Subharmonic functions on discrete structures

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Dedicated to Carlos Berenstein and Daniele Struppa

Summary. We define the Laplacian on an arbitrary set with a not necessarily symmetric weight function and discuss the Dirichlet problem and other classical topics in this setting.
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1. Introduction

The problem of describing the shape of a three-dimensional object is important in many applications. Images in medicine and industry are often three-dimensional nowadays.

One should be able to store the description of a shape in a computer and be able to compare it with other shapes, using some measure of likeness. One approach to shape description is to introduce a triangulation of the surface of the object and then map this triangulation to a sphere. The position of a point on the surface is then a function on the sphere, and can be expanded in terms of spherical harmonics. This approach, initiated by Brechbühler et al. (1995), was the background of my talk at the Berenstein conference, and it leads to the study of harmonic, or more generally subharmonic, functions on a graph or rather a directed graph with weight functions that need not be symmetric; indeed the Dirichlet problem is not interesting in the symmetric case.

It turns out that the values of harmonic functions often cluster together in an undesirable way, and to get rid of this clustering is a special problem of importance in the shape-description project of Ola Weistrand (MS). This means that, although there is a homeomorphism from the surface of the object to a sphere, the modulus of continuity of the inverse mapping is terribly large. There are various remedies, one being to use different weights in the definition of harmonicity.

This paper is an introduction to the study of harmonic and subharmonic functions on discrete structures. The Dirichlet problem will be studied and explicit solutions in some simple cases will be given. In other cases, however, explicit formulas corresponding to well-known solutions in the classical setting are apparently not known.

Harmonic functions on discrete structures were introduced by Phillips & Wiener (1923). They solved the Dirichlet problem for harmonic functions on subsets of $\mathbf{Z}^n$ using the minimum of a positive definite quadratic form. Blanc (1939) solved the Dirichlet problem on a circular net in the plane by classical arguments from linear algebra. Duffin (1953) developed a discrete potential theory, including the study of a fundamental solution for the Laplacian in $\mathbf{Z}^3$.

In theoretical physics there has been an interest in a discrete calculus during several years as witnessed for instance by Novikov & Dynnikov (1997) and Guo & Wu (2003).

Recently several studies of a discrete Laplace operator have appeared, for instance that of Chung & Yau (2000), who solve the Dirichlet problem in terms of eigenfunctions for the Laplacian, and that of Kenyon (2002), where the weight functions are symmetric. A variant of the discrete calculus is when the edges of a graph are equipped with a metric, as in the research of Favre & Jonsson (2004) and Baker & Rumely (2004).

2. Subharmonic functions

Let X be an arbitrary set. We shall say that f is a *structural function on X* if f is defined on the Cartesian product $X^2 = X \times X$, has complex values, and is such that the set $\{y \in X; f(x,y) \neq 0\}$ is finite for every $x \in X$. The set of all structural functions is an algebra with addition defined pointwise:

$$(f+g)(x,z) = f(x,z) + g(x,z), \qquad (x,z) \in X^2, \tag{2.1}$$

and multiplication defined as

$$(f \diamond g)(x,z) = \sum_{y \in X} f(x,y)g(y,z), \qquad (x,z) \in X^2. \tag{2.2}$$

If X is finite, this is just a matrix algebra, but it is convenient to use the functional notation even then. The multiplication generalizes both pointwise multiplication and convolution. Indeed, if $f(x,y) = F(x,y)\delta(x,y)$ and $g(x,y) = G(x,y)\delta(x,y)$ with the Kronecker delta, then $(f \diamond g)(x,z) = F(x)G(x)\delta(x,z)$; if X is an abelian group and $f(x,y) = F(x-y)$, $g(x,y) = G(x-y)$, then

$$(f \diamond g)(x,z) = \sum_y F(x-y)G(y-z) = (F * G)(x-z),$$

the convolution product of F and G.

We shall say that a structural function $\lambda\colon X^2 \to \mathbf{R}$ is a *weight function* if $\lambda \geqslant 0$ and $\sum_y \lambda(x,y) > 0$ for every $x \in X$.

The inequality

$$f(x)\sum_{y\in X}\lambda(x,y)\leqslant\sum_{y\in X}\lambda(x,y)f(y) \tag{2.3}$$

has a sense for all weight functions λ and all functions $f\colon X\to\mathbf{R}$ and even for functions with values in $[-\infty,+\infty[$ if we define $0\cdot(-\infty)=0$.

Let us define the *Laplacian* $\Delta f=\Delta_\lambda f$ of f at a point x as

$$\Delta f(x)=\sum_{y\in X}\lambda(x,y)(f(y)-f(x)),\qquad x\in X. \tag{2.4}$$

We shall say that $f\colon X\to\mathbf{R}$ is *subharmonic at* x if $\Delta f(x)\geqslant 0$, and that it is *subharmonic in* X if $\Delta f(x)\geqslant 0$ for all $x\in X$. As usual we shall say that f is *superharmonic* if $-f$ is subharmonic, and *harmonic at* x (*in* X) if $\Delta f(x)=0$ (for all $x\in X$, respectively). All this depends of course on the choice of weight function.

