in a domain which is pseudoconvex in the sense of Hartogs, minus the logarithm of the distance from a point to the boundary is pseudoconvex, i.e., plurisubharmonic in today's terminology.

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[5]Oka called a domain *pseudoconvex* if it satisfies the continuity theorem of Hartogs. Nowadays one often uses plurisubharmonicity to define pseudoconvexity: a domain is called pseudoconvex if there exists a plurisubharmonic function which tends to $+\infty$ at the boundary.

[6]In Oka's terminology, *plan caractéristique* means a complex line.

[7]A domain of holomorphy can be defined as a domain of existence of a holomorphic function, i.e., a domain such that there exists a holomorphic function which cannot be extended to any larger domain over $\mathbf{C}^n$.

[8]Oka [1953], Bremermann [1954], and Norguet [1954] later solved the Levi problem in any finite number of variables.

Toshio Nishino[9] has explained Oka's line of reasoning when he defined the new class as follows. Oka was familiar with Levi's condition from 1910; see (4.1) and (4.2) below. It is nonlinear in ρ. Oka wanted to find a linear condition on ρ which implies Levi's condition. Now (2.6) with f replaced by ρ, i.e., plurisubharmonicity of ρ, is such a condition: we get (4.2) from (2.6) making the special choice $b = (-\partial\rho/\partial z_2, \partial\rho/\partial z_1)$. And Oka wanted to impose linearity for a very precise reason. A pseudoconvex domain can have a very complicated boundary; it can be fractal in nature and boundary points need not be accessible along paths. Therefore it seems impossible to construct directly a holomorphic function whose domain of existence is the given domain. It is a natural idea to approximate the domain by pseudoconvex domains having a smooth boundary. To construct such domains, it seems reasonable to consider the distance function d_Ω, $d_\Omega(z)$ denoting the distance from z to $\partial\Omega$. Since d_Ω is not smooth in general, it is natural to form mean values of it over neighborhoods of a given point; cf. (2.7). But such mean values remain in a certain class only if the class is a convex cone; hence the importance of finding a linear condition instead of the nonlinear Levi condition (4.2).

Lelong defined a plurisubharmonic function as one that takes finite values or minus infinity and is bounded from above in any relatively compact subdomain. The function is not allowed to be minus infinity identically. Moreover it shall be subharmonic or $-\infty$ on every complex plane of dimension one. Lelong did not impose upper semicontinuity but deduced it as a consequence from his definition; the proof appeared in [1945]. In the most common definition today upper semicontinuity is imposed, just as Oka did. However, interestingly enough, in the recent study of Poletsky's holomorphic currents (see section 16), it appears that Lelong's original definition will be useful. As a motivation for his introduction of this "classe remarquable" Lelong mentioned only "l'extension de la méthode sousharmonique" [1942a:398]. He developed in his notes [1942a, b] mean-value properties and properties with respect to passage to the limit. In a "mémoire" [1945], he presented more fully the properties of this class of functions. And fifty years later [1995] he described from his point of view the development of the theory of plurisubharmonic functions and closed positive currents during the first twenty years, from 1942 to 1962.

The intuition behind the new class of functions proved to be fruitful. Oka and Lelong had defined a class of great importance and the development during the following fifty years was to be spectacular.

4. Domains of holomorphy and pseudoconvex domains

A holomorphic function defined in a domain in $\mathbf{C}^n$ can be holomorphically extended to a unique maximal domain. This is a result which goes back to Weierstrass in one complex variable and it is equally valid, with a similar proof, in several

[9] Personal communication, October 9, 1997.

variables. However, we cannot let the term *domain* here denote a connected open subset of $\mathbf{C}^n$: we must understand the word as meaning domain *over* $\mathbf{C}^n$. For $n = 1$ this is the classical notion of a Riemann surface, and for several variables it is not that different. Any convergent power series defines a holomorphic function in a neighborhood of a point, in other words a germ of a holomorphic function, and the maximal domain of existence is simply the connectivity component containing that germ in the space of all germs. This means that the material needed to build up the maximal domain is provided by the function itself, i.e., by its Taylor expansions and all possible continuations of these.

It was noted in the beginning of the twentieth century that the maximal domain of a holomorphic, even meromorphic, function must possess a certain convexity property. Levi [1910:80] proved that an open set in $\mathbf{C}^2$ which is the domain of existence of a meromorphic function must satisfy a differential condition at every boundary point. If the boundary $\partial\Omega$ has the equation $\rho(x_1, x_2, y_1, y_2) = 0$, where $x = x_1 + ix_2$, $y = y_1 + iy_2$ are two complex variables, and Ω is on the side where ρ is negative, then it is necessary that the inequality

$$\begin{aligned}\left(\frac{\partial^2\rho}{\partial x_1^2}+\frac{\partial^2\rho}{\partial x_2^2}\right)\left(\left(\frac{\partial\rho}{\partial y_1}\right)^2+\left(\frac{\partial\rho}{\partial y_2}\right)^2\right)+\left(\frac{\partial^2\rho}{\partial y_1^2}+\frac{\partial^2\rho}{\partial y_2^2}\right)\left(\left(\frac{\partial\rho}{\partial x_1}\right)^2+\left(\frac{\partial\rho}{\partial x_2}\right)^2\right)&\\ -2\left(\frac{\partial^2\rho}{\partial x_1\partial y_1}+\frac{\partial^2\rho}{\partial x_2\partial y_2}\right)\left(\frac{\partial\rho}{\partial x_1}\frac{\partial\rho}{\partial y_1}+\frac{\partial\rho}{\partial x_2}\frac{\partial\rho}{\partial y_2}\right)&\\ -2\left(\frac{\partial^2\rho}{\partial x_1\partial y_2}-\frac{\partial^2\rho}{\partial x_2\partial y_1}\right)\left(\frac{\partial\rho}{\partial x_1}\frac{\partial\rho}{\partial y_2}-\frac{\partial\rho}{\partial x_2}\frac{\partial\rho}{\partial y_1}\right)&\geqslant 0\end{aligned}\tag{4.1}$$

hold at all points of the boundary of Ω. If we let z_1 and z_2 be the complex variables and use the notation (2.2) (which Levi did not do) we get a more compact formula:

$$\frac{\partial^2\rho}{\partial z_1\partial\overline{z}_1}\left|\frac{\partial\rho}{\partial z_2}\right|^2-2\,\mathrm{Re}\,\frac{\partial^2\rho}{\partial z_1\partial\overline{z}_2}\frac{\partial\rho}{\partial\overline{z}_1}\frac{\partial\rho}{\partial z_2}+\frac{\partial^2\rho}{\partial z_2\partial\overline{z}_2}\left|\frac{\partial\rho}{\partial z_1}\right|^2\geqslant 0\tag{4.2}$$

on $\partial\Omega$.

Nowadays it is customary to reformulate (4.1) and (4.2), called the *Levi condition,* as follows. If ρ is a defining function for a domain Ω, then (4.2) says that

$$\sum_{j,k=1}^{2}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}(a)b_j\overline{b}_k\geqslant 0 \text{ when } a\in\partial\Omega \text{ and } \sum_{j=1}^{2}\frac{\partial\rho}{\partial z_j}(a)b_j=0.\tag{4.3}$$

The last condition in (4.3) means that $b \in \mathbf{C}^2$ is tangent to the boundary of Ω. The geometrical significance of (4.3) is that the boundary bends away from the best fitting second-degree analytic curve in the good direction, i.e., inward. In contrast to (4.2), (4.3) is easy to generalize to several variables. As we remarked in the last section, the step from (4.2) to (4.3) was instrumental in Oka's work.

A year later [1911:70] Levi proved a kind of converse: if the corresponding strong condition is satisfied, i.e., with strict inequality in (4.3) for all $b \neq 0$, then there is a neighborhood ω of a such that $\omega \cap \Omega$ is the domain of existence of a holomorphic function. The full converse in $\mathbf{C}^2$ was proved by Oka [1942] as we have already mentioned in section 3.

The situation for plurisubharmonic functions is very different from that for holomorphic functions. For every domain ω in $\mathbf{C}^n$ there exists a domain Ω containing ω such that all plurisubharmonic functions in ω can be extended to Ω and such that Ω is maximal with this property; Cegrell [1983b]. However, this maximal domain is not unique. Several sufficient conditions have been established for a domain to be the domain of existence of some plurisubharmonic function, for example by Bedford and Burns [1978] and Cegrell [1983b], but there seems to be no known geometric characterization of such domains.

As mentioned in section 2, all functions of the type $c \log |h|$ with c a positive constant and h a holomorphic function are plurisubharmonic. Bremermann [1956a:82] proved that every plurisubharmonic function in a pseudoconvex domain is the upper semicontinuous envelope of the upper limit of a sequence of such special functions.[10] Earlier, Lelong had noted already in [1941:116] that the result is true for one complex variable, and he had also proved [1952:198] that it is equivalent, in any number of variables, to approximation in the L^1_{loc} topology by maxima of finitely many functions of the form $c \log |h|$.

Bremermann used in an essential way the solution of the Levi problem. I cannot resist the temptation to sketch his elegant proof: it combines the notions of plurisubharmonic function, pseudoconvex domain, and domain of holomorphy with the classical formula for the radius of convergence of a power series, and goes back to the pioneering study of Hartogs [1906] touched upon in section 3. If a plurisubharmonic function f is given in a domain Ω in $\mathbf{C}^n$, then we can construct a domain over Ω (now called a *Hartogs domain*) in the space of $n + 1$ complex variables:

$$\Omega(f) = \{(z,t) \in \Omega \times \mathbf{C};\ |t| < e^{-f(z)}\}, \tag{4.4}$$

and any holomorphic function h in $\Omega(f)$ admits a partial Taylor expansion

$$h(z,t) = \sum_{j=0}^{\infty} h_j(z) t^j,$$

which converges for $|t| < R(z)$, z being fixed, where $R(z)$ is given by the formula for the radius of convergence of a power series in one variable:

$$-\log R(z) = \limsup_{j\to\infty} \frac{1}{j} \log |h_j(z)|, \qquad z \in \Omega.$$

[10]Bochner and Martin [1948:145] had in fact stated that a plurisubharmonic function of class C^2 is of this type in any domain, but Bremermann showed by an example that this is not so.

Since h is assumed holomorphic for $|t| < e^{-f(z)}$, the series must converge for at least these values of t, which means that $-\log R(z) \leqslant f(z)$. On the other hand, the series defines a holomorphic extension of h to the open set $\Omega((-\log R)^*)$, where the star denotes the upper semicontinuous envelope, cf. (7.2). So if h can be chosen such that it does not admit a holomorphic extension beyond the boundary of Ω, then $(-\log R)^* = f$. Thus the given function f has a representation of the kind we wanted. We can choose h such that it does not admit a holomorphic extension precisely when $\Omega(f)$ is a domain of holomorphy, and, in view of the solution of the Levi problem, this is the case when $\Omega(f)$ is pseudoconvex. This, in turn, is true if Ω is pseudoconvex and f plurisubharmonic. In fact, given a pseudoconvex domain Ω, f is plurisubharmonic there if and only if $\Omega(f)$ is pseudoconvex: the theory of plurisubharmonic functions is therefore equivalent to the theory of pseudoconvex Hartogs domains $\Omega(f)$.

5. Integration on analytic varieties

In 1957, Lelong published a paper [1957a] on integration on an analytic subvariety of $\mathbf{C}^n$. This was the starting point of an important development in complex geometry and it led up to the theory of Lelong numbers of closed positive currents.

The idea is in fact simple to express although it relies on the deep theory of analytic varieties. An analytic variety (also called an analytic set) is defined locally as the common zero set of a family of holomorphic functions defined near a given point. However, at most points the variety is in fact a manifold, and on a manifold it is possible to integrate using local coordinates to move the problem to a space $\mathbf{C}^p$ of appropriate dimension, where Lebesgue measure is defined. (Alternatively, one could use $2p$-dimensional Hausdorff measure in $\mathbf{C}^n$.) Lelong's idea was to extend integration from the regular points to all points of the variety.

Let A be an analytic subvariety of an open set Ω in $\mathbf{C}^n$, and let A_{reg} denote the set of all its regular points, i.e., the set of all points in a neighborhood of which A is a manifold of the top dimension p. The complement with respect to A is called the singular set and we shall denote it by A_{sing}. So far we can only say that A_{reg} is open in A, possibly empty, and that A_{sing} is a closed subset of A. But the theory of analytic varieties tells us that A_{reg} is in fact a complex manifold of pure dimension p (i.e., all its component are of the same dimension p) and that A_{sing} is an analytic variety of top dimension less than p. This enables us to use induction, and it is an important fact in the construction of the extension of integration over A_{reg}.

Lelong started from integration over A_{reg}: if A_{reg} is of dimension p, we can define a current t by

$$t(\varphi) = \int_{A_{\mathrm{reg}}} \varphi, \qquad \varphi \in \mathcal{D}(\Omega \smallsetminus A_{\mathrm{sing}}).$$

Here $\mathcal{D}(\omega)$ denotes the space of differential forms whose coefficients are test functions in ω. As already mentioned, t is defined by Lebesgue measure of dimension

$2p$ in $\mathbf{C}^p$, to which we can transport the form φ. Only those forms which are of bidegree (p,p) can give a nonzero integral. Now it is sometimes possible to extend a current t defined in an open set ω to a current T in a larger set Ω by defining the extension as

$$T(\varphi) = \lim_{\alpha} t(\alpha\varphi), \qquad \varphi \in \mathcal{D}(\Omega),$$

where α is a smooth function which is zero in a neighborhood of $\Omega \smallsetminus \omega$ so that t can be applied to $\alpha\varphi \in \mathcal{D}(\omega)$, and where the limit is taken over functions α which satisfy $0 \leqslant \alpha \leqslant 1$ and tend to one on all compact sets in ω. This is in fact the easiest way to extend currents. If this is possible we call T the *simple extension* of t. Lelong proved that the current of integration t over A_{reg} admits indeed a simple extension from $\omega = \Omega \smallsetminus A_{\mathrm{sing}}$ to all of Ω; the extension T is a positive closed current and it has measure coefficients. A simple extension of a closed current is not always closed, but in the present case this is true. If the function α is equal to one except in an ε-neighborhood of A_{sing}, the mass of the boundary of the current αt must lie in that neighborhood. Now the p-dimensional volume of an ε-neighborhood of A_{sing} can be controlled. It follows that the mass of $d(\alpha t)$ is locally not worse that a constant times ε.

To conclude, Lelong's investigation showed that we can write simply

$$T(\varphi) = \int_{A_{\mathrm{reg}}} \varphi, \qquad \varphi \in \mathcal{D}(\Omega),$$

with the important additional information that T, the extension of t, is closed in Ω.[11]

An elegant proof which "differs slightly from that of Lelong" was published by de Rham [1969], who had sketched his proof at a conference already in 1957. Federer [1965] gave another proof in the algebraic case.