Given a point $x\in X$, the points y such that $\lambda(x,y)>0$ may be called *neighbors* of x; note that this relation need not be symmetric. We have a directed graph where there is an arrow from x to all its neighbors, and we thus compare the value at x with a weighted mean value over all neighbors of x.

Example 2.1. Extreme examples are the following. If $\lambda(x,y)=\delta(x,y)$, then every function is harmonic; $\Delta_\delta=0$. If X is finite and $\lambda(x,y)=1$ for all x,y, then only the constant functions are subharmonic.

Example 2.2. Let $X=\mathbf{Z}$ and $\lambda(x,y)=\delta(x+1,y)$. Then a function is subharmonic exactly when it is increasing.

Example 2.3. The solutions to a discrete analogue of the heat equation,

$$u(x,s+1)-u(x,s)=\kappa\bigl(u(x-1,s)-2u(x,s)+u(x+1,s)\bigr)$$

are λ-harmonic if we define $\lambda((x,s),(y,t))=\kappa$ when $(y,t)=(x\pm1,s-1)$, $\lambda((x,s),(y,t))=1-2\kappa$ when $(y,t)=(x,s-1)$, and zero otherwise. Here we assume that $0\leqslant\kappa\leqslant\frac{1}{2}$. Another discrete analogue of the heat equation is

$$u(x,s)-u(x,s-1)=\kappa\bigl(u(x-1,s)-2u(x,s)+u(x+1,s)\bigr);$$

here we define $\lambda((x,s),(y,t))=\kappa$ when $(y,t)=(x\pm1,s)$, $\lambda((x,s),(y,t))=1$ when $(y,t)=(x,s-1)$, and zero otherwise. Here any $\kappa\geqslant 0$ will do. The two equations have very different properties.

A natural interpretation of the Laplacian is in terms of random walks: $\lambda(x,y)$ is then the transition probability from x to y, assuming that $\sum_y\lambda(x,y)=1$; see, e.g., Chung & Yau (2000). If the weight function λ is symmetric, i.e., $\lambda(x,y)=\lambda(y,x)$, then there is an interpretation of harmonic functions as potentials in an electric circuit. We let $1/\lambda(x,y)$ be the resistance of the link between x and y. Then harmonicity at a point x means exactly that Kirchhoff's law holds: the sum of all outgoing currents from x is equal to the sum of all incoming currents to x. However, for symmetric weight functions the Dirichlet problem as we formulate it is not interesting.

Let $X = \mathbf{Z}^2$ and let $\lambda(x, y) = 1$ if $\|y - x\|_1 = 1$ and zero otherwise. Then a function is harmonic at a point x if and only if its value at x is equal to the arithmetic mean of its values at its four neighbors $(x_1 \pm 1, x_2)$, $(x_1, x_2 \pm 1)$. We shall call such functions $\mathbf{Z}^2$*-harmonic* and define the $\mathbf{Z}^2$-Laplacian as

$$\Delta_{\mathbf{Z}^2} u(x) = u(x_1+1, x_2) + u(x_1-1, x_2) + u(x_1, x_2+1) + u(x_1, x_2-1) - 4u(x).$$

We shall compare this with the classical Laplacian,

$$\Delta_{\mathbf{R}^2} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Example 2.4. A second degree polynomial $u(x, y) = ax^2 + 2bxy + cy^2$, where $(x, y) \in \mathbf{R}^2$ or $(x, y) \in \mathbf{Z}^2$, has Laplacians $\Delta_{\mathbf{R}^2} u = 2(a + c)$, $\Delta_{\mathbf{Z}^2} u = 2(a + c)$, respectively. It is therefore harmonic simultaneously in the two settings: if and only if $a + c = 0$.

Example 2.5. The exponential functions behave differently in the two cases. An exponential function $\mathbf{R}^2 \ni (x, y) \mapsto e^{\alpha x + \beta y}$, where α and β are complex numbers, is harmonic in the classical sense if and only if $\beta = \pm i\alpha$. In fact, the $\mathbf{R}^2$-Laplacian is

$$\Delta_{\mathbf{R}^2} e^{\alpha x + \beta y} = (\alpha^2 + \beta^2) e^{\alpha x + \beta y}.$$

In particular we have the functions $\cos(\beta x) e^{-\beta y}$, $\beta \in \mathbf{R}$, which are harmonic and tend rapidly to zero in the upper half plane if β is a large positive number. By way of contrast, the exponential function $\mathbf{Z}^2 \ni (x, y) \mapsto e^{\alpha x + \beta y}$ is $\mathbf{Z}^2$-harmonic if and only if $\cosh \alpha + \cosh \beta = 2$; its $\mathbf{Z}^2$-Laplacian is

$$\Delta_{\mathbf{Z}^2} e^{\alpha x + \beta y} = (2 \cosh \alpha + 2 \cosh \beta - 4)\, e^{\alpha x + \beta y}.$$