This result has been generalized in several directions, for instance by El Mir [1984], who proved that the extension T to Ω of a closed positive current t in $\omega \subset \Omega$ is closed if $\Omega \smallsetminus \omega = f^{-1}(-\infty)$ is the polar set of some function $f \in PSH(\Omega)$ (see section 7) and t has finite mass near every point in $\Omega \smallsetminus \omega$.

6. Weighted estimates for the Cauchy–Riemann operator

Since the logarithm of the modulus of a holomorphic function is plurisubharmonic, it is natural to estimate the growth of holomorphic functions by inequalities $|h(z)| \leqslant e^{\varphi}$ with a plurisubharmonic φ, thus with an inequality in L^∞ norm $\|he^{-\varphi}\|_\infty \leqslant 1$. However, duality behaves much better in L^2 than in L^∞, so in

[11] Earlier Stoll [1952:145] had defined integration over the zero set A (of real dimension $2n-2$) of an analytic function by this formula and actually shown [1952:153] that a differential form defined in $\mathbf{C}^n$ can be integrated over the regular points. But it seems that Stokes' theorem, corresponding to the property of the current of integration being closed, was not available in his calculus.

order to work with adjoint mappings it is convenient to use instead L^2 norms: $\|he^{-\varphi/2}\|_2 \leqslant 1$. The constructed objects are often holomorphic functions, and for them an estimate of the mean value yields pointwise estimates. Therefore L^2 methods are often useful even though we want pointwise estimates in the end. In 1965, Hörmander published a paper where he solved the Cauchy–Riemann equation using Hilbert-space methods, i.e., the theory of closed linear operators in Hilbert spaces. The Hilbert spaces are L^2 spaces with plurisubharmonic weight functions. That such weight functions came to be used was natural in view of what we just remarked. Andreotti and Vesentini [1965] had a similar approach, also using L^2 methods, to prove vanishing theorems for compactly supported cohomology on manifolds. The basic a priori estimates had been proved earlier by Morrey [1958] and Kohn [1963].

The use of L^2 methods, especially those in Hörmander's paper [1965] and his monograph [1966], revolutionized the methods of constructing holomorphic functions.

The results have the following form. Let f be a differential form of bidegree (p,q), $q \geqslant 1$, in a pseudoconvex domain Ω in $\mathbf{C}^n$ and satisfying an L^2 estimate $\|fe^{-\varphi/2}\|_2 \leqslant 1$, where $\varphi \in PSH(\Omega)$. If we impose the necessary condition $d''f = 0$, then there exists a solution u of bidegree $(p, q-1)$ to $d''u = f$ satisfying an estimate $\|ue^{-\psi/2}\|_2 \leqslant 1$ for a certain ψ. Ideally one would like to have $\psi = \varphi$, and Hörmander's results came very close to this. In [1965:105] he had $\varphi = \psi + \varkappa + \log q$, where ψ is an arbitrary strongly plurisubharmonic function and where $\varkappa$ is a continuous function such that $e^{\varkappa}$ is a lower bound for the plurisubharmonicity of ψ, i.e., such that the Levi form $L_\psi(z;b)$ of ψ (cf. (2.6)) has $e^{\varkappa(z)}|b|^2$ as a minorant. A later variant in the case $(p,q) = (0,1)$ is this: in his monograph *Notions of Convexity,* [1994:258] Hörmander chose $\varphi = g + (a-2)\log(1+|z|^2) + \log a$ and $\psi = g + a\log(1+|z|^2)$, where g is an arbitrary plurisubharmonic function and a a positive number. This choice of φ and ψ goes back to an improvement due to Bombieri [1970] of a result of Hörmander [1966]. It was further sharpened by Skoda [1977:318] in the case $\Omega = \mathbf{C}^n$.

The last-mentioned choice of weight-functions φ and ψ gives rise to the following result, now known as the Hörmander–Bombieri theorem: If Ω is pseudoconvex and $f \in PSH(\Omega)$ is such that e^{-f} is integrable in some neighborhood of a point $a \in \Omega$, then to every positive ε there exists a holomorphic function h in Ω such that $h(a) = 1$ and $\|he^{-f/2}(1+|z|)^{-n-\varepsilon}\|_2$ is finite; Bombieri [1970:275], Hörmander [1994:258]. The interest here is that h must be zero at every point such that e^{-f} is not integrable near that point. Letting now a vary, we see that the set of points of nonintegrability of e^{-f} is an analytic variety. The Hörmander–Bombieri theorem thus provides a strong link between the local behavior of plurisubharmonic functions and globally defined analytic varieties. This link was further investigated by Kiselman [1992, 1994b].

Another important development was that Ohsawa and Takegoshi [1987] could prove sharp extension theorems in L^2 norms for holomorphic functions using the L^2-methods of Hörmander, Andreotti and Vesentini. This remarkable result says that if h is a holomorphic function defined in $\Omega \cap \mathbf{C}^{n-1}$ and $\psi \in PSH(\Omega)$, then every holomorphic function $h \in \mathcal{O}(\Omega \cap \mathbf{C}^{n-1})$ such that

$$\|he^{-\psi}\|_{L^2(\Omega \cap \mathbf{C}^{n-1})} < +\infty$$

can be extended to a holomorphic function $H \in \mathcal{O}(\Omega)$ satisfying an estimate

$$\|He^{-\psi}\|_{L^2(\Omega)} \leqslant C\|he^{-\psi}\|_{L^2(\Omega \cap \mathbf{C}^{n-1})},$$

where C is a constant depending only on the diameter of Ω. Here the same weight function ψ is used in both sides and no regularity is imposed on it. A simplified proof was given by Berndtsson [1996].

7. Small sets: pluripolar sets and negligible sets

Let us denote by $P(f) = f^{-1}(-\infty)$ the set of points where a function takes the value minus infinity. If E is a subset of a domain Ω in $\mathbf{R}^n$, we shall say that it is *polar* if there is a subharmonic function in Ω which is not identically minus infinity and such that $E \subset P(f)$. According to Lelong [1945:307] this usage goes back to Brelot. The polar sets are of Lebesgue measure zero, but they are even smaller, and there is a well-developed theory of Newtonian capacity; in this theory the polar sets are exactly the zero sets for the capacity as was proved by Cartan [1942, 1945].

Lelong [1945:307] called a set *polaire* if it is contained in $P(f)$ for some plurisubharmonic function f (a global definition), but later [1957b:264] he changed the definition to a local one: he called a set $\mathbf{C}^n$*-polaire* in a domain Ω in $\mathbf{C}^n$ if to every point $a \in \Omega$ there is a connected open neighborhood V and a function $f \in PSH(V)$ such that $X \cap V \subset P(f)$, and, of course, f is not identically minus infinity. Nowadays the term *pluripolar* is the most common. If the function can be chosen to be plurisubharmonic in all of Ω, we shall call the set *globally pluripolar.*

The pluripolar sets form a family of small sets; they are of Lebesgue measure zero and also of Newtonian capacity zero, but their exact character remained mysterious. It was for instance not known whether a set which is pluripolar is also globally pluripolar. This was called the "first problem/question of Lelong" by Sadullaev [1981: §9] and Bedford [1993:58]. The corresponding problem for subharmonic functions got its solution because of Newtonian capacity: a small piece of a polar set can be defined by a subharmonic function defined not only in a neighborhood of a point but in the whole space; the whole set is therefore a countable union of globally polar sets. Since the capacity is countably subadditive we are done. But it was not known whether there exists a corresponding capacity for pluripolar sets.

Another family of small sets appears in the passage to the limit. If we have a sequence (f_j) of plurisubharmonic functions in a domain Ω and the sequence is locally bounded from above, then

$$f(z) = \limsup_{j\to\infty} f_j(z), \qquad z \in \Omega, \tag{7.1}$$

is not necessarily plurisubharmonic—it is not necessarily upper semicontinuous. We form the upper semicontinuous envelope of f:

$$f^*(z) = \limsup_{w \to z} f(w), \qquad z \in \Omega. \tag{7.2}$$

One can prove without difficulty that f^* is plurisubharmonic. We have corrected f to get f^* and the question arises whether we can characterize the sets $\{z \in \Omega;\ f(z) < f^*(z)\}$. These sets, which are of Lebesgue measure zero and of Newtonian capacity zero, were studied by Lelong [1961]. He later called them *négligeables* [1966:276]. It is easy to see that every globally pluripolar set is negligible: just take $f_j(z) = f(z)/j$; Lelong [1966:280]. On the other hand it is easy to see that a negligible set which appears in a situation where f^* is pluriharmonic is pluripolar; Lelong [1966:281]. Are all negligible sets pluripolar? This was stated as an open problem by Lelong [1966:276], and was called the "second problem/question of Lelong" by Sadullaev [1981: §12] and Bedford [1993:59]. Cartan [1942, 1945:102, 105] had solved the corresponding problem for subharmonic functions.

These problems were around for a long time, and we shall now sketch how they were solved. The first problem, whether (locally) pluripolar implies globally pluripolar, was solved in 1976 by Josefson [1978]. His approach was completely elementary (and very brave). The argument goes as follows. A piece of the given pluripolar set is contained in the polar set of a plurisubharmonic function defined in some small ball. Now this function is the upper semicontinuous envelope of an upper limit of a sequence of logarithms of holomorphic functions in the ball according to a famous result of Bremermann mentioned in section 4. These holomorphic functions, in turn, can be approximated by polynomials, which of course are globally defined—but the partial sums of their Taylor expansions cannot serve as approximants. By tinkering with these polynomials, Josefson obtained a globally defined plurisubharmonic function whose polar set contains the given set, or at least a certain fraction of it. By "tinkering" I mean that the coefficients of the polynomials are changed in a sophisticated way, which involves solving very large systems of equations, the equations for the coefficients of the polynomials. Josefson's paper contains a complicated result on large systems of linear equations, intertwined with deliberations on small sets. Siciak [1983:301] and Hörmander [1994:287] isolated the result on systems of equations and made Josefson's proof more transparent. However, the most satisfying explanation for the choice of polynomials was provided by Alexander and Taylor [1984] who showed that "the Tchebycheff polynomials themselves already do the job": if E is a relatively compact subset of the unit ball B in $\mathbf{C}^n$, we look for polynomials P of degree at most j that minimize $\sup_E |P|$ while $\sup_B |P| \geqslant 1$. Taking E as the set in $\frac{1}{2}B$ where a given plurisubharmonic function takes large negative values, we get polynomials which are small there, and they become the building blocks for a plurisubharmonic function F in $\mathbf{C}^n$ which has large negative values in E.

The second problem, whether negligible sets are pluripolar, was solved by Bedford and Taylor [1982]. They constructed a capacity which is countably subadditive. The sets of zero capacity are precisely the pluripolar sets, which implies that their construction also gives a new proof of Josefson's theorem. Their capacity is defined as follows:

$$C_n(K,\Omega) = \sup_u \left[\int_K (dd^c u)^n ; u \in PSH(\Omega),\ 0 < u < 1 \right]. \tag{7.3}$$

Here $(dd^c)^n$ is the complex Monge–Ampère operator, which the authors defined for all locally bounded plurisubharmonic functions. We shall describe that important development in section 12. They proved that all negligible sets have capacity zero and therefore are pluripolar. So finally the identity of several important classes of sets associated already from the beginning with the family of plurisubharmonic functions was established.

8. The analogy with convexity

All convex functions are plurisubharmonic, so the latter class is a generalization of the former, just like pseudoconvex sets generalize convex open sets. But there are also strong analogies between the two classes.[12] Bremermann [1956b] was the first to publish a systematic study of the analogy between the two classes of functions, as well as between convex domains and pseudoconvex domains. He established many results for the latter classes in analogy with already known results for convex functions or domains.

Lelong [1952:205] proved that a plurisubharmonic function which is independent of the imaginary parts of the variables is necessarily convex. Thus if f is plurisubharmonic and satisfies $f(z) = f(x+iy) = g(x)$ for x in a convex domain in $\mathbf{R}^n$ and for all $y \in \mathbf{R}^n$, then f is convex.[13] In the same paper he showed that this implies the closely related result that a pseudoconvex domain which is a tube, i.e., such that $z' \in \Omega$ if $z \in \Omega$ and $\operatorname{Re} z = \operatorname{Re} z'$, is convex. Stein [1937] had proved this for domains of holomorphy. Lelong [1952:211] obtained results also in the more difficult case of tubes of finite length, i.e., when Ω is the set of all $z = x + iy$ with $x \in \omega \subset \mathbf{R}^n$ and $|y| < R$.

There is, however, a simple classical result for convex sets and functions which has no direct analogue for plurisubharmonic functions. This is that the image under any linear mapping of a convex set is a convex set. To express the

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[12]If we were to follow the idea that has led to the term *plurisubharmonic,* convex functions of several variables would be called *pluriconvex.* Radó [1937:III] pointed out that convex functions of one variable may be called *sublinear,* so we might push the analogy one step further and call convex functions of several variables *plurisublinear* or *plurisubaffine.*

[13]For the earlier history of this topic, including Riesz's convexity theorem and Thorin's extension of it, we refer to Gårding [1997:32].

corresponding result for functions we need the notion of marginal function of a convex function f of two groups of variables $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$. The *marginal function* g of f is by definition

$$g(x) = \inf_{y \in \mathbf{R}^m} f(x, y), \qquad x \in \mathbf{R}^n. \tag{8.1}$$

It is easy to prove that g is convex in $\mathbf{R}^n$ if f is convex in $\mathbf{R}^n \times \mathbf{R}^m$; indeed this follows easily from the fact that a projection of a convex set is convex. Now simple examples show that a projection of a pseudoconvex set is not necessarily pseudoconvex. Thus if $\Omega \subset \mathbf{C}^n \times \mathbf{C}^m$ is a given and $\omega \subset \mathbf{C}^n$ is defined as the set of all points x such that $(x, y) \in \Omega$ for some $y \in \mathbf{C}^m$, we cannot say that ω is pseudoconvex if Ω is. Equivalently, if we define in analogy with (8.1),

$$g(x) = \inf_{y \in \mathbf{C}^m} f(x, y), \qquad x \in \mathbf{C}^n, \tag{8.2}$$

it does not follow that g is plurisubharmonic if f is. Kiselman [1978] found a simple condition under which we can conclude that g defined as in (8.2) is plurisubharmonic. The condition is that f shall be independent of the imaginary part $\operatorname{Im} y$ of $y \in \mathbf{C}^m$, thus $f(x, y) = f(x, y')$ if $\operatorname{Re} y = \operatorname{Re} y'$. More generally we define

$$g(x) = \inf_{y \in \pi^{-1}(x)} f(x, y), \qquad x \in \pi(\Omega), \tag{8.3}$$

where π is the projection defined by $\pi(x, y) = x$, where Ω is a pseudoconvex open subset of $\mathbf{C}^{m+n}$, which, like f, is independent of $\operatorname{Im} y$, and where, this time for simplicity only, each fiber $\pi^{-1}(x)$ is supposed to be connected.[14] The result was called the *minimum principle*, and applications of it were published for instance by Lelong [1983a:484] and Kiselman [1979, 1992, and 1994a]. A special case appeared in Kiselman [1967:14]. A more general but rather complicated minimum principle, although known in 1978, was published in 1994; Kiselman [1994b].

In complex analysis, radial functions, i.e., functions independent of the argument of a complex variable, appear quite often in the study of growth problems, like

$$F(x, z) = \sup_{t \in \mathbf{R}} u\big(x, ze^{it}\big), \qquad (x, z) \in \mathbf{C}^n \times \mathbf{C}.$$

They are covered by the minimum principle by a simple change of variable $z = e^y$:

$$f(x, y) = \sup_{t \in \mathbf{R}} u\big(x, e^{y+it}\big), \qquad (x, y) \in \mathbf{C}^n \times \mathbf{C}.$$

[14] More care is needed if the fibers $\pi^{-1}(x)$ are not connected, for then g cannot in general be defined in a subset of $\mathbf{C}^n$ but only on a non-schlicht domain over $\mathbf{C}^n$. However, with a suitable modification of (8.3), the conclusion is still valid.

Chafi and Loeb generalized the minimum principle to a Lie-group setting. Let G be a connected complex Lie group and $G_{\mathbf{R}}$ a connected closed real form of G, i.e., a closed and connected Lie subgroup of G such that $\mathrm{Lie}(G) = \mathrm{Lie}(G_{\mathbf{R}}) + i\,\mathrm{Lie}(G_{\mathbf{R}})$, where Lie denotes the Lie algebra of a Lie group. (In the case discussed above, $G = \mathbf{C}^m$ and $G_{\mathbf{R}} = \mathbf{R}^m$.) Let Ω be an open subset of $\mathbf{C}^n \times G$ which is invariant under the action of $G_{\mathbf{R}}$ from the right, and let $\Phi \in PSH(\Omega)$ also be invariant under the action of $G_{\mathbf{R}}$ from the right, i.e., $\Phi(x, gh) = \Phi(x, g)$, $(x, g) \in \Omega$, $h \in G_{\mathbf{R}}$. Then, under certain hypotheses, $\varphi(x) = \inf_{g\in G} \Phi(x, g)$ is plurisubharmonic in the projection of Ω. Chafi [1983] showed that the hypotheses are satisfied if $G = \mathrm{GL}(p, \mathbf{C}) \times \mathbf{C}^q$ and $G_{\mathbf{R}}$ is the real form $G_{\mathbf{R}} = \mathrm{U}(p) \times \mathbf{R}^q$. Loeb [1985] extended the result to more general groups.

The minimum principle for Lie groups was used in the proof of a long-standing conjecture in quantum field theory, the so-called extended future tube conjecture. To state this result, let us first define the *future tube*

$$T = \{(z_0, z_1, z_2, z_3) \in \mathbf{C}^4;\ \mathrm{Im}\, z_0 > 0,\ (\mathrm{Im}\, z_0)^2 > (\mathrm{Im}\, z_1)^2 + (\mathrm{Im}\, z_2)^2 + (\mathrm{Im}\, z_3)^2\}.$$

It is invariant under the action of a group G, viz. the connectivity component containing the identity of the complex Lorentz group $L(\mathbf{C}) = \mathrm{O}(1, 3, \mathbf{C})$. The *extended future tube* is

$$G \cdot T^n = \{gz;\ z \in T^n,\ g \in G\},$$

where G acts diagonally on the elements of T^n. The *extended future tube conjecture,* which was open for thirty years, asserts that $G \cdot T^n$ is pseudoconvex for $n \geqslant 3$. Zhou presented a proof of it in January, 1997, revised in [1998]. Heinzner [1998] later in 1997 gave another proof, based on results of Zhou.

Berndtsson [forthc.] found a generalization of the minimum principle in a different direction. He replaced (8.2) by

$$e^{-g(x)} = \int_{\mathbf{R}^m} e^{-f(x,y)}\, d(\mathrm{Re}\, y), \qquad x \in \mathbf{C}^n, \tag{8.4}$$

and proved that if $f \in PSH(\mathbf{C}^n \times \mathbf{C}^m)$ is independent of $\mathrm{Im}\, y$, then g is plurisubharmonic. This generalizes Prekopa's theorem for convex functions as well as the minimum principle described above; one gets (8.2) as a limiting case from (8.4) by substituting pf and pg for f and g and letting the positive number p tend to infinity.

Hörmander published an announcement [1955] of certain theorems on the Laplace transformation in spaces of distributions, which has now, forty years later [forthc.], been expanded into a comprehensive theory for the Legendre transformation operating on partially plurisubharmonic functions and its relation to the distributional Laplace transformation. Entire functions, such as the Fourier–Laplace transforms of distributions, can naturally be estimated using functions which are concave in the real directions and partially plurisubharmonic. For such functions it is natural to take the maximum over $\mathbf{R}^n$ and then the minimum over $i\mathbf{R}^n$. This gives a kind of saddle-point transformation, a generalization of the classical Legendre transformation, and it is involutive under certain conditions. The Laplace transformation maps distributions in a class defined by one such function φ isomorphically onto the class of distributions defined by the Legendre transform of φ. This theory generalizes results by Gel′fand and Shilov [1953] and McKennon [1976].

9. Lelong numbers

To describe the behavior of a plurisubharmonic function near a given point, the most important parameter is a quantity which has become known as the Lelong number. In this section we shall sketch its appearance and later generalization. Then, in section 11, we shall touch upon a finer local object, the tangent cone to a function or current, which may or may not exist.

The result described in this section combine ideas from potential theory and the theory of holomorphic functions. They go back to Poincaré [1899]. He studied a holomorphic function F and the logarithm of its absolute value $\log|F|$. He concluded [1899:159] that $\log|F|$ is equal to a harmonic function plus a potential of a “variété attirante” C (the zero set of F) whose “matière attirante” has density equal to one. (This has a sense only for functions with simple zeros.) In present-day terms, we would express Poincaré's result by saying that the Lelong number of the variety C is one at all regular points.

Let a closed positive current t of bidimension (p,p) (bidegree $(n-p,n-p)$) be given, for instance the current of integration on an analytic variety of complex dimension p. Let μ be the measure $t\wedge\beta^p/p!$, where β is the differential form

$$\beta=\frac{i}{2}d'd''|z|^2=\frac{1}{4}dd^c|z|^2=\frac{i}{2}\sum dz_j\wedge d\bar z_j. \tag{9.1}$$

The mean density in the ball $a+rB$ is the mass of μ in that ball divided by the volume of the ball of the same radius in $\mathbf{C}^p$, the corresponding dimension, thus

$$\nu_t(a,r)=\frac{\mu(a+rB)}{\lambda_{2p}(rB\cap\mathbf{C}^p)}; \tag{9.2}$$

the mean density is thus actually a mass divided by a volume. Lelong [1957a:260] proved that this quotient is an increasing function of r. Thus the limit

$$\nu_t(a)=\lim_{r\to0}\nu_t(a,r)=\lim_{r\to0}\frac{\mu(a+rB)}{\lambda_{2p}(rB\cap\mathbf{C}^p)} \tag{9.3}$$

exists. For varieties of codimension one, he proved the corresponding statement already in [1950]. Later Thie [1967:271] called this limit the *Lelong number* when t

is the current of integration on an analytic variety and proved that it is an integer in that case, and equal to the multiplicity of the variety at the point a.

It was conjectured for some time that the set of points where the Lelong number exceeds or equals a certain level, thus the superlevel set

$$E_c(t) = \{z \in \Omega;\ \nu_t(z) \geqslant c\},$$

is an analytic variety. This means that the Lelong number is upper semicontinuous for the analytic Zariski topology. Probably the first appearance in print of the conjecture was in Harvey and King [1972:52]. Thie's result [1967] showed that the conjecture is true in the case of the current of integration on an analytic variety. A partial but important result was proved by Skoda [1972:406]: to any $c > 0$ there exists an analytic variety X of dimension at most p such that

$$E_c(t) \subset X \subset E_{c'}(t), \text{ where } c' = c(1 - p/n). \tag{9.4}$$

Skoda first constructed a so-called canonical potential that has the same Lelong numbers everywhere as the current t and then used the Hörmander–Bombieri theorem (mentioned in section 6) that the set X of nonintegrability of e^{-f} is an analytic variety. The set of nonintegrability is contained in one superlevel set of ν_f and contains another.

The conjecture was proved by Siu in [1974] (announced in [1973]). His proof depended heavily on Hörmander's L^2 estimates for the Cauchy–Riemann operator, and, for currents of higher degree ($n-p > 1$), on Federer's theory of slicing [1969]. We shall sketch later a proof of the result in the case $p = n - 1$.

We have defined the Lelong number as the density at a point of a certain measure. If a plurisubharmonic function is given, it is natural to consider its Laplacian, or rather a multiple of it:

$$\mu = \frac{1}{2\pi}\Delta f = \frac{2}{\pi}\sum_1^n \frac{\partial^2 f}{\partial z_j \partial \overline{z}_j}.$$

This is the *Riesz mass* of f. Its density can be defined in terms of the behavior of f near the point in question. The higher the density, the faster f tends to minus infinity at the point. The mass of μ in a ball can be expressed in terms of the derivatives of a function $u(x,t)$, the mean value of f over the sphere $x + e^t S$ of radius $r = e^t > 0$. (It is convenient to use t rather than r as a variable here, because u is convex in t.) Using Green's formula we can write

$$\mu(x + rB) = \frac{1}{2\pi}\int\limits_{x+rB} \Delta f = \frac{1}{2\pi}\int\limits_{x+rS} \frac{\partial f}{\partial r} dS = \frac{1}{2\pi}\frac{\partial u}{\partial t}\frac{dt}{dr}\int\limits_{rS} dS = \frac{1}{2\pi r}\frac{\partial u}{\partial t}\int\limits_{rS} dS.$$

This calculation was done by Lelong already in [1950] in the case when $f = \log|h|$ for a holomorphic h (although he did it on f and not on u). The relation between u and the Lelong number is simple: since the quotient

$$\frac{\text{area}(rS^{2n-1})}{\text{volume}(rB^{2n-2})} = 2\pi r$$

regardless of the dimension, the mean density (9.2) is equal to the slope $\partial u/\partial t$ at the point $t = \log r$, and the Lelong number (9.3) is its limit as $t \to -\infty$.

Now if we extend u to complex values of t by putting $u(x,t) = u(x, \operatorname{Re} t)$, we obtain a function which is plurisubharmonic as a function not only of x but of (x,t) where it is well defined, i.e., for points (x,t) satisfying $(x,t) \in \Omega \times \mathbf{C}$ and $\operatorname{Re} t < \log d_\Omega(x)$, $d_\Omega(x)$ being the Euclidean distance from x to the complement of Ω. By the minimum principle of Kiselman [1978] (see section 8), the function

$$f_\tau(x) = \inf_t \big(u(x,t) - \tau \operatorname{Re} t; \operatorname{Re} t < -q(x)\big), \qquad x \in \Omega,$$

is plurisubharmonic in Ω for any $\tau \geqslant 0$. We assume here that q is a given plurisubharmonic function satisfying $q \geqslant -\log d_\Omega$ and with $\nu_q = 0$ everywhere (for instance with finite values). One can calculate the Lelong number of f_τ, and it turns out that it is given by a simple formula: $\nu_{f_\tau}(x) = \max(\nu_f(x) - \tau, 0) = (\nu_f(x) - \tau)^+$. We now apply Skoda's result (9.4) to the function $g = f_\tau/(c-\tau)$ with $0 < \tau < c$: there is an analytic variety X_τ such that

$$E_c(f) = E_1(g) \subset X_\tau \subset E_{1/n}(g) = E_{c_\tau}(f),$$

where $c_\tau = \tau + (c-\tau)/n \leqslant c$. If we let τ tend to c, we get $c_\tau \to c$ and we see that

$$E_c(f) = \bigcap_{0<\tau<c} X_\tau.$$

The right-hand side is an intersection of analytic varieties and therefore itself an analytic variety. This short proof of Siu's theorem in the case $p = n-1$ was given by Kiselman [1979].

The Lelong number at the origin of the function $\log|z|$ is 1, and we can say that the Lelong number compares the behavior of a plurisubharmonic with that function, or, what amounts to the same thing, with $\max_j(\log|z_j|)$. However, we might as well compare it with a function like $\max_j(c_j \log|z_j|)$ for any choice of positive numbers c_j. This was the idea behind the so-called *refined Lelong numbers* presented by Kiselman in 1986 and published in [1987, 1992, 1994a]. One forms the mean value of a plurisubharmonic function over a polycircle:

$$v(x,y) = \fint_{|z_j|=|e^{y_j}|} f(x+z),$$

where the bar through the integral sign indicates mean value. The usual Lelong number is obtained when we choose all y_j equal. The limit

$$\nu_f(x,y) = \lim_{t\to-\infty} \frac{v(x,ty)}{t}, \qquad x \in \Omega,$$

exists for all y with $y_j > 0$ and is a concave function of y. Therefore the methods of convexity theory can be applied to the function $\nu_f(x, \cdot)$. For instance it is a consequence of the concavity of this function that all refined Lelong numbers at a fixed point are comparable in the sense that $\nu_f(x, y) \leqslant C_{y,z}\nu_f(x, z)$ for some constant depending on $y, z \in \mathbf{R}^n$, $y_j, z_j > 0$, but independent of f. Siu's theorem is valid, with the same proof as sketched above, as was proved in 1986 by Kiselman [1992, 1994a].

Demailly introduced an even larger class of Lelong numbers in his paper [1987a]. Let φ be a continuous plurisubharmonic function such that the sublevel set $\{x \in \Omega;\ \varphi(x) < R\}$ is relatively compact in Ω for some $R > 0$. Demailly defined

$$\nu(T, \varphi, r) = (2\pi)^{-p} \int\limits_{\varphi<r} T \wedge (dd^c \max(\varphi, s))^p,$$

where T is a current of bidimension (p, p), and $s < r$. If T is closed, Stokes' formula shows that this number is in fact independent of s. The function $r \mapsto \nu(T, \varphi, r)$ is increasing and its limit

$$\nu(T, \varphi) = \lim_{r\to-\infty} \nu(T, \varphi, r)$$

is called the *generalized Lelong number* of T with respect to the weight function φ. When $\varphi(z) = \log|z - a|$ we get the usual Lelong number $\nu_T(a)$; when $\varphi(z) = \max_j(y_j^{-1} \log|z_j - a_j|)$ and $T = dd^c f$ we get the refined Lelong number $\nu_f(a, y)$ just discussed. Demailly's definition is very supple and lends itself to a very short and natural proof of the coordinate invariance of Lelong numbers, a result obtained earlier by Siu [1974] but with a longer proof.

Siu's theorem is valid for the generalized Lelong numbers. In his proof, Demailly first constructed the canonical potential of the current following Lelong [1964] and Skoda [1972]. From that point on, the proof is as sketched above in the proof of Siu's theorem for the usual Lelong numbers.