The function $h(x, y) = \cos(\alpha x) e^{-\beta y}$, where $\alpha, \beta \in \mathbf{C}$, is harmonic if and only if $\cos \alpha + \cosh \beta = 2$, thus if and only if

$$e^{-\beta} = 2 - \cos \alpha \pm \sqrt{(2 - \cos \alpha)^2 - 1}.$$

If we take α real here and choose the minus sign, we note that $e^{-\beta}$ satisfies $3 - \sqrt{8} \leqslant e^{-\beta} \leqslant 1$, so that the functions tend to zero, but not as rapidly as in the real case. Choosing $\alpha = \pi$ we get $e^{-\beta} = 3 - \sqrt{8}$ and

$$h(x, y) = \cos(\pi x) e^{-\beta y} = (-1)^x \left(3 - \sqrt{8}\right)^y, \qquad (x, y) \in \mathbf{Z}^2,$$

with the fastest possible decay as $y \to +\infty$ for this class of functions. Therefore $\mathbf{R}^2$-harmonic functions cannot be well approximated by $\mathbf{Z}^2$-harmonic ones. We can get a faster decay as $y \to +\infty$ only if we allow growth in the x direction:

$h(x,y) = e^{i\alpha x-\beta y}$ is harmonic for large positive β and a suitable α, but then α cannot be real, implying that h is unbounded on the real axis.

We shall say that a weight function λ is *normalized* if $\sum_{y\in X}\lambda(x,y) = 1$ for all $x \in X$. To any weight function λ we associate a normalized weight function

$$\lambda'(x,y) = \frac{\lambda(x,y)}{\sum_{z\in X}\lambda(x,z)}, \qquad (x,y) \in X^2. \tag{2.5}$$

If λ is normalized, the Laplacian may be written

$$\Delta f(x) = \sum_{y\in X}(\lambda(x,y) - \delta(x,y))f(y), \qquad x \in X.$$

We note that $I+\Delta$ is an increasing operator: $f \leqslant g$ implies $(I+\Delta)f \leqslant (I+\Delta)g$.

Proposition 2.6. *Assume that u is subharmonic with respect to two weight functions λ and μ. Let $c\colon X \to \mathbf{R}$ be a function with nonnegative values. Then u is subharmonic with respect to λ' (defined by (2.5)), $\lambda(x,y)+c(x)\mu(x,y)$, and $\lambda\diamond\mu'$. Moreover it is subharmonic with respect to $\lambda(x,y)+g(x)\delta(x,y)$ for any real-valued function g such that $g(x)+\lambda(x,x) \geqslant 0$ and $g(x)+\sum_{y\in X}\lambda(x,y) > 0$ for all $x \in X$. (We may thus choose $g(x) = -\lambda(x,x)$ except when $\lambda(x,y) = 0$ for all $y \neq x$.)*

The proofs are straightforward. We note that the Laplacian of $\lambda\diamond\mu$ is given by

$$\Delta_{\lambda\diamond\mu}u(x) = \sum_y \lambda(x,y)\Delta_\mu u(y) + \Delta_\lambda u(x) \tag{2.6}$$

if μ is normalized. If both λ and μ are normalized, this can be written succinctly as

$$I + \Delta_{\lambda\diamond\mu} = (I + \Delta_\lambda)\circ(I + \Delta_\mu). \tag{2.7}$$

Example 2.7. Let X be the infinite strip $\{x \in \mathbf{Z}^2; x_2 = 0,1\}$ with only two pixels in the vertical direction. We define the weight function for $(x,y) \in X$ by $\lambda(x,y) = 1$ when $y = (x_1 \pm 1, x_2)$; $\lambda(x,y) = \kappa$ when $y = (x_1, 1-x_2)$ and zero otherwise. Here $\kappa \geqslant 0$ is a kind of coupling constant. Thus

$$\Delta u(x) = u(x_1-1,x_2) + \kappa u(x_1,1-x_2) + u(x_1+1,x_2) - (2+\kappa)u(x), \quad x \in X.$$

Let $\tau \geqslant 1$ be the largest solution to the equation $\tau^2 - 2(\kappa+1)\tau + 1 = 0$ and put $\gamma = \log\tau$. The function

$$U(x) = (2x_2-1)\tau^{x_1} = (2x_2-1)e^{\gamma x_1}, \qquad x \in X, \tag{2.8}$$

is harmonic.

By combining U and its reflection in the x_1 variable and then performing a translation we get further examples for all choices of real constants a and C,

$$U_0(x) = (2x_2-1)\cosh\gamma(x_1-a) + C(x_1-a), \qquad x \in X, \tag{2.9}$$

and

$$U_1(x) = (2x_2 - 1)\sinh\gamma(x_1 - a) + C(x_1 - a), \qquad x \in X, \tag{2.10}$$

with odd symmetry in the line $x_1 = a$. This is an infinitely long finger, and the solution will serve as a building block for the case of a finitely long finger.

3. The Dirichlet problem on finite sets

Given a weight function λ on a set X we define the *boundary* of X, denoted by ∂X, as the set of all points $x \in X$ such that $\lambda(x, y) = 0$ for all $y \neq x$. Its complement $X^\circ = X \smallsetminus \partial X$ is the *interior* of X. We note that a point x has a neighbor different from x if and only if x is in the interior.