10. The growth at infinity of entire functions

In complex analysis in one variable, the entire functions of finite order are remarkable because of a nice structure theorem that holds for them. If an entire function h has zeros $a_j \neq 0$ that tend to infinity rapidly enough, it can be written

$$h(z) = z^m e^{g(z)} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right), \qquad z \in \mathbf{C},$$

where g is another entire function. The product converges if the zeros are sufficiently sparse, but in general it does not of course. To improve convergence, Weierstrass introduced the so called *primary factors* or *elementary factors*

$$E_q(z) = (1 - z)\exp(z + z^2/2 + z^3/3 + \cdots + z^q/q), \qquad z \in \mathbf{C}, \tag{10.1}$$

which have a zero at $z = 1$. For $|z| < 1$ they can be expanded in a Taylor series

$$\log E_q(z) = -\frac{1}{q+1} z^{q+1} - \frac{1}{q+2} z^{q+2} - \cdots, \qquad |z| < 1, \tag{10.2}$$

and they satisfy the estimate

$$|1 - E_q(z)| \leqslant |z|^{q+1}, \qquad |z| < 1,$$

cf. Rudin [1966:293]. Weierstrass showed that any entire function h can be written

$$h(z) = z^m e^{g(z)} \prod_{j=1}^{\infty} E_{q_j}(z/a_j), \qquad z \in \mathbf{C},$$

if the numbers q_j are chosen large enough; it is sufficient that

$$\sum_{j=1}^{\infty} \left(\frac{R}{|a_j|} \right)^{q_j+1}$$

converges for every $R > 0$. The sequence (q_j) is of course not uniquely determined, but if the function is of finite order (cf. (10.5) below), we can choose $q_j = q$ to be independent of j; we take the smallest possible q (the *genus*) to get uniqueness and find that the function admits a representation

$$h(z) = z^m e^{g(z)} \prod_{j=1}^{\infty} E_q(z/a_j), \qquad z \in \mathbf{C}.$$

In 1964, Lelong published a remarkable result for plurisubharmonic functions which is an analogue of this classical theorem (announced already in [1953]). Earlier Kneser [1938:25] and Stoll [1953:230] had published similar results for $\log |h|$ with h entire or meromorphic of finite order in $\mathbf{C}^n$, but their integral formula represents $\log |h|$ only in a ball free from zeros and poles of h.

A plurisubharmonic function f defines a current θ of bidegree $(1,1)$,

$$2id'd''f = \theta. \tag{10.3}$$

If $f = \log |h|$ for an entire function, θ represents its zeros. The current θ has measure coefficients. Lelong associates with any current θ of bidegree $(n-p, n-p)$ two other currents, the *projective trace* ν and the *trace* σ, defined as

$$\nu = \pi^{-p} \theta \wedge \alpha^p, \qquad \sigma = \frac{1}{p!} \theta \wedge \beta^p,$$

where

$$\alpha = \frac{i}{2} d'd'' \log |z|^2, \qquad \beta = \frac{i}{2} d'd'' |z|^2.$$

It should be remarked that when θ comes from a plurisubharmonic function as in (10.3), thus with $p = n - 1$, then

$$\sigma = \frac{1}{(n-1!)}\theta \wedge \beta^{n-1} = \frac{1}{n!}\Delta f \cdot \beta^n,$$

so σ is essentially the Laplacian of f.

The fundamental solution of the Laplacian can be expanded in a Taylor series in z around a fixed point a:

$$|a - z|^{2-2n} = |a|^{2-2n} + P_1(a, z) + P_2(a, z) + \cdots,$$

where the P_j are homogeneous polynomials in $z, \overline{z}$. Lelong [1964:375] defined kernels

$$e_n(a, z, q) = -|a - z|^{2-2n} + |a|^{2-2n} + P_1(a, z) + \cdots + P_q(a, z)$$

in analogy with (10.1). Then he defined a *canonical potential of genus* q *of* θ as

$$I_q(z) = k_n \int e_n(a, z, q) d\sigma(a), \qquad z \in \mathbf{C}^n,$$

where q is the smallest integer such that

$$\int_1^\infty t^{-q} d\nu(t) < +\infty.$$

If E is the fundamental solution for the Laplacian, $\Delta E = \delta$, it is well-known that we have $\Delta(E * \mu) = \mu$ for any distribution μ such that the convolution $E * \mu$ has a good sense. But if $\mu = \Delta u$ is the Riesz mass of a plurisubharmonic function in $\mathbf{C}^n$, $n \geqslant 2$, then the convolution can be defined only in the uninteresting case $\mu = 0$, so some modification of E is always necessary to get convergence—in contrast to the one-variable theory, where we can have $q = 0$.

It is not so remarkable that the canonical potential I_q satisfies the Poisson equation $\Delta I_q = \sigma$. In fact, this is to be expected from the construction in analogy with the one-variable considerations above. The interesting, almost mysterious, fact is that I_q is in fact plurisubharmonic; it has so to speak no reason to be. This is what Lelong proved in [1964], and it shows that methods from classical potential theory, using concepts like the Newtonian potential, can work sometimes also in the plurisubharmonic category. Thus $2id'd''I_q$ is a closed positive current; moreover it is equal to θ and $V = I_q$ is in fact the solution to the equation $2id'd''V = \theta$ with the smallest possible growth at infinity.

For entire functions, Ronkin [1968, 1971:396ff] investigated a variant of Lelong's result: he integrated not over the whole $(2n-2)$-dimensional zero set of an entire function h but over an $(n-1)$-dimensional subset of it:

$$\{z \in \mathbf{C}^n;\ h(z) = 0 \text{ and } |z_1|/\tau_1 = |z_2|/\tau_2 = \cdots = |z_n|/\tau_n\},$$

where the τ_j are fixed positive numbers.

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We shall now describe another development concerning functions of finite order that started in 1965 and continued into the 1990s. To investigate the behavior of a function at infinity we make a dilation

$$S_r f(z) = f(rz), \qquad z \in \mathbf{C}^n,\ r > 0, \tag{10.4}$$

and see what happens when $r \to +\infty$. Of course this will lead in general to infinite limits, so we need to dampen the growth of $S_r f$. But to do so we must impose some restriction on the growth of f; a classical such restriction is to assume that f is equal to $\log|h|$ for an entire function h of finite order ρ and finite type σ, or at least satisfies the corresponding growth estimate, i.e.,

$$f(z) \leqslant \tau + \sigma|z|^\rho, \qquad z \in \mathbf{C}^n, \tag{10.5}$$

for some constant τ. Then it is natural to define

$$T_r f(z) = \frac{f(rz)}{r^\rho}, \qquad z \in \mathbf{C}^n,\ r > 0. \tag{10.6}$$

The *limit set of f at infinity* with respect to the order ρ (or of h, if f is of the form $f = \log|h|$) is the set of all limits in the topology of $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ that can be obtained from the family $(T_r f)_{r>0}$ as $r \to +\infty$. The limit set was introduced by Azarin [1976] for subharmonic functions, and was investigated in the case of plurisubharmonic functions by Sigurdsson [1986]. The modified dilation T_r is viewed as a linear operator $T_r\colon L^1_{\mathrm{loc}}(\mathbf{C}^n) \to L^1_{\mathrm{loc}}(\mathbf{C}^n)$ or $T_r\colon \mathcal{D}'(\mathbf{C}^n) \to \mathcal{D}'(\mathbf{C}^n)$. The set $\mathit{PSH}(\mathbf{C}^n)$ (now excluding the constant $-\infty$) is a closed cone in each of these spaces, and they induce the same topology there. So $\mathit{PSH}(\mathbf{C}^n)$ is a complete metric space. We can think of $\{T_r f;\ 0 < r < +\infty\}$ as a curve in that space, and we are interested in what happens when r tends to $+\infty$ (or when r tends to 0, see the next section). If f is not identically $-\infty$ we consider its forward orbit, i.e., $\{T_r f;\ r \geqslant 1\}$. It is relatively compact in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ if and only if (10.5) holds, as was shown by Sigurdsson [1991:293]. It is clear that if φ belongs to the limit set, then so does $T_r\varphi$ for any r; we express this fact by saying that the limit set is *T-invariant.* It is also easy to see that the limit set must be connected and compact in $\mathit{PSH}(\mathbf{C}^n)$ for the topology induced by $L^1_{\mathrm{loc}}(\mathbf{C}^n)$.

Concerning this limit set an important development has occurred during the decades we survey.

The *indicator* of f is by definition the function

$$p_f(z) = \limsup_{t \to +\infty} \frac{f(tz)}{t^\rho}, \qquad z \in \mathbf{C}^n.$$

It is in general not plurisubharmonic, but its upper regularization p_f^* is; cf. (7.2).

The indicator of f gives an upper bound of the functions in the limit set: it is clear that every function φ in the limit set of f must satisfy $\varphi \leqslant p_f^*$. More precisely, p_f^* is the supremum of all the φ in the limit set. The indicator was defined already by Pólya [1929] for functions of one complex variable. It is also clear from the definition that the indicator as well as its regularization are positively homogeneous of order ρ, i.e., $p(tz) = t^\rho p(z)$ for all $z \in \mathbf{C}^n$ and all positive numbers t. Is any such function the regularized indicator of some entire function? In other words, if f is a given plurisubharmonic function which is positively homogeneous, does there exist an entire function h of exponential type such that $p_{\log|h|}^* = f$? Lelong [1965, 1966] solved the problem for $\rho = 1$ in the special case of complex homogeneity, i.e., when $f(tz) = |t|f(z)$ for all $z \in \mathbf{C}^n$ and all complex numbers t. His solution relied on the solution of the Levi problem for a balanced pseudoconvex domain, viz. $\{z;\ f(z) < 1\}$.

The general problem was solved for $\rho = 1$ by Kiselman [1967] and for $\rho > 0$ by Martineau [1966, 1967]. The first author transformed the problem by means of the Borel transformation into the Levi problem for open subsets of projective space,[15] whereas the second used Hörmander's L^2 methods for the Cauchy–Riemann operator.

Sigurdsson [1986:262] proved a refined indicator theorem in that he showed that it is possible to impose a precise growth condition on the entire function h; this enabled him to characterize the indicators of Fourier transforms of distributions with compact support [1986:290]. Hörmander and Sigurdsson [1998] continued the study of the asymptotic behavior of Fourier transforms of compactly supported distributions, a truly elusive class of entire functions.

If a plurisubharmonic function behaves sufficiently well at infinity, its limit set is a singleton. One then says that the function is of *completely regular growth.* The notion was introduced for one complex variable by B. Levin and A. Pfluger in the 1930s. It was studied in several variables by Ronkin, Azarin, P. Z. Agranovič, and others from 1958, and especially during the 1970s, not only in the whole space but also in cones. Ronkin's monograph [1992] is a standard reference; Lelong and Gruman included a chapter on the subject in their monograph [1986]. The notion proved to be highly relevant for the characterization of surjective convolution operators in $\mathbf{C}^n$; see the survey by Krivosheev and Napalkov [1992].

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[15] For functions which admit a linear minorant, the proof can be simplified: it is enough to work in $\mathbf{C}^n$. In a letter to Martineau dated July 11, 1966, Kiselman asked whether a positively homogeneous plurisubharmonic function always admits a linear minorant. Martineau replied on a card dated August 31, 1966: "Je m'étais posé votre question autrefois et j'avais cru répondre négativement mais je suis actuellement incapable de retrouver mes raisons." Martineau's intuition was right but the answer to the question was provided much later: Znamenskiĭ [1993] constructed a nonzero analytic functional with three convex supports without a common point. The indicator of the Fourier transform of such a functional cannot admit a linear minorant.

The more precise question of which sets of plurisubharmonic functions are limit sets of entire functions was solved by Sigurdsson [1986:252] and Hörmander and Sigurdsson [1989]. In the first paper Sigurdsson proved that the limit set of a plurisubharmonic function is also the limit set of some entire function. In the second paper, the authors proved that a given subset M of $PSH(\mathbf{C}^n)$ which is compact, connected and T-invariant is the limit set of some plurisubharmonic function of order ρ and finite type if and only if there does not exist any proper open subset V of M satisfying $T_r\overline{V} \subset V$ for some $r > 1$. Let us make two remarks to illuminate this result. First, a sufficient condition is that each pair of points in M can be joined by a polygonal path (in particular, convexity is sufficient). Second, if each function in M is homogeneous of order ρ, i.e., $f(tz) = t^\rho f(z)$ for all positive t, then Sigurdsson [1986:243] proved that it is enough that M be compact, connected and T-invariant.

11. The existence of a tangent cone

To investigate the behavior of a function at a point—which we may take as being the origin—we consider the dilation (10.4) and see what happens when $r \to 0$. This will lead in general to a limit minus infinity in the interesting cases, i.e., when $f(0) = -\infty$, so we need to modify the function for each r somewhat like in (10.6). We may define

$$U_r f(z) = f(rz) - \sup_{rB} f, \qquad |z| < \frac{1}{r} d_\Omega(0),$$

where B is the unit ball and $d_\Omega(z)$ denotes the Euclidean distance from z to the complement of Ω. This means that we are looking at f with a microscope magnifying $1/r$ times, but have adjusted the level by an additive constant so that $\sup_B U_r f = 0$. Every sequence $(U_{r_j} f)_j$ contains a subsequence converging in $L^1(B)$ and even in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$. We say that g belongs to the *limit set of f at the origin* if there is a sequence $(r_j)_j$ with $r_j \to 0$ as $j \to +\infty$ and such that $U_{r_j} f$ tends to g in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$.

Now all this can be done more generally for currents. If t is a current we define its push-forward $(h_r)_* t$ under the mapping $h_r(z) = z/r$. The current $\lim_{r\to 0}(h_r)_* t$, if it exists, was called the *tangent cone to t at the origin* by Harvey [1977:332]. A special case is when $t = dd^c f$ and f is a plurisubharmonic function. Then $(h_r)_* t = dd^c(U_r f)$. Therefore the tangent cone of the positive closed current $dd^c f$ exists at the origin if and only if the limit set of f at the origin is a singleton.

In case t is the current of integration on an analytic variety, the tangent cone was known to exist, and Harvey formulated the conjecture that the tangent cone to a strongly positive closed current always exists [1977:332, Conjecture 1.32]. This conjecture was disproved in 1988 by Kiselman [1991]. The proof was very much along the lines of Sigurdsson [1986:243], where he had constructed a plurisubharmonic function with a prescribed limit set at infinity consisting of functions which are homogeneous of a certain order. In fact a plurisubharmonic function can do less mischief at the origin than at infinity—but still be sufficiently badly behaved to have a limit set which is not a singleton. Given any subset M of $PSH(\mathbf{C}^n)$ which is closed and connected for the topology induced by $L^1_{\text{loc}}(\mathbf{C}^n)$ and which consists of functions g which satisfy $\sup_B g = 0$ and

$$g(tz) = C \log |t| + g(z), \qquad t \in \mathbf{C}, z \in \mathbf{C}^n,$$

there exists a plurisubharmonic function such that its limit set at the origin is M. The logarithmic homogeneity replaces the homogeneity $f(tz) = t^\rho f(z)$ in Sigurdsson's case. As soon as M contains more than one element we get a counterexample to Harvey's conjecture.

Later Blel, Demailly and Mouzali [1990] studied the more general case of closed positive currents of arbitrary degree. They found conditions which ensure the existence of a tangent cone. One such condition is that the quotient

$$\frac{\nu_t(a,r) - \nu_t(a)}{r}, \qquad 0 < r < r_0,$$

where $\nu_t(a,r)$ and $\nu_t(a)$ are defined by (9.2) and (9.3), be integrable at the origin. Conversely it was shown that if this is not so, then it is possible to construct a current without a tangent cone by the methods already known. Thus the condition is sharp: as soon as $\nu_t(a,r)$ does not tend to its limit $\nu_t(a)$ sufficiently fast, the construction works, meaning roughly that there is enough mass to move around to build up a non-singleton limit set. Blel [1993] generalized Kiselman's theorem on prescribed limit sets to the case of currents of arbitrary degree; the conditions are the same, viz. that the set shall be closed and connected and consist of conical currents (i.e., satisfying an obviously necessary homogeneity condition).

12. The complex Monge–Ampère operator

A fundamental problem for harmonic functions, the classical Dirichlet problem, consists in finding a harmonic function with prescribed values at every point on the boundary of a given domain. In several complex variables, the class of harmonic functions is not invariant under holomorphic mappings or even under complex linear mappings, so the classical Dirichlet problem is not relevant; one tries instead to find a solution in some other class of functions. The pluriharmonic functions, on the other hand, form an invariant class under biholomorphic mappings, but if one tries to solve the Dirichlet problem for them (even in a nice domain like a ball), one soon finds that there is in general no solution. Bergman [1948] was the first to try to remedy this situation by introducing classes that are larger than the pluriharmonic class. He called them *extended classes* and he defined several such classes in which the Dirichlet problem has a unique solution. However, he did not define the classes for all domains, and, moreover, they depend on the domain considered. In the bicylinder, for instance, the extended class coincides with the doubly harmonic functions, and in a domain which is biholomorphically equivalent to a bicylinder, one has to apply the same biholomorphism to the class of doubly harmonic functions [1948:525]. The problem of defining an invariant class which allows unique solvability in the Dirichlet problem remained. Bremermann attacked it using plurisubharmonic functions.

In his seminal paper [1959], Bremermann used the Perron method on plurisubharmonic functions to define a solution to the Dirichlet problem. Invariance under biholomorphic mappings is then automatic. He proved [1959:250] that the Dirichlet problem in a bounded, strongly pseudoconvex domain Ω with boundary of class C^2 is solved for arbitrary continuous boundary data $\varphi \in C(\partial\Omega)$ by the supremum of all plurisubharmonic functions defined in the closure of Ω and majorized by $\varphi(z)$ at every point $z \in \partial\Omega$. Later J. B. Walsh [1968] proved that the solution is continuous in $\overline{\Omega}$. Of course we can turn the problem upside down and define instead a plurisuperharmonic solution. The Perron–Bremermann method thus yields two solutions u, v with $u, -v \in PSH(\Omega)$ and $u \leqslant h \leqslant v$, where h is the harmonic solution to the problem.

A solution obtained by this method is plurisubharmonic and also automatically maximal in the sense defined at the end of section 2. Bremermann observed (although somewhat implicitly) [1959: 250, 273] that if a maximal plurisubharmonic function is of class C^2, then it satisfies the differential equation $\det(\partial^2 u/\partial z_j \partial\overline{z}_k) = 0$. Conversely, Kerzman [1977:164] proved that this equation is also sufficient for a plurisubharmonic function of class C^2 to be maximal. Thus the solutions of the homogeneous complex Monge–Ampère equation (equation (2.4) with $g = 0$) are precisely the maximal plurisubharmonic functions, at least in the C^2 case. Equation (2.4) was identified as an equation having biholomorphically invariant solutions to the Dirichlet problem, and the methods of nonlinear partial differential operators could be put to work.

The foundation for the definition of the complex Monge–Ampère operator $(dd^c)^n$ operating on more general functions was laid by Chern, Levine and Nirenberg when they proved [1969:125] an estimate for the Monge–Ampère which turned out to be very useful. We quote for simplicity their estimate in the improved form given by Klimek [1991: Proposition 3.4.2]: for any open set $\Omega \subset \mathbf{C}^n$ and any compact set $K \subset \Omega$ there exists a compact set $L \subset \Omega \smallsetminus K$ and a constant C such that for all $u_1, ..., u_n \in PSH(\Omega) \cap C^2(\Omega)$ we have

$$\int_K dd^c u_1 \wedge \cdots \wedge dd^c u_n \leqslant C \sup_L |u_1| \cdots \sup_L |u_n|. \tag{12.1}$$

Thanks to this and similar inequalities, the Monge–Ampère operator could be shown to admit extensions to wider classes of functions.

Bedford and Taylor did pioneering work on the complex Monge–Ampère operator which they published in a series of papers. In the first [1976], they defined the operator $(dd^c)^n$ on continuous plurisubharmonic functions using (12.1). More importantly, they defined for the first time $(dd^c u)^n$ for $u \in PSH(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ using induction [1976:11]. Let χ be a differential form of type $(n-k, n-k)$ with smooth coefficients and compact support in Ω. Then

$$\int (dd^c u)^k \wedge \chi = \int u(dd^c u)^{k-1} \wedge dd^c \chi.$$

More explicitly, this formula is proved first for smooth functions u and then used as a definition in the general case: if we have already defined $(dd^c u)^{k-1}$ and proved that it is a positive current, then the right-hand side makes sense and defines $(dd^c u)^k$ in the left-hand side as a positive current. They also established a "minimum principle" for their definition, viz. that

$$\inf_{\partial\Omega}(u - v) = \inf_{\overline{\Omega}}(u - v)$$

for functions $u, v \in C(\overline{\Omega})$ which are plurisubharmonic in Ω and satisfy $(dd^c u)^n \leqslant (dd^c v)^n$ [1976:3]. This shows immediately that the Dirichlet problem cannot have more than one solution in the class considered. The result was later extended and called "comparison theorems" [1982:14] and these comparison theorems or comparison principles have been used in all work on the Dirichlet problem ever since. To top off their paper they proved [1976:43] that the nonhomogenous Dirichlet problem

$$u \in PSH(\Omega), \quad (dd^c u)^n = \mu \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega, \tag{12.2}$$

is solvable in a bounded strongly pseudoconvex domain Ω with $u \in C(\overline{\Omega})$, with arbitrary continuous boundary values $\varphi \in C(\partial\Omega)$, and with $\mu = f\beta_n$ for arbitrary nonnegative $f \in C(\overline{\Omega})$. Here $\beta_n = \beta^n/n!$ is the volume form in $\mathbf{C}^n$, β being defined by (9.1).

Bedford and Taylor gave in [1982] solutions to long standing open problems in complex analysis. These depended on convergence properties of the Monge–Ampère, e.g., the following [1982:27]: let $(u_j^i)_{j\in\mathbf{N}}$, $i = 1, ..., k$, be k sequences of functions in $PSH(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ that are uniformly bounded on compact subsets of Ω. Suppose there exist $u^1, ..., u^k \in PSH(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ such that $\lim_{j\to\infty} u_j^i = u^i$ exists almost everywhere in Ω, $i = 1, ..., k$, and that all but one of the sequences are either increasing or decreasing. Then

$$\lim_{j\to\infty} dd^c u_j^1 \wedge \cdots \wedge dd^c u_j^k = dd^c u^1 \wedge \cdots \wedge dd^c u^k$$

in the topology of currents. In particular, the Monge–Ampère operator $(dd^c)^n$ is stable under monotone limits of locally bounded functions provided the limit itself is locally bounded. The fine, or pluri-fine, topology is the weakest topology in an open set Ω for which all plurisubharmonic functions are continuous. In [1987], the authors were able to give sharp statements of some of the convergence theorems proved in [1982] in terms of the fine topology. As an example, in the situation just mentioned, $(dd^c u_j)^n$ tends to $(dd^c u)^n$ weak* in the fine topology, which means that

$$\int \psi (dd^c u_j)^n \to \int \psi (dd^c u)^n, \qquad j \to \infty,$$

for every bounded, finely continuous function ψ with compact support.

However, the Monge–Ampère operator is not stable under some commonly used topologies as was discovered by Cegrell [1983a], who proved that the Monge–Ampère operator is discontinuous for the topologies of L^p, $1 \leqslant p < +\infty$: there exists a bounded sequence (u_j) of plurisubharmonic functions of two variables such that $u_j \to u$ in L^p for all p, $1 \leqslant p < +\infty$, but such that $(dd^c u_j)^2$ does not tend to $(dd^c u)^2$ as defined by Bedford and Taylor. Later Lelong [1983b] extended this result and proved that continuous plurisubharmonic functions with vanishing Monge–Ampère are actually dense in the space of all locally bounded plurisubharmonic functions for the topology of $L^1_{\mathrm{loc}}(\Omega)$, $\Omega \subset \mathbf{C}^n$, $n \geqslant 2$.

A sharp sufficient condition for the convergence of the Monge–Ampère was established by Xing [1996]. He introduced a capacity analogous to that defined in (7.3),

$$C_{n-1}(K, \Omega) = \sup_u \left[\int_K (dd^c u)^{n-1} \wedge dd^c |z|^2; u \in PSH(\Omega),\ 0 < u < 1 \right],$$

for K a compact subset of Ω, and

$$C_{n-1}(E, \Omega) = \sup_K \big(C_{n-1}(K, \Omega); K \text{ compact and contained in } E \big), \qquad E \subset \Omega.$$

Let us say that u_j tends to u *in C-capacity on E* if for every positive δ,

$$C(\{z \in E;\ |u_j(z) - u(z)| > \delta\}) \to 0, \text{ as } j \to \infty.$$

Xing's result is that if $u_j \to u$ in C_{n-1}-capacity on each compact subset of Ω, then $(dd^c u_j)^n$ tends to $(dd^c u)^n$ in the sense of currents. Convergence in C_{n-1}-capacity is very close to being necessary. One example out of several is the following: under the additional assumption that all u_j are equal outside some compact subset of Ω and that either $u_j \leqslant u$ for every j or $u_j \geqslant u$ for every j, he proved that $u_j \to u$ in C_{n-1}-capacity if and only if $(dd^c u_j)^n \to (dd^c u)^n$ in the sense of currents.

Caffarelli, Kohn, Nirenberg, and Spruck [1985] proved fundamental results on the regularity of solutions to the complex Monge–Ampère equation. More precisely, they proved solvability in the Dirichlet problem (12.2) in $C^\infty(\Omega)$ when the data are C^∞ and Ω strongly pseudoconvex (actually for a more general situation than (12.2)). Solutions of (12.2) are, however, not regular up to the boundary: Bedford and Fornaess [1979] gave an example of a bounded strongly pseudoconvex domain Ω with smooth boundary such that the solution u to (12.2) with $\mu = f\beta_n$, $f \in C_0^\infty(\Omega)$, $\varphi = 0$, is in $C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ but not in $C^2(\overline{\Omega})$. The flatter the boundary of a pseudoconvex domain is, the less regular is the behavior of the solution: Coman [1997] proved that if the solution u to (12.2) is always Hölder continuous in $\overline{\Omega}$ when $\mu = 0$ and φ is Hölder continuous, then Ω must be of finite type; the converse holds in $\mathbf{C}^2$. And for real analyticity even interior regularity is tricky: Cegrell and Sadullaev [1992] showed that there is a strongly pseudoconvex domain Ω with real analytic boundary and a real analytic function φ on $\partial\Omega$ such that the solution u to (12.2) with $\mu = 0$ is not real analytic in Ω.

The fact that the Monge–Ampère can be applied to any bounded plurisubharmonic function in the theory of Bedford and Taylor makes it natural to single out a class of measures μ such that (12.2) has a bounded solution u. Cegrell and Persson [1992] proved that if $\mu = f\beta_n$ with $f \in L^2(\Omega)$, then there is a continuous solution. Kołodziej [1996] extended this to $f \in L^p(\Omega)$, $p > 1$. A general fact, proved by Kołodziej [1995], is that if (12.2) has a bounded solution for a certain measure μ, then it can also be solved for every μ' which is majorized by μ (while keeping the boundary value φ). He also established some sufficient and some necessary conditions for solvability. It is for instance necessary that μ be dominated by capacity in the sense that, for some constant M,

$$\mu(K) \leqslant MC_n(K, \Omega)$$

for any compact subset K of Ω, where $C_n(K, \Omega)$ is the relative capacity defined by (7.3). The necessity follows from the very definition of the relative capacity. However, this condition is not sufficient. Kołodziej therefore replaced the condition by a stronger one, $\mu(K) \leqslant F(C_n(K, \Omega))$ for a suitable function F. He obtained a positive result with $F(x) = x(\log(1 + 1/x))^{-n-\varepsilon}$ for any positive ε, a result which is close to best possible [1998]. Recently Xing [MS] obtained a characterization of the class of measures μ such that $\mu = (dd^c u)^n$ for some bounded plurisubharmonic function u.

It was known that the Monge–Ampère cannot be applied without problem to unbounded plurisubharmonic functions: the first example to illustrate this was due to Shiffman and Taylor and published in Siu [1975:451–453]. Kiselman [1984] presented simple examples of functions which are smooth outside a complex hyperplane but whose Monge–Ampère has infinite mass near that hyperplane. Actually this is not any more remarkable than the observation that a convex function may have an infinite real Monge–Ampère mass in an unbounded subset of $\mathbf{R}^n$. In fact, the complex Monge–Ampère of $u(z) = f(\log|z_1|, ..., \log|z_n|)$ in the set $\{z \in \mathbf{C}^n;\ 0 < |z_j| < 1\}$ is essentially equal to the real Monge–Ampère[16] of f in the unbounded set $\{x \in \mathbf{R}^n;\ x_j < 0\}$, taking f to be a smooth convex function. These examples indicated that the extension of the results on the nonhomogeneous Monge–Ampère equation $(dd^c u)^n = \mu$ to the case where μ is a general measure remained (and remains) problematic.

The solvability of (12.2) has been successfully studied in more general domains. Two classes of pseudoconvex domains are of interest. A domain is *hyperconvex* if there exists a negative plurisubharmonic function which tends to zero

[16] The real Monge–Ampère of f is by definition the function $\det(\partial^2 f/\partial x_j \partial x_k)$.

at its boundary. Sibony [1987] introduced and investigated the notion of a B-regular compact set. We may call a bounded pseudoconvex open set *B-regular* if its boundary is B-regular. If the boundary is of class C^1 a necessary and sufficient condition is that the restriction mapping $C(\overline{\Omega}) \cap PSH(\Omega) \to C(\partial\Omega)$ be surjective; [1987:306]. If (12.2) can be solved for arbitrary continuous boundary values φ, then obviously Ω has to be B-regular; conversely, Błocki [1996:736] extended the result of Bedford and Taylor [1976] for strongly pseudoconvex domains to B-regular domains, using in his proof improvements due to Demailly [MS]. In the more general case of a hyperconvex domain one has to give something up, and Błocki gave a very satisfying answer: if we assume that φ is the restriction of a function in $C(\overline{\Omega}) \cap PSH(\Omega)$, then (12.2) can be solved for all $f \in C(\overline{\Omega})$, $f \geqslant 0$ [1996:744]; if not, we can find a supersolution, replacing the first equation in (12.2) by the inequality $(dd^c u)^n \geqslant \mu = f\beta_n$ while keeping the boundary values $u = \varphi$ [1996:745]. In particular Błocki's result implies that there is a plurisubharmonic function in any hyperconvex domain such that the product of all n eigenvalues of its Levi form is everywhere at least one—in B-regular domains the eigenvalues can all be taken to be at least one; Sibony [1987:306].