Given a point $a \in X$ we define $N^0(a) = \{a\}$ and then inductively

$$N^{k+1}(a) = \{y; \lambda(x, y) > 0 \text{ for some } x \in N^k(a)\}, \qquad k \in \mathbf{N}.$$

The union of all the $N^k(a)$ will be called the λ-*component* of a and be denoted by $C(a)$. Clearly $C(a) = \{a\}$ if and only if a is a boundary point.

We shall say that X is *boundary connected* if $C(a)$ intersects ∂X for every $a \in X^\circ$. (In particular ∂X is nonempty if X is boundary connected and nonempty.) We shall say that X is *connected* if $C(a) = X$ for all $a \in X^\circ$. (Here ∂X may be empty.)

Proposition 3.1. *If X is finite and boundary connected, then* $\sup_X u = \sup_{\partial X} u$ *for all subharmonic functions u on X. The converse holds.*

Proof. Take a point $a \in X$ such that $u(a) = \sup_X u = A$. Then u must take the value A also at all points y such that $\lambda(a, y) > 0$, in other words in the set $N^1(a)$. Continuing this argument, we see that $u = A$ at all points in $N^k(a)$ and so in their union $C(a)$. By hypothesis $C(a)$ contains a point in ∂X. We are done.

For the converse we note that the characteristic function $\chi_{C(a)}$ is subharmonic in X and if $C(a)$ is contained in X°, then it is zero on the boundary but takes the value one at a.

Example 3.2. If X is boundary connected and u is harmonic and constant on ∂X (in particular if ∂X consists of just one point), then u is constant in all of X. It might be surprising that one point suffices to keep u constant. If $X = [0, m]_{\mathbf{Z}}$ with $\partial X = \{0\}$ and $\lambda(x, y) = 1$ when $x \in X^\circ$ and $y \in X$ with $y = x \pm 1$ and zero otherwise, then this result applies. (The harmonicity at $x = m$ forces $u(m-1)$ to be equal to $u(m)$.) If on the other hand X is the infinite interval $[0, +\infty[_{\mathbf{Z}}$, then all functions $u(x) = ax + b$ are harmonic.

Proposition 3.3. *If X is connected (finite or infinite), then a subharmonic function which attains its supremum at an interior point must be constant. The converse holds.*

Proof. Assume that there exists a point $a \in X$ such that $u(a) = \sup_X u = A$. Then, just as in the previous proof, u must take the value A also at all points in $C(a)$. But by assumption this is all of X.

For the converse we note again that $\chi_{C(a)}$ is subharmonic. If $C(a) \neq X$ it is not constant.

We shall now take a look at the Dirichlet problem in finite sets X. This is about finding u such that $\Delta u = f$ in X and $u = g$ on ∂X for given functions f and g. Since a function is always harmonic at a boundary point, we must have $f = 0$ on ∂X; equivalently, we let f be given in X° and require only that $\Delta u = f$ in the interior X°.

From linear algebra we know that a system of equations $\Delta u(x) = f(x)$ has a solution if f satisfies as many independent linear conditions as there are independent functions ρ satisfying

$$\sum_{x \in X} \rho(x)\big(\lambda(x,y) - \delta(x,y)\big) = 0, \qquad y \in X. \tag{3.1}$$

More precisely, if (3.1) holds, then we must have $\sum_x \rho(x)f(x) = 0$. (We assume λ to be normalized here.) We can always take $\rho(x) = \delta(a,x)$ if a is a boundary point. This gives the condition that $f(a)$ must vanish as we already noted. But there may be other such functions ρ. For instance if λ is symmetric, $\lambda(x,y) = \lambda(y,x)$, then we may take ρ equal to 1 identically, and we conclude that the equation $\Delta u = f$ can be solved only if $\sum_{x\in X} f(x) = 0$. In this case the set X is not boundary connected: $C(a)$ is contained in X° for all $a \in X^\circ$.

Even though the equation $\Delta u(x) = f$ is a finite system of linear equations it is convenient to express conditions for solvability in terms of subsolutions, just as in the real case. The resulting theorem is more general than those of Phillips & Wiener (1923) and Blanc (1939).

Theorem 3.4. *Let X be a finite set and λ a weight function on X. Assume that X is boundary connected. Let two functions be given: $f \geqslant 0$ in X° and g on ∂X. Assume also that the Dirichlet problem has a subsolution, i.e., that there exists a function w such that $\Delta w \geqslant f$ in X° and $w \leqslant g$ on ∂X. Then the Dirichlet problem $\Delta u = f$ in X°, $u = g$ on ∂X, has a unique solution.*

We note that if f is identically zero, then there is always a subsolution: we may take w as a constant not exceeding $\inf_{x\in\partial X} g(x)$. Also, if X is a subset of $\mathbf{Z}^2$ and we define $\lambda(x,y) = 1$ when $\|y-x\|_1 = 1$ and the five points y with $\|y-x\|_1 \leqslant 1$ all lie in X, and $\lambda(x,y) = \delta(x,y)$ otherwise, then there is always a subsolution: we may take $w(x) = c_1\|x\|_2^2 - c_2$ for sufficiently large constants c_1 and c_2.