In ongoing research one tries to define the Monge–Ampère in as large classes of functions as possible while still keeping some key properties. Bedford and Taylor introduced the class of plurisubharmonic functions u such that locally there exists a plurisubharmonic function v such that $-(-v\varphi(-v))^{1/n} < u$ for some decreasing function $\varphi\colon [1, +\infty[\to \mathbf{R}$ such that

$$\int_1^\infty \frac{\varphi(x)}{x} dx < +\infty;$$

Bedford [1993:67]. They proved that for functions u_j, u in this class, $(dd^c u_j)^n$ behaves well when the u_j decrease to u. Cegrell [1998] found another class of functions on which the Monge–Ampère can be applied successfully: he showed that $(dd^c u)^n$ can be defined for a plurisubharmonic function u in a bounded hyperconvex open set Ω if $u = \lim_{j\to\infty} u_j$ for a decreasing sequence (u_j) of bounded negative plurisubharmonic functions in Ω which tend to zero at the boundary of Ω and are such that $\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty$ and $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$. Here $1 \leqslant p < +\infty$. He then defined $(dd^c u)^n$ as the weak limit of $(dd^c u_j)^n$, which exists and is independent of the choice of sequence. The comparison principle mentioned above is valid, and the Dirichlet problem (12.2) can be solved for measures μ belonging to a precisely described class.

13. The global extremal function

Siciak introduced in [1961, 1962] an extremal function of several complex variables analogous to the Green function for the unbounded component of the complement of a compact set in the complex plane and with pole at infinity. He emphasized that the Green function plays a primary role in the theory of interpolation and approximation of holomorphic functions of one variable by polynomials. Indeed his function was to play a similar role in several variables and his article became the starting point of a rich development.

The extremal function $z \mapsto \Phi(z, E, b)$ depends on a given subset E of $\mathbf{C}^n$ and a given function b defined on E. Siciak's original definition used Lagrange interpolation of the values $\exp b(p^\nu)$ to define a polynomial taking those values at certain points p^ν in E, then choosing the points in an extremal way (in analogy with the Fekete points in one variable) and finally passing to the limit. A consequence was the Bernstein–Walsh inequality for polynomials P of degree at most j,

$$|P(z)| \leqslant \|P\|_E\, \Phi(z, E, 0)^j, \qquad z \in \mathbf{C}^n, \tag{13.1}$$

where the norm is the supremum norm on E.

Siciak proved that the sublevel sets of the extremal function, i.e., the sets

$$E_R = \{z;\, \Phi(z, E, 0) < R\}, \qquad R > 1,$$

determine the possible holomorphic extensions of a given function f on a compact set E. More precisely, assuming $\Phi(\,\cdot\,, E, 0)$ to be continuous, f was shown to admit a holomorphic extension to the open set E_R if and only if

$$\limsup_{j\to\infty} \|f - \pi_j\|_E^{1/j} \leqslant 1/R,$$

where π_j is a polynomial of degree at most j which best approximates f on E [1962:346]. This was a striking generalization of the corresponding one-dimensional result, due to Bernstein (in the case of an interval), and Walsh and Russell; cf. J. L. Walsh [1935:79].

Later (13.1) was taken as the definition, i.e., one usually defined

$$\Phi(z, E, 0) = \sup_{j\geqslant 1} \sup_P \big(|P(z)|^{1/j};\, \|P\|_E \leqslant 1\big), \qquad z \in \mathbf{C}^n, \tag{13.2}$$

where P varies in the space of polynomials of degree at most j.

Zahariuta [1975:382] introduced an extremal function defined in terms of plurisubharmonic functions

$$V_E(z) = \sup_u \big(u(z);\, u \in \mathcal{L}, u \leqslant 0 \text{ on } E\big), \qquad z \in \mathbf{C}, \tag{13.3}$$

where $\mathcal{L}$ denotes the class of plurisubharmonic functions with logarithmic growth, i.e.,

$$\mathcal{L} = \{u \in PSH(\mathbf{C}^n);\, \sup_z \big(u(z) - \log(1 + |z|)\big) < +\infty\}. \tag{13.4}$$

Liouville's theorem for plurisubharmonic functions says that a nonconstant plurisubharmonic function cannot grow slower than a positive constant times $\log|z|$.

Therefore $\mathcal{L}$ is called the class of plurisubharmonic functions of *minimal growth.* It is a subclass of $PSH(\mathbf{C}^n)$ of great interest. The upper semicontinuous envelope V_E^*, where the star is defined by (7.2), is either plurisubharmonic (when E is not pluripolar) or identically $+\infty$ (when E is pluripolar).

If we use (13.2) to define $\Phi(z, E, 0)$, it is obvious that $\log \Phi(\,\cdot\,, E, 0) \leqslant V_E$ for any set E. Zahariuta proved that $V_K^* = \log \Phi(\,\cdot\,, K, 0) = V_K$ if K is a compact set such that V_K^* is zero on K [1976a:146]. Siciak [1981, 1982:23] proved that $V_K = \log \Phi(\,\cdot\,, K, 0)$ for general compact sets K. A fourth proof, using Hörmander's L^2 methods, was given by Demailly [MS]. Thus a definition that had its origin in interpolation problems in one complex variable came to be directly expressed using plurisubharmonic functions.

A striking characterization of algebraic varieties in terms of the global extremal function was established by Sadullaev [1982]. Let a connected analytic variety A in an open subset of $\mathbf{C}^n$ be given as well as a compact subset K of A, and assume that K is not pluripolar in A. Then V_K is locally bounded in A if and only if A is a piece of an algebraic variety.

It is not easy to calculate V_E. Sadullaev [1985] determined V_K when K is a ball in $\mathbf{R}^n \subset \mathbf{C}^n$ and noted that it is not a smooth function. More generally, Lundin [1985] determined V_K when K is a convex, symmetric, compact subset of $\mathbf{R}^n \subset \mathbf{C}^n$. From the special form of V_K in Lundin's case, one can see easily that the sublevel sets $\{z \in \mathbf{C}^n;\, V_K^*(z) < c\}$, $c \in \mathbf{R}$, are convex. It is a general result of Lempert that these sublevel sets are convex if K is any convex compact subset of $\mathbf{C}^n$; we shall come back to that question in section 15. Lempert's result relies on a beautiful description of V_K (published in Momm [1996:160]) when K is strongly convex and has real analytic boundary, viz.

$$(13.5) \qquad V_K(z) = \inf_{r,f} \big(\log r;\, r > 1,\ f(r) = z\big), \qquad z \in \mathbf{C}^n \smallsetminus K,$$

where f varies in the class of all holomorphic mappings of the complement of the closed unit disk into $\mathbf{C}^n$ such that $f(t)/|t|$ is bounded and f has a continuous extension to the unit circle, mapping it into K.

Bedford and Taylor [1986] gave precise estimates for the measure $(dd^cV_K)^n$ when K is compact and contained in $\mathbf{R}^n$ and gave an exact expression for it when K is convex and symmetric.

Zeriahi [1996] investigated the global extremal function on nonsingular algebraic varieties and extended results in $\mathbf{C}^n$ to that case. To treat the more general case of analytic spaces he introduced an axiomatic approach [MS] in that he replaced the class $\mathcal{L}$ by a class of functions satisfying certain axioms.

The global extremal function has had a great significance in many results on approximation and the problems of isomorphisms between spaces of holomorphic functions, and even in real analysis; see, e.g., Pawłucki and Pleśniak [1986] and the surveys by Klimek [1991] and Zahariuta [1994].

The notion of capacity appeared in classical potential theory as a measure of the size of sets in $\mathbf{R}^n$, and was a model for the capacity of a metal conductor to hold electric charges: how many coulombs can you put into the conductor while not letting the tension exceed one volt? An early attempt to generalize this notion to several variables was the Γ-capacity of Ronkin [1971]. It is built up from the logarithmic capacity in $\mathbf{C}$ using induction over the dimension, and is not invariant under biholomorphic mappings. Zahariuta [1975] and Siciak [1981] studied the functionals

$$\gamma(E) = \limsup_{|z|\to+\infty} \bigl(V_E(z) - \log|z|\bigr) \text{ and } c(E) = \exp(-\gamma(E)), \qquad E \subset \mathbf{C}^n. \tag{13.6}$$

In fact, for $n = 1$, $c(E)$ is the classical logarithmic capacity of E, so it was natural to expect that the behavior of the extremal function at infinity would reflect important properties of the set. The functional c was called a capacity by analogy (e.g., by Zahariuta [1975:383]), without claiming that it is actually a capacity in Choquet's sense.

Choquet introduced an axiomatic approach to capacities in his immensely influential paper [1955]. He defined a *capacity* as a functional $\varphi\colon \mathcal{E} \to [-\infty, +\infty]$ which is defined on an arbitrary family $\mathcal{E}$ of subsets of a topological space X and which is increasing and continuous on the right [1955:174]. He then defined the interior capacity related to φ as

$$\varphi_*(A) = \sup_E \bigl(\varphi(E); E \in \mathcal{E}, E \subset A\bigr), \qquad A \subset X,$$

with the modification that $\varphi_*(A) = \inf_E(\varphi(E); E \in \mathcal{E})$ when there is no element of $\mathcal{E}$ contained in A, and the exterior capacity as

$$\varphi^*(A) = \inf_\omega \bigl(\varphi_*(\omega); \omega \text{ open}, \omega \supset A\bigr), \qquad A \subset X.$$

He called a set *capacitable* if the interior and exterior capacities agree on it. The continuity on the right means precisely that $\varphi(E) = \varphi^*(E)$ for all $E \in \mathcal{E}$, and clearly $\varphi_*(E) = \varphi(E)$ when $E \in \mathcal{E}$, so all elements of $\mathcal{E}$ are capacitable. For which other sets A does the equation $\varphi_*(A) = \varphi^*(A)$ hold? Before him it was not known whether all Borel set are capacitable for the classical Newtonian capacity; Cartan [1945:94]. Choquet solved the problem affirmatively. His famous theorem of capacitability [1955:223] says that every K-analytic set is capacitable for every capacity in a very large class. The class of K-analytic sets contains all Borel sets in $\mathbf{R}^n$ and in particular the sets $\{x \in \mathbf{R}^n; u(x) < u^*(x)\}$, where $u = \limsup u_j$, (u_j) being a sequence, locally bounded from above, of subharmonic functions.

Soon afterwards Choquet streamlined his definition. Specialized to the case of the family of all compact subsets of a Hausdorff space X, his new definition read as follows [1959:84]: an *abstract capacity* (later to become known as a *Choquet capacity*) is an increasing functional f defined on all subsets of X with values in $[-\infty, +\infty]$ and satisfying

$$f\bigl(\textstyle\bigcap K_j\bigr) = \lim f(K_j) \text{ and } f\bigl(\textstyle\bigcup A_j\bigr) = \lim f(A_j) \tag{13.7}$$

for every decreasing sequence $(K_j)_{j\in\mathbf{N}}$ of compact sets and every increasing sequence $(A_j)_{j\in\mathbf{N}}$ of arbitrary subset of X. In his new theory, he called a set A *f-capacitable* if $f(A) = \sup f(K)$, the supremum being taken over all compact sets K contained in A. All K-Suslin sets (in many cases the same as K-analytic sets) are capacitable for all abstract capacities. Links between the two systems of axioms are provided by two facts: (i) the exterior capacity associated to a capacity in his theory in [1955] is always an abstract capacity (Brelot [1959:59]); and (ii) an abstract capacity in the sense of [1959] is a capacity in the sense of [1955] when $\mathcal{E}$ is the family of compact sets, provided the underlying space is locally compact. For a full account of the history of potential theory, see Brelot [1954, 1972]. Choquet has presented his personal reflections on the birth of capacity theory in [1986].

Kołodziej [1988] proved the remarkable result that the functional c defined in (13.6) actually satisfies Choquet's axioms (13.7)—the difficult point being the first condition on decreasing sequences of compact sets. Therefore all theorems on abstract capacities can be applied to this functional: Borel sets can be approximated from the inside by compact sets and from the outside by open sets. He later discovered new fundamental properties of extremal functions [1989] and showed his result in [1988] to be an easy consequence of them.

El Mir [1980] proved that given $f \in PSH(\Omega)$ and a relatively compact open subset ω of Ω, there exists a globally defined function $F \in \mathcal{L}$ such that $F \leqslant h\circ f$ in ω, where h is a convex increasing function on the real axis with $h(x) = -\log(-x)$, $x < -1$. In particular $\omega \cap P(f) \subset P(F)$, yielding an improvement of Josefson's theorem. The Lelong number of F is zero at every point. This is reasonable, since, in view of Siu's theorem, the set of all points $z \in \omega$ where $\nu_f(z) \geqslant c > 0$ is an analytic subvariety of ω which need not be extendable to a subvariety of $\mathbf{C}^n$. We can express this by saying that the singularities of f cannot in general be extended, but the singularities of $h\circ f$ are weakened by h and can be extended to $\mathbf{C}^n$. Alexander and Taylor [1984] simplified El Mir's proof and relaxed his choice of the function h; it is enough, they showed, that h, still required to be convex and increasing, satisfy

$$\int_{-\infty}^{-1} |x|^{-1-1/n}|h(x)|dx < +\infty.$$

For example, we can let $h(x) = -(-x)^\alpha$, $x < -1$, for some α satisfying $0 < \alpha < 1/n$. The weakest possible conditions on h do not seem to be known.

However, the functional γ in (13.6) just gives the crudest information about the behavior at infinity of V_E. It is possible to study the behavior in a much finer way, e.g., by compactifying $\mathbf{C}^n$ to a projective space, and studying the behavior of V_E near the hyperplane at infinity. This is what Bedford and Taylor [1988] did. Functions in the class $\mathcal{L}$ are not well-behaved at infinity in general, but a subclass is, and Bedford and Taylor studied an important class of plurisubharmonic functions whose singularities (called logarithmic singularities) could be handled successfully [1989]. In analogy with the Robin constant in classical potential theory (cf. (15.2) below), Taylor [1983:320] introduced the function

$$\rho_u(z) = \limsup_{\mathbf{C}\ni t\to\infty} \big(u(tz) - \log^+|tz|\big), \qquad z \in \mathbf{C}^n \smallsetminus \{0\},\ u \in \mathcal{L}, \tag{13.8}$$

and Bedford and Taylor [1988:133] called ρ_u the *Robin function* of u. Obviously ρ_u is complex homogeneous of degree zero on $\mathbf{C}^n \smallsetminus \{0\}$, so we may regard it as a function on projective space of dimension $n-1$. The function $\rho_u^* + \log|z|$ is plurisubharmonic on $\mathbf{C}^n$, and the functions u such that ρ_u^* is not identically $-\infty$ form a class $\mathcal{L}_\rho \subset \mathcal{L}$ for which the Robin function has a good sense. Bedford and Taylor developed a calculus for $\mathcal{L}_\rho$, and they could show for instance that any polar set is contained in the polar set of some function in $\mathcal{L}_\rho$ [1988:165].

Hartogs proved a theorem on separately analytic functions [1906:12]. Terada [1967] weakened its hypotheses, using Chebyshev polynomials in the proof. In subsequent studies, the extremal function V_E has played an important role in the proofs of generalizations of Hartogs' theorem; see, e.g., Siciak [1969], Nguyen Thanh Van and Zeriahi [1983], and Shiffman [1989].

14. The relative extremal function

An extremal function which has become known as the *relative extremal function* was introduced by Siciak [1969:154]. Given an open set Ω in $\mathbf{C}^n$ and a compact subset E of Ω he defined a function $h = u^*_{E,\Omega}$, where the star denotes the upper semicontinuous envelope defined by (7.2), and where

$$u_{E,\Omega}(z) = \sup_u \big(u(z); u \in PSH(\Omega), u \leqslant 0 \text{ on } E, u \leqslant 1 \text{ in } \Omega\big), \qquad z \in \Omega. \tag{14.1}$$

The definition makes sense of course for any subset E of Ω. Siciak noted that $h = u^*_{E,\Omega}$ is extremal in the sense that any plurisubharmonic function v which is $\leqslant m$ on E and $\leqslant M$ in Ω must satisfy $v \leqslant m + (M-m)h$ in Ω; the function h serves "as a version of the Two Constants Theorem" for plurisubharmonic functions.

A motivation for Siciak's studies was Hartogs' theorem on separate analyticity. He considered sets in the form of a cross, $X = (\Omega_1 \times K_2) \cup (K_1 \times \Omega_2)$, where K_j is a compact set in a domain of holomorphy (or a Stein manifold) Ω_j, $j = 1, 2$, and established the existence of holomorphic extensions of separately analytic[17] functions defined on such sets. The conclusion was that every separately analytic function on X can be extended to a holomorphic function in

$$\Omega = \{(z,w) \in \Omega_1 \times \Omega_2;\ u^*_{K_1,\Omega_1}(z) + u^*_{K_2,\Omega_2}(w) < 1\}.$$

Actually Siciak proved some special cases of that result in [1969], whereas Zahariuta [1976b: 64] proved the result just quoted, assuming a certain regularity of

[17] By this we mean that $f(\,\cdot\,, w)$ is analytic in Ω_1 for every $w \in K_2$ and $f(z, \cdot\,)$ is analytic in Ω_2 for every $z \in K_1$.

K_j. This is just one of several generalizations of Hartogs' theorem on separately analytic functions. Siciak returned to the subject in [1981].

Zahariuta [1974: §3] used the sublevel sets of the function $u^*_{E,\Omega}$ to define open and compact sets

$$\Omega_\alpha = \{z \in \Omega;\ u^*_{K,\Omega}(z) < \alpha\}, \quad K_\alpha = \{z \in \Omega;\ u^*_{K,\Omega}(z) \leqslant \alpha\}.$$

He proved that they are associated to interpolation of Hilbert spaces. Suppose Hilbert spaces H_1 and H_0 are given satisfying

$$\mathcal{O}(\overline{\Omega}) \subset H_1 \subset \mathcal{O}(\Omega) \subset \mathcal{O}(K) \subset H_0 \subset AC(K),$$

where $\mathcal{O}(\Omega)$ is the space of holomorphic functions in Ω, $\mathcal{O}(K)$ the inductive limit of $\mathcal{O}(\omega)$ for all open neighborhoods ω of a compact set K, and finally $AC(K)$ is the Banach space obtained by taking the closure of $\mathcal{O}(K)$ in $C(K)$. Then, under certain regularity assumptions,

$$\mathcal{O}(K_\alpha) \subset H^\alpha \subset \mathcal{O}(\Omega_\alpha), \qquad 0 < \alpha < 1,$$

where H^α is the interpolation between H_0 and H_1 defined using a basis (e_j) which is common for H_1 and the closure of H_1 in H_0, and determined by the requirement that $\|e_j\|_{H^\alpha} = e^{\alpha a_j}$ if $\|e_j\|_{H_0} = 1$ and $\|e_j\|_{H_1} = e^{a_j}$, $j \in \mathbf{N}$. Thus interpolation in Hilbert spaces approximates very well the interpolation between K and Ω provided by $u^*_{K,\Omega}$. Concerning the notion of bases which are common for two spaces, let us remark that Zahariuta had established the existence of bases common to $\mathcal{O}(\Omega)$ and $\mathcal{O}(K)$ in one and several variables already in [1967]. The theme was further developed by Nguyen Thanh Van, who seems to have been the first to point out [1972:230] that common bases can be used to obtain holomorphic extensions of separately analytic functions.

Just as in the case of the global extremal function, the relative extremal function can serve to define a capacity. Bedford [1980a, 1980b] expressed the functional C_n defined in (7.3) in terms of the relative extremal function as

$$C_n(K, \Omega) = \int_\Omega (dd^c u^*_{K,\Omega})^n$$

for a compact subset K of Ω, a strongly pseudoconvex domain in a Stein manifold. In their fundamental paper [1982], which we have quoted already in sections 7 and 12, Bedford and Taylor continued this study and extended the arguments. They proved [1982:32] that the measure $(dd^c u^*_{K,\Omega})^n$ is supported by K. Actually C_n plays the role of an inner capacity, so they defined

$$C_n(E, \Omega) = \sup_K \big(C_n(K, \Omega); K \text{ is a compact set contained in } E\big) \tag{14.2}$$

for any subset E of Ω, and an outer capacity

$$C_n^*(E,\Omega) = \inf_U \big(C_n(U,\Omega); U \text{ is an open set containing } E\big).$$

They proved [1982:23] that C_n^* satisfies Choquet's axioms (13.7); it follows that, for compact sets, $C_n^*(K,\Omega) = C_n(K,\Omega)$ as defined by (7.3). Thus the functional many authors had called a "capacity" was proved to actually be a Choquet capacity. We mentioned in section 7 some of the important consequences. Alexander and Taylor [1984] proved sharp inequalities between the relative capacity C_n of (7.3), (14.2) and the capacity c defined in (13.6). In particular, for a relatively compact subset E of Ω, $C_n(E,\Omega) = 0$ if and only if $c(E) = 0$.

As Zahariuta's result showed, it is natural to think of the sublevel sets Ω_α, K_α as a kind of interpolation between K and Ω. In particular, if both Ω and K are convex, one would expect the sublevel sets to be convex, too. This is, however, a highly nontrivial result and was proved by Lárusson, Lassere and Sigurdsson [forthc.] using Poletsky's theory of holomorphic currents (see section 16).

15. Green functions

The classical Green function in a domain in $\mathbf{C}$ is zero on the boundary of the domain and has a logarithmic pole at one point. Lempert [1981, 1983] introduced an analogous function in several variables: his function is plurisubharmonic in Ω, tends to zero at the boundary of Ω and has a logarithmic pole at a given point $a \in \Omega$. Moreover it solves the homogeneous Monge–Ampère equation in $\Omega \smallsetminus \{a\}$, in other words, it is a maximal plurisubharmonic function in the open set $\Omega \smallsetminus \{a\}$ [1981:430, 1983:516]. Actually Lempert's construction was done for a strongly convex set. Klimek [1985] replaced Lempert's construction by a Perron–Bremermann approach: he took the supremum of all negative plurisubharmonic functions u in Ω with a logarithmic singularity at a given point a, thus

$$(15.1)\quad g_\Omega(z,a) = \{u \in PSH(\Omega);\ u < 0 \text{ and } u(z) \leqslant \log|z-a| + O(1),\ z \to a\}.$$

Demailly [1987b] proved that Klimek's definition yields a solution $z \mapsto g_\Omega(z,a)$ to the Dirichlet problem (12.2) with $\mu = (2\pi)^n \delta_a \beta_n$ and $\varphi = 0$. He called the solution "fonction de Green pluricomplexe." This was exactly Lempert's point of departure in the convex case, but Demailly got the result in any hyperconvex domain (see section 12 for the definition). Of course an extension of the definition of the Monge–Ampère operator is a necessary prerequisite when one wants to study right-hand sides of the indicated kind in (12.2), but when a function is bounded outside a relatively compact set this is possible in a nice sense; Sibony [1985:191], Demailly [1985: §4, 1993].

Demailly's solution defines a reproducing kernel for the pluriharmonic functions in a hyperconvex domain. He also proved that the Green function $g_\Omega(z,a)$ is continuous when z varies in $\overline{\Omega}$ and the pole a varies in Ω [1987b:534]. However, the function is not of class C^2 up to the boundary as was shown by Bedford and

Demailly [1988] (cf. the result of Bedford and Fornaess mentioned in section 12), nor is it in general symmetric, in constrast to the classical Green function. It follows from the non-symmetry that $a \mapsto g_\Omega(z,a)$ is not plurisubharmonic in general.

Lempert [1985] introduced a fascinating transformation of solutions of the homogeneous Monge–Ampère equation and showed that the interior Dirichlet problem (15.1) of defining g_Ω is equivalent to the exterior Dirichlet problem (13.3) of defining the global extremal function V_K if K is strongly convex with real analytic boundary, in fact if K is only lineally convex in a strong sense (satisfies a curvature condition). More precisely,

$$V_K(\zeta) = -g_\Omega(\gamma^{-1}(\zeta), 0), \qquad \zeta \in \mathbf{C}^n \smallsetminus K^\circ,$$

where

$$\Omega = \{z \in \mathbf{C}^n;\ \textstyle\sum z_j\zeta_j \neq 1 \text{ for all } \zeta \in K\}$$

is the dual complement of K, and where γ is the mapping (actually a diffeomorphism) of $\overline{\Omega} \smallsetminus \{0\}$ onto $\mathbf{C}^n \smallsetminus K^\circ$ defined by

$$\gamma(z) = \left(\frac{\partial g_\Omega(z,0)/\partial z_j}{\sum_k z_k \partial g_\Omega(z)/\partial z_k}\right)_{j=1}^{n};$$

Lempert [1985:882], Momm [1996:161]. Note that γ depends on g_Ω, so the relation between V_K and g_Ω is not given by a simple change of coordinates. Lempert's results in [1981] implies, as was shown by Momm [1994:54], that the sublevel sets of $g_\Omega(\,\cdot\,,a)$ are convex when Ω is convex. As we have seen, that theorem could be translated into the corresponding result for V_K; cf. (13.5).

Momm [1994] investigated the behavior of $g_\Omega(\,\cdot\,,0)$ near the boundary of Ω (Ω is assumed to contain the origin) and proved that its growth there is related to the extremal function

$$v_H(\zeta) = \sup\big(u(\zeta); u \in \mathcal{L},\ u \leqslant H\big),$$

H being the supporting function of Ω:

$$H(\zeta) = \sup_{z\in\Omega} \operatorname{Re} \textstyle\sum z_j\overline{\zeta}_j, \qquad \zeta \in \mathbf{C}^n,$$

more precisely to the starshaped set where v_H and H are equal.

The results on g_Ω have been refined and generalized in several ways. The behavior of $g_\Omega(z,a)$ near a can be studied using a quantity $C_\Omega(\xi)$ for ξ belonging to the unit sphere of $\mathbf{C}^n$, the ξ*-directional harmonic capacity*, introduced by Nivoche [1994] and defined by

$$-\log C_\Omega(\xi) = \limsup_{\mathbf{C}\ni\lambda\to 0} \big(g_\omega(a+\lambda\xi, a) - \log|\lambda|\big), \qquad \xi \in \mathbf{C}^n,\ |\xi| = 1.$$

The function $-\log C_\Omega$ is analogous to the Robin function (13.8) of Bedford and Taylor.

Nivoche [1995] proved that the Green function can be defined using holomorphic functions with a zero of high order at the point a. This is a result analogous to the equation $V_K = \log \Phi_K$ discussed in section 13. She also proved that the ξ-directional harmonic capacity can be obtained from the behavior of the derivatives of these holomorphic functions.

The classical Green function $G_\Omega(\,\cdot\,, a)$ is harmonic in $\Omega \smallsetminus \{a\}$, tends to zero at the boundary of Ω, and has a singularity $-|z-a|^{2-2n}$ at the pole a. The *Robin constant* for (Ω, a) is by definition

$$\Lambda(a) = \lim_{z\to a} \left(G_\Omega(z,a) + |z-a|^{2-2n}\right), \qquad a \in \Omega, \tag{15.2}$$

and is a measure of the singularity at a. It tends to $+\infty$ as a tends to the boundary of Ω. Yamaguchi [1989] discovered a surprising relationship between the classical potential theory of $\mathbf{R}^{2n}$ and the pluripotential theory of $\mathbf{C}^n$, viz. that the logarithm of the Robin constant is a strongly plurisubharmonic function in Ω. Levenberg and Yamaguchi [1991] gave a new proof of this result, derived an explicit expression for the Levi form of $\log \Lambda$, and studied the metric defined by it—in several cases it was shown to be complete.

Carlehed compared the pluricomplex Green function g_Ω with the classical Green function G_Ω. For a strongly pseudoconvex domain Ω in $\mathbf{C}^n$, $n \geqslant 2$, with boundary of class C^2 he proved the inequality

$$\operatorname{diam}(\Omega)^{2n-4} G_\Omega(z,a) \leqslant |z-a|^{2n-4} G_\Omega(z,a) \leqslant c g_\Omega(z,a) < 0, \qquad z \in \overline{\Omega},\ a \in \Omega,$$

where c is a positive constant depending on Ω only [1998]. For the bidisk the estimate breaks down. When Ω is the unit ball he proved that $G_\Omega < (n-1)2^{3-2n} g_\Omega$, where the constant is best possible [1997].

A generalization has been attracting attention during the last years: Green functions with more than one pole. They were introduced by Lelong—even in Banach spaces [1987, 1989]. He fixed a number of points $a_1, ..., a_k$ in Ω and a number of positive numbers $\nu_1, ..., \nu_k$, called weights. Then the pluricomplex Green function $z \mapsto g_\Omega(z, A)$ with poles in $A = \{(a_j, \nu_j)\}$ is the supremum of all negative plurisubharmonic functions in Ω such that $u(z) - \nu_j \log|z-a_j|$ is bounded from above near a_j, $j = 1, ..., k$. Lelong proved [1989:337] that

$$\sum_{j=1}^{k} \nu_j g_\Omega(z, a_j) \leqslant g_\Omega(z, A) \leqslant \min_{j=1,\ldots,k} \nu_j g_\Omega(z, a_j), \qquad z \in \Omega.$$

When $n = 1$, we have equality to the left here because of the linearity of the Laplace operator, but when $n \geqslant 2$ this is no longer true: the Monge–Ampère is nonlinear! Equality on the left holds in a closed subset of Ω. Lelong observed that this subset may be nonempty but conjectured that its interior is empty [1989:338]. However, Carlehed [1998] proved that this is not so in the bidisk for a particular choice of two or more poles.