As already noted before the statement of the theorem, there are cases when there is no solution (hence no subsolution).

Proposition 3.5. *Under the hypotheses in Theorem 3.4, the unique solution to the Dirichlet problem is given as the supremum of all subsolutions: define*

$$u(x) = \sup_w \big(w(x); \Delta w \geqslant f \text{ in } X^\circ \text{ and } w \leqslant g \text{ on } \partial X\big).$$

Then u solves the Dirichlet problem: $\Delta u = f$ in X°, $u = g$ on ∂X.

Proof. We define an operator T by the equations $T(w) = w + \Delta w - f$ in X°, $T(w) = g$ on ∂X. We assume here that λ is normalized, i.e., that $\sum_y \lambda(x, y) = 1$, which is no restriction. If w is a subsolution, then $T(w) \geqslant w$. Then also $T(w)$ is a subsolution in view of the fact that $I + \Delta$ is increasing: $(I + \Delta)T(w) \geqslant (I+\Delta)w = T(w)+f$, which implies that $\Delta T(w) \geqslant f$. The boundary condition is of course satisfied.

Let now u be the supremum of all subsolutions. Since there exists at least one subsolution, $u > -\infty$. On the other hand $u < +\infty$, for a subsolution can never exceed $\sup_{x\in\partial X} g(x)$ in view of Proposition 3.1. Thus u has real values. We claim that u itself is a subsolution. Let w be any subsolution. Then $u \geqslant w$ and, since $I + \Delta$ is increasing, also $(I + \Delta)u \geqslant (I + \Delta)w \geqslant w + f$. Taking the supremum over all w we get $(I + \Delta)u \geqslant \sup_w w + f = u + f$, which shows that u is a subsolution.

Now also $T(u) = u + \Delta u - f$ is a subsolution; hence $T(u) \leqslant u$. But we already proved that $T(u) \geqslant u$. Therefore $T(u) = u$, which implies that $\Delta u = f$.

Proof of Theorem 3.4. Only uniqueness remains to be considered. Let us assume that we have two solutions u and v. Then their difference $u - v$ is harmonic and Proposition 3.1 shows that $\sup_X(u - v) = \sup_{\partial X}(u - v)$. But $u - v = g - g = 0$ on the boundary. Hence $u - v \leqslant 0$ in X; on interchanging the roles of u and v we obtain $v - u \leqslant 0$ in X.

Proposition 3.6. *There is a comparison principle: if X is finite and boundary connected, and if u and v satisfy $\Delta u \geqslant \Delta v$ in X° and $u \leqslant v$ on ∂X, then $u \leqslant v$ in all of X.*

Proof. The function $w = u - v$ is subharmonic and satisfies $w \leqslant 0$ on the boundary. Hence also in the interior by Proposition 3.1.

Remark 3.7. We can get the solution as the limit of a sequence. We may start with $u_0 = w$, any subsolution, and then define $u_{j+1} = T(u_j)$, $j \in \mathbf{N}$. The sequence (u_j) is increasing as we have seen, and all the u_j are subsolutions. It is bounded from above, for in view of Proposition 3.1 it can never exceed $\sup_{x\in\partial X} g(x)$. Hence the sequence admits a limit u. If we pass to the limit in the definition $u_{j+1} = u_j + \Delta u_j - f$ we obtain $\Delta u = f$. The boundary condition

$u = g$ on ∂X is preserved. The convergence may be very slow, however. Better start with a very good approximation.

Example 3.8. Let X be the cube $\{0,1\}^n$ in $\mathbf{Z}^n$, and let the boundary consist of the two points 0 and $(1,1,\dots,1)$. Let the weight function be $\lambda(x,y) = 1$ if x is an interior point and $\|y-x\|_1 = 1$, and zero otherwise. Then the solution to the Dirichlet problem with zero Laplacian and boundary values $u(0) = 0$, $u(1,1,\dots,1) = 1$ is

$$u(x) = C\sum_{j=0}^{k-1}\binom{n-1}{j}^{-1} \text{ where } k = \sum_{j=1}^{n} x_j \text{ and } C = \left[\sum_{j=0}^{n-1}\binom{n-1}{j}^{-1}\right]^{-1}.$$

Here

$$\binom{n-1}{j}^{-1} = \frac{j!(n-1-j)!}{(n-1)!}$$

is the inverse of the binomial coefficient. For example, in the six-dimensional cube we have $u(0,0,0,0,0,1) = \frac{5}{13}$ and $u(0,0,0,0,1,1) = \frac{6}{13}$, the other values being easily obtained from these two and the symmetry.