Dineen and Gaughran [1993] introduced a Green function with infinitely many poles a_j, $j \in \mathbf{N}$, in a domain Ω in a Banach space. The poles have to converge to a point in Ω, and the weights have to form a convergent series $\sum \nu_j < +\infty$. A more general Green function was studied by Zeriahi [1997], who defined

$$g_\Omega(z;\psi) = \sup_u \bigl(u(z); u \in PSH(\Omega), u \leqslant 0, \nu_u(a) \geqslant \nu_\psi(a) \text{ for all } a \in \Omega\bigr), \quad z \in \Omega,$$

where $\nu_u(a)$ denotes the Lelong number of a function u at a point $a \in \Omega$ defined by (9.3) and ψ is a given plurisubharmonic function in Ω satisfying certain conditions of which the most important is that its polar set $P(\psi)$ be compact. He proved that $z \mapsto g_\Omega(z;\psi)$ is the unique solution of (12.2) with $\mu = (2\pi)^n \sum_{a\in\Omega} \nu_\psi(a)^n \delta_a$ and boundary values $\varphi = 0$; his method works also for arbitrary φ after a simple change in the definition of $g_\Omega(z;\psi)$. Lárusson and Sigurdsson [1998] defined an even more general Green function,

$$g_{\Omega,\alpha}(z) = \sup_u \bigl(u(z); u \in PSH(\Omega), u \leqslant 0, \nu_u(a) \geqslant \alpha(a) \text{ for all } a \in \Omega\bigr), \quad z \in \Omega,$$

α being any nonnegative function in Ω. They showed that this general Green function behaves well under finitely branched covering maps.

Coman [MS] and, independently, Edigarian and Zwonek [forthc.] have computed explicitly the Green function for the unit ball with two poles and equal weights at the poles. They proved that it is of class $C^{1,1}$ outside the poles, real analytic in certain regions of the ball, and nicely foliated in each of these regions by a one-parameter family of complex curves to which the restrictions of $g_\Omega(\,\cdot\,, A)$ are harmonic. Edigarian and Zwonek deduced the formula from a study of the behavior of the Green function under holomorphic mappings, viz. the result by Lárusson and Sigurdsson [1998]. The problem with two poles for the unit ball is equivalent to finding the Green function with one pole for the convex set

$$E = \{w \in \mathbf{C}^2;\ |w_1|^2 + |w_2| < 1\};$$

the mapping $z \mapsto (z_1, z_2^2)$ maps the unit ball in z-space onto the oval E in w-space. Most points have two preimages, which explains the appearance of two poles. For more general domains, it seems that little is known about this nonlinear problem, for instance concerning the existence of foliations.

16. Plurisubharmonic functions as lower envelopes

The Perron–Bremermann method consists in taking the supremum of a family of plurisubharmonic functions subject to some kind of control from above. The upper semicontinuous envelope of such a supremum is then plurisubharmonic. Thus we construct a function by approaching it from below. It is, however, also possible to approach a function from above, and we shall now take a look at some of the efforts of that kind that have been made during the last decades.

For comparison, let us consider first the construction of the convex envelope of a function. Let f be any function with real values defined on $\mathbf{R}^n$. Then the largest convex minorant of f is given by

$$G_f(x) = \inf\Big(\textstyle\sum_1^k \lambda_j f(x_j); \sum_1^k \lambda_j x_j = x,\ \lambda_j > 0,\ \sum_1^k \lambda_j = 1\Big), \qquad x \in \mathbf{R}^n,$$

where the infimum is taken over all representations of x as a barycenter of finitely many points $x_1, ..., x_k$. This is how we arrive at the envelope from above. There is also a construction from below:

$$H_f(x) = \sup\big(A(x); A \text{ affine},\ A \leqslant f\big), \qquad x \in \mathbf{R}^n,$$

where the supremum is taken over all affine minorants $A(x) = \xi \cdot x + C$ of f. This is the lower semicontinuous and convex envelope of f. If f itself is semicontinuous from below, then $G_f = H_f$. So we see that there is a convex envelope which can be approached both from below and from above. Similarly, we can define the plurisubharmonic envelope of a function: it is simply the upper regularization of the supremum of all plurisubharmonic minorants of f. Is there a way to construct this envelope from above?

Already Bremermann [1959:269] showed that the solution to (12.2) with $\mu = 0$ can be obtained by solving the Dirichlet problem in the intersection of Ω with complex lines. He first solved the one-dimensional Dirichlet problem for all lines, using the original boundary values φ, and took the lower envelope ψ of them. Then he took a disk D in one of the lines L and solved the Dirichlet problem in $D \cap \Omega$ as well as in all disks $(D+c) \cap \Omega$ parallel to D, where c varies in the space orthogonal to L, now using the values of ψ as boundary values. One has to repeatedly solve the problem in a dense family of disks D, each time replacing ψ by the infimum of ψ and the new solution. Since it is possible to use a countable family of disks, this process defines a sequence of functions which is obviously decreasing; Bremermann proved that its limit is equal to the function he had already obtained as the upper envelope of plurisubharmonic functions majorized by φ on the boundary.

Gamelin [1976] worked in the setting of uniform algebras, an important example of which is the algebra of those continuous functions on a compact set which are holomorphic in its interior. If A is a uniform algebra on a compact space X, he defined

$$Y = \{(x, \zeta) \in X \times \mathbf{C};\ |\zeta| \leqslant \exp(-f(x))\},$$

where $f\colon X \to\,]-\infty, +\infty]$ is any lower semicontinuous function. He let B be the uniform algebra on Y generated by polynomials in ζ with coefficients in A. The maximal ideal space M_B of B then has the form

$$M_B = \{(\varphi, \zeta) \in M_A \times \mathbf{C};\ |\zeta| \leqslant \exp(-\widetilde{f}(\varphi))\}$$

for some function $\widetilde{f}$. The problem of finding $\widetilde{f}$ is a generalization of the problem of describing the envelope of holomorphy of the set $\Omega(f)$ in (4.4). One of Gamelin's descriptions of $\widetilde{f}$ involves Jensen measures. Let us give the definition: a *Jensen measure* on X for a point $\varphi \in M_A$ is a probability measure σ on X such that $\log|g(\varphi)| \leqslant \int \log|g| d\sigma$ for all $g \in A$. He proved that

$$\widetilde{f}(\varphi) = \inf_{\sigma} \left[\int_X f\, d\sigma; \sigma \text{ is a Jensen measure on } X \text{ for } \varphi \right], \qquad \varphi \in M_A.$$

So $\widetilde{f}$ is approached from above; other descriptions of it involved approaches from below.

Gaveau [1977] discussed differential operators

$$\Delta_a = \sum_{j,k=1}^{n} a_{jk}(z) \frac{\partial^2}{\partial z_j \partial \overline{z}_k}, \qquad z \in \Omega, \tag{16.1}$$

where the a_{jk} are smooth functions that form a positive definite Hermitian matrix at every point. He observed a fact from matrix algebra, viz.

$$\inf_a (\Delta_a u)^n = n^n \det \left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right), \qquad u \in C^2(\Omega) \cap PSH(\Omega), \tag{16.2}$$

where the infimum is taken with respect to all constant matrices (a_{jk}) which are Hermitian and positive definite with determinant 1. (The matrix for which the infimum is attained depends on the point considered, and the infimum is the same for variable a_{jk}.) This implies that (12.2) with $\mu = f\beta_n$ is equivalent, formally, to a dynamic programming problem in the sense of Bellman:

$$u \in PSH(\Omega), \quad \inf_a \Delta_a u = n f^{1/n} \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega,$$

where the infimum is over all matrices $(a_{jk}(z))$ satisfying the conditions mentioned. (If we know that $u \in C^2(\Omega)$, the equivalence is not only formal: it follows from what we just remarked.) Inspired by this, Gaveau proved that the solution to (12.2) can be obtained as an infimum over a larger class, $u = \inf w_\sigma$, where the $w_\sigma(z)$ are defined by Brownian motion starting from z and subject to certain Kähler controls (but not necessarily solving any of the equations $\Delta_a u = n f^{1/n}$).

Lempert [1983:517] constructed plurisubharmonic functions by taking the infimum over a family of mappings of the unit disk into a domain, a method which is a precursor of Poletsky's theory of analytic disks and holomorphic currents.

Bedford [1985] worked with a family of differential operators in a domain Ω similar to (16.1) but in divergence form,

$$P_a u = \sum_{j,k=1}^{n} \frac{\partial}{\partial z_j} \left(a_{jk}(z) \frac{\partial u(z)}{\partial \overline{z}_k} \right), \qquad z \in \Omega, \tag{16.3}$$

where the a_{jk} now are measurable functions and form an Hermitian matrix which satisfies $\varepsilon I \leqslant (a_{jk}(z)) \leqslant I/\varepsilon$ almost everywhere in Ω for some positive ε and has determinant $\det(a_{jk}(z)) = 1$ almost everywhere. Moreover, each a_{jk} is assumed to be antiholomorphic in z_j and holomorphic in z_k when the other variables are fixed; in particular a_{jj} must be constant. (This implies that $P_a u = \Delta_a u$ if u is of class C^2.) He then considered the Dirichlet problem

$$P_a u = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega, \tag{16.4}$$

with $f = 0$ and a given function $\varphi \in C^1(\partial\Omega)$, assuming that Ω is bounded and strongly pseudoconvex with boundary of class C^2. It is known that (16.4) has a unique solution $u_a \in L^2_1(\Omega)$, that is, u has first-order derivatives in $L^2(\Omega)$. The equation $P_a u = f$ is interpreted to mean that

$$-\sum_{j,k=1}^{n} \int_\Omega \frac{\partial \psi}{\partial z_j} a_{jk} \frac{\partial u}{\partial \overline{z}_k} = \int_\Omega \psi f, \qquad \psi \in \mathcal{D}(\Omega),$$

which has a good sense if $f \in L^2(\Omega)$ and $\partial u/\partial \overline{z}_k$ exists as a distribution and is in $L^2(\Omega)$. Bedford then took the infimum of all solutions u_a when a varies under the conditions mentioned and proved that it is equal to the Perron–Bremermann solution to the problem.

It is also possible to use differential operators $P_a = \Delta_a$ with constant coefficients, as was shown by Levenberg and Okada [1993], who also used (16.2). It is now, however, not possible to get the Perron–Bremermann solution by solving the problem (16.4) and taking the infimum over all solutions u_a. The punishment for keeping the a_{jk} constant is that the process has to be repeated after a balayage procedure. In this way, the authors get a sequence (u_m), $m \in \mathbf{N}$, of solutions, where each u_m is obtained from u_{m-1} by a balayage procedure in a ball, and then taking the infimum over all balls and all matrices a, much like in Bremermann's construction mentioned above. The sequence is decreasing, and its limit was shown to be equal to the Perron–Bremermann function u satisfying $(dd^c u)^n = f$ in Ω, $u \leqslant \varphi$ on $\partial\Omega$, this time with an arbitrary bounded $f \in C(\Omega)$, $f \geqslant 0$, Ω being pseudoconvex. If, in addition, Ω is B-regular with smooth boundary, they obtained $u = \varphi$ on the boundary, thus a solution to (12.2).

In Bremermann's approach mentioned above he used disks contained in complex lines. Poletsky [1991] replaced these disks by more general ones and obtained a very satisfying result. To describe it, let $\mathcal{O}(\overline{D}, \Omega)$ denote the set of all mappings that are defined and holomorphic in a neighborhood of the closed unit disk $\overline{D} \subset \mathbf{C}$ and with values in a domain $\Omega \subset \mathbf{C}^n$; this is the family of *analytic disks* in Ω. Let g be any upper semicontinuous function in Ω. Then its lower envelope

$$Eg(z) = \inf_f \left[\frac{1}{2\pi} \int_0^{2\pi} g(f(e^{it}))dt; f \in \mathcal{O}(\overline{D}, \Omega), f(0) = z \right], \qquad z \in \Omega,$$

is plurisubharmonic and equal to the supremum of all plurisubharmonic minorants of g: the approaches from above and below give the same result.

An axiomatic theory encompassing these constructions was proposed and studied by Poletsky [1993] in a highly original paper. He considered functionals, called holomorphic currents, defined on analytic disks. The holomorphic currents have envelopes that, under certain conditions, are plurisubharmonic functions. More precisely, a *holomorphic current* is a mapping from the set of analytic disks into a space of measures on $\overline{D}$; every such measure μ defines a subharmonic function u_μ in D by means of the Riesz representation formula, viz. the sum of the potential of $\mu|_D$ and the Poisson integral of $\mu|_{\partial D}$. The *envelope* of Φ is the function

$$E\Phi(z) = \inf_f \left(u_\mu(0); f \in \mathcal{O}(\overline{D}, \Omega), f(0) = z\right), \qquad z \in \Omega.$$

The holomorphic currents are subject to certain axioms, and moreover, are supposed to possess a property called approximate upper semicontinuity. Then, says Poletsky's main theorem [1993:102], the envelope $E\Phi$ is plurisubharmonic in Ω. Among the many interesting corollaries of his theorem, let us mention the following beautiful characterization of the polynomial hull of a pluriregular compact subset of $\mathbf{C}^n$ [1993:87, Corollary 7.1]; cf. Lárusson and Sigurdsson [1998: Theorem 7.4]. A point z belongs to the polynomial hull of K if and only if there exists a number R such that for every positive ε and every neighborhood ω of K there is an analytic disk $f \in \mathcal{O}(\overline{D}, \mathbf{C}^n)$ with $f(0) = z$, $|f| \leqslant R$, and $f(e^{it}) \in \omega$ for all $t \in [0, 2\pi]$ outside a set of Lebesgue measure less than ε.

A *disk functional* is any mapping $H: \mathcal{O}(\overline{D}, \Omega) \to [-\infty, +\infty[$, and its *envelope* is the function $EH: \Omega \to [-\infty, +\infty[$ defined by

$$EH(z) = \inf_f \left(H(f); f \in \mathcal{O}(\overline{D}, \Omega), f(0) = z\right), \qquad z \in \Omega.$$

A holomorphic current gives rise to a disk functional: we go from the analytic disk f to the measure $\mu = \Phi(f)$ and then to the value $u_\mu(0)$ defined above. Actually only three disk functionals have been studied extensively; Lárusson and Sigurdsson [1998] associated them with the names of Poisson (using boundary values), Riesz (using masses defined by pullbacks of plurisubharmonic functions), and Lelong (using multiplicities of mappings). The authors analyzed them carefully and proved that their envelopes are plurisubharmonic. Moreover, they showed that the conclusions are valid if the open subset Ω of $\mathbf{C}^n$ is replaced by a manifold in a very large class of manifolds. In a similar vein, Edigarian [1997] studied the pluricomplex Green function and gave several equivalent definitions of it, proving that also that function can be constructed as an infimum as well as a supremum.

Although now almost forty years old, the idea to construct plurisubharmonic functions using lower envelopes of various families is still being exploited. The methods are varied and in a state of rapid development.

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