Example 3.9. Let us look at a narrow rectangle in $\mathbf{Z}^2$ with boundary at two vertices: $X_m = \{x \in \mathbf{Z}^2; x_1 = 0,\dots,m,\ x_2 = 0,1\}$. Let the boundary be $\partial X_m = \{(0,0),(m,0)\}$ and let the weight function be $\lambda(x,y) = \delta(x,y)$ if $x \in \partial X_m$; $\lambda(x,y) = 1$ if $y = (x_1 \pm 1, x_2)$ and $x,y \in X_m$; and $\lambda(x,y) = \kappa$ when $y = (x_1, 1-x_2)$, $x_1 = 1,\dots,m-1$ or $(x,y) = ((0,1),(0,0))$ or $(x,y) = ((m,1),(m,0))$. Then the function U_1 defined by (2.10) is harmonic at all points in X_m except at the two vertices $(0,1)$ and $(m,1)$ where we have changed the weight function compared with the infinite strip in Example 2.7. This is true for all choices of the constants a and C. We now choose $a = \frac{1}{2}m$ to get odd symmetry in the line $x_1 = \frac{1}{2}m$. If $\kappa > 0$, there is a unique constant C such that U_1 is harmonic at $(0,1)$ (and by the symmetry also at $(m,1)$). This value of C is

$$C = -\tfrac{1}{2}(\tau-1)\big(\tau^{m/2} + \tau^{-m/2-1}\big) = -2\sinh(\tfrac{1}{2}\gamma)\cosh\tfrac{1}{2}\gamma(m+1),$$

and the boundary values are

$$U_1((0,0)) = -U_1((m,0)) = \sinh\tfrac{1}{2}\gamma m + m\sinh(\tfrac{1}{2}\gamma)\cosh\tfrac{1}{2}\gamma(m+1),$$

where τ and $\gamma = \log\tau$ are defined as in Example 2.7. After a normalization we get the solution $u = U_1/U_1((m,0))$ to the Dirichlet problem $\Delta u = 0$ in X_m, $u = -1, 1$ at the two boundary points $(0,0)$ and $(m,0)$ respectively. How much does it deviate from the affine function $v(x) = -1 + 2x_1/m$, which is harmonic at all points $x \in X_m$ except at the two vertices $(0,1)$ and $(m,1)$?

4. Clustering of values

We shall now study a phenomenon which can be expressed as clustering of values of harmonic functions.

On a long narrow object, clustering can be severe as shown by the following example. But by relaxing the coupling between strategically chosen parts of the structure, the clusters can be resolved.

Example 4.1. Let us consider $X_m = \{x \in \mathbf{Z}^2; 0 \leqslant x_1 \leqslant m,\ 0 \leqslant x_2 \leqslant 1\}$, but now with the boundary $\partial X = \{(0,0),(0,1)\}$. We prescribe the boundary values $g(0,0) = -1$, $g(0,1) = 1$. The weight function shall be $\lambda(x,y) = 1$ if x is an interior point and $y = (x_1 \pm 1, x_2) \in X_m$; $\lambda(x,y) = \kappa \geqslant 0$ if x is an interior point and $y = (x_1, 1 - x_2)$; otherwise $\lambda(x,y) = 0$. We would like to illustrate what happens if we vary the weight by varying the coupling constant κ. When $\kappa = 0$ there is no coupling between the points $(x_1, 0)$ and $(x_1, 1)$ and the solution to the Dirichlet problem is $u(x) = 2x_2 - 1$.

In view of the symmetry, the solution to the Dirichlet problem with zero Laplacian must satisfy $u(x_1, 0) = -u(x_1, 1)$. We define u as AU_0 in (2.9), where $C = 0$ and where a and A are constants to be determined. This function is harmonic everywhere in X_m except possibly when $x_1 = m$. We determine a so that it becomes harmonic also at the two points $(m, 0)$, $(m, 1)$; this happens if and only if $a = m + \frac{1}{2}$. To give the function the value 1 at $(0, 1)$ we define

$$A = \frac{1}{U_0((0,1))} = \frac{1}{\cosh \gamma(m + \frac{1}{2})} = \frac{2\tau^{-m-1/2}}{1 + \tau^{-2m-1}}.$$

The function $v(x) = (2x_2 - 1)\tau^{-x_1}$ is harmonic in the infinite set $X = \mathbf{N} \times \{0, 1\}$. Its restriction to $0 \leqslant x_1 \leqslant m$ is close to being harmonic in X. In fact, it is harmonic at every point $x \in X$ with $x_1 < m$; at the point $(m, 1)$ it is subharmonic, and at the point $(m, 0)$ superharmonic:

$$\begin{aligned}\Delta v(m,1) &= -\Delta v(m,0) \\ &= v(m-1,1) + \kappa v(m,0) - (1+\kappa)v(m,1) = (1 - 1/\tau)\tau^{-m},\end{aligned}$$

which is a small number for large m. So it is natural to guess that v is a good approximation of u.[1] In fact,

$$u(j,1) = a_j = \tau^{-j} + \frac{\tau^{-2m-1}(\tau^j - \tau^{-j})}{1 + \tau^{-2m-1}}.$$

The relative deviation of the true solution u from the comparison function v is

[1] One could perhaps believe that it would be enough to change v at the points $(m, 1)$ and $(m, 0)$ so that it becomes subharmonic or superharmonic in all of X (to be able to use the comparison principle). But this is not possible! One must change the value at several points.

$$\frac{u(j,1)-v(j,1)}{v(j,1)} = \frac{a_j - \tau^{-j}}{\tau^{-j}} = \frac{\tau^{-2m-1}(\tau^{2j}-1)}{1+\tau^{-2m-1}} \leqslant \tau^{2(j-m)-1} \leqslant \frac{1}{\tau}.$$

It is clear that the solution u to the Dirichlet problem decays exponentially and that the values lie extremely close to each other if for instance $\kappa = 1$ and $m = 100$. Then $\tau = 2+\sqrt{3} \approx 3.73$ and $\tau^{-100} \approx 6 \cdot 10^{-58}$. If κ is small, the decay is slower: $\kappa = .01$ yields $\tau \approx 1.15$ and $\tau^{-100} \approx 7 \cdot 10^{-7}$. In this simple example it is thus possible to dissolve the clusters by relaxing the coupling.

Example 4.2. Let us look at an example in three dimensions. Let Y_m be the set of points $x = (x_1, x_2, x_3)$ in $\mathbf{Z}^3$ satisfying $0 \leqslant x_1 \leqslant m$, $0 \leqslant x_2, x_3 \leqslant 1$. Its boundary shall be $\{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\}$, the four points in Y_m with $x_1 = 0$. For an interior point x we take $\lambda(x,y) = 1$ if $y \in Y_m$, $\|y-x\|_1 = 1$, $y_1 = x_1$; we take $\lambda(x,y) = \kappa$ if $y \in Y_m$, $\|y-x\|_1 = 1$, $y_2 = 1-x_2$; and zero otherwise. We define boundary values $g(0,0,0) = g(0,0,1) = -1$, $g(0,1,0) = g(0,1,1) = 1$. Then the solution to the Dirichlet problem with zero Laplacian satisfies $-u(x_1,0,x_3) = u(x_1,1,x_3) = a_{x_1}$, where the sequence (a_j) is the same as in the previous example. Thus the phenomenon of rapid exponential decay occurs also in three dimensions when there is a narrow finger; in this case consisting of four voxels at each level $x_1 =$ constant. Of course this finger can be a part of a larger set provided the latter is symmetric around the plane $x_2 = \frac{1}{2}$.

5. The Dirichlet problem on infinite sets

Let us take a look at a Dirichlet problem on infinite sets.

Theorem 5.1. *Let X be a finite or infinite set and λ a weight function on X. We shall consider a Dirichlet problem,*

$$\Delta u = f \text{ in } X^\circ, \quad u = g \text{ on } Y,$$

where Y is a subset of X, not necessarily contained in the boundary, and where f and g have real values. Assume that X can be written as a disjoint union $X = \bigcup_0^N X_j$ or $X = \bigcup_0^\infty X_j$ with $X_0 \subset Y$. In the finite case we assume that X_N is contained in the boundary. Assume moreover that for every $y \in X_{j+1}$ either y belongs to Y or there is a unique $x \in X_j$ such that $\lambda(x,y) > 0$, that

$$\{z; \lambda(x,z) > 0\} \subset X_0 \cup \cdots \cup X_j \cup \{y\},$$

and that every $x \in X_j$ occurs in this way. (There is thus a bijection between $X_{j+1} \smallsetminus Y$ and X_j.) Then the Dirichlet problem has a unique solution.

Proof. We define first $u = g$ in X_0. Then, if u has already been defined in

$$X_0 \cup \cdots \cup X_j$$

satisfying $\Delta u = f$ in $X_0 \cup \cdots \cup X_{j-1}$ (the empty set if $j = 0$) and $u = g$ in $Y \cap (X_0 \cup \cdots \cup X_j)$, we define u at a point $y \in X_{j+1}$ so that it satisfies $\Delta u = f$ at every point in X_j. Indeed, if $y \in Y$ we just define $u(y) = g(y)$; if not, we need to solve the equation

$$\lambda(x, y)(u(y) - u(x)) + \sum_{z \neq y} \lambda(x, z)(u(z) - u(x)) = f(x),$$

where x is uniquely determined from y. The equation has only one unknown $u(y)$; the coefficient in front of it is nonzero by hypothesis and all the z that occur in the sum belong to $X_0 \cup \cdots \cup X_j$. We do this for every $y \in X_{j+1}$. The extended function is now defined in $X_0 \cup \cdots \cup X_{j+1}$ and satisfies $\Delta u = f$ in $X_0 \cup \cdots \cup X_j$, so the induction step is completed, and the induction goes on in the infinite case. In the finite case it stops, and we note that we need not require that u be harmonic in X_N since this is automatic in view of our assumption that X_N is contained in the boundary.

Example 5.2. Let $X = \{x \in \mathbf{Z}^2; x_2 \geqslant 0\}$ with $\lambda(x, y) = 1$ if $\|y - x\|_1 = 1$, $x, y \in X$, and zero otherwise. Let $Y = \{x \in X; x_2 = 0\}$. (Note that Y is not contained in the boundary; the latter is in fact empty.) Here we can define $X_j = \{x \in X; x_2 = j\}$, $j \in \mathbf{N}$. The theorem shows that the Dirichlet problem $\Delta u = f$ in X, $u = g$ on Y has a unique solution. This problem is equivalent to constructing a solution (with another weight function) in $\mathbf{Z}^2$ which is symmetric in the second variable.

Example 5.3. Let $X = \{x \in \mathbf{Z}^2; x_1 \geqslant |x_2|\}$ with $\lambda(x, y) = 1$ if $\|y - x\|_1 = 1$, $x, y \in X$, and zero otherwise. Let $Y = \{x \in X; x_1 = |x_2|\}$. (The boundary is empty also in this case.) We define now $X_j = \{x \in X; x_1 = j\}$. The theorem says that there is a unique solution to the Dirichlet problem. This problem is equivalent to solving the Dirichlet problem with a function u which is symmetric in the sense that $u(x_1, x_2) = u(x_2, x_1)$ and $u(x_1, x_2) = u(-x_2, -x_1)$ and prescribed on the diagonals.

6. The Poisson kernel

The Poisson kernel in two real variables is

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, \qquad (x, y) \in \mathbf{R} \times \mathbf{R}^+.$$

It satisfies the convolution equations

$$P_y * P_{y'} = P_{y+y'}, \qquad y, y' \in \mathbf{R}^+,$$

where the convolution, denoted by $*$, is in the x-variable only. Its Fourier transform with respect to x is

$$\widehat{P_y}(\xi) = \int_{\mathbf{R}} P_y(x) e^{-i\xi x} dx = e^{-y|\xi|}, \qquad y \in \mathbf{R}^+, \ \xi \in \mathbf{R},$$

from which the convolution property is also evident; $\widehat{P_y} = \widehat{P_1}^y$ for all $y > 0$. If we define P_0 as the Dirac measure at the origin, then the convolution equation extends to all $y \geqslant 0$.

The Poisson kernel in $\mathbf{Z}^2$ shall be defined in a similar way. It is a function $Q_y(x)$ of $(x, y) \in \mathbf{Z} \times \mathbf{N}$ which is $\mathbf{Z}^2$-harmonic at every point (x, y) with $y \geqslant 1$ and satisfies $Q_y * Q_{y'} = Q_{y+y'}$. Its Fourier transform

$$\widehat{Q_y}(\xi) = \sum_{x \in \mathbf{Z}} Q_y(x) e^{-ix\xi}, \qquad \xi \in \mathbf{R},$$

satisfies $\widehat{Q_y} = \widehat{Q_1}^y$, $y \in \mathbf{N}$.

If we use the harmonicity of Q at points (x, y) with $y = 1$ and the condition that $Q_0 = Q_1^0 = \delta$, $Q_2 = Q_1 * Q_1$, then $\widehat{Q_1}$ must satisfy an equation of the second degree,

$$\widehat{Q_1}(\xi)^2 + e^{-i\xi}\widehat{Q_1}(\xi) + e^{i\xi}\widehat{Q_1}(\xi) + 1 = 4\widehat{Q_1}(\xi),$$

so that

$$\widehat{Q_1}(\xi) = 2 - \cos\xi \pm \sqrt{(2 - \cos\xi)^2 - 1}, \qquad \xi \in \mathbf{R},$$

where we may choose the sign $\pm$ for each $\xi \in \mathbf{R}$ in a measurable way. If we always choose the negative sign, we get a decreasing function of y,

$$\widehat{Q_y}(\xi) = \left(2 - \cos\xi - \sqrt{(2 - \cos\xi)^2 - 1}\right)^y, \qquad y \in \mathbf{N}, \quad \xi \in \mathbf{R}.$$

We can state that Q_y is the inverse Fourier transform of $\widehat{Q_y} = \widehat{Q_1}^y$:

$$Q_y(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{Q_1}(\xi)^y e^{i\xi x} d\xi, \qquad x \in \mathbf{Z}, \quad y \in \mathbf{N}.$$

So there is an explicit formula for $\widehat{Q_y}$ but hardly for Q_y. However, $Q_y(x)$ is very close to $P_y(x)$.

7. Fundamental solutions

In two real variables a well-known fundamental solution for the Laplacian is

$$E(x, y) = \frac{1}{4\pi} \log(x^2 + y^2), \qquad (x, y) \in \mathbf{R}^2.$$

We would like to determine a fundamental solution of the $\mathbf{Z}^2$-Laplacian.

Lemma 7.1. *Let a be a nonnegative number and define*

$$E_a(x, y) = \log(a + x^2 + y^2) = \log(a + r^2), \qquad (x, y) \in \mathbf{Z}^2,$$

where $r = \sqrt{x^2 + y^2}$. It is subharmonic in all of $\mathbf{Z}^2$ if and only if $a \geqslant \frac{1}{2}$.

Proof. We find that $\Delta E_a = \log \frac{N}{D}$, where

$$N = r^8 + 4ar^6 + (6a^2 + 4a - 2)r^4 + (4a^3 + 8a^2 + 4a)r^2 + (a+1)^4 + 16x^2y^2$$

and

$$D = (a + r^2)^4 = r^8 + 4ar^6 + 6a^2r^4 + 4a^3r^2 + a^4.$$

Now this function is nonnegative if and only if $N/D \geqslant 1$; equivalently $N - D \geqslant 0$. We get

$$N - D = (4a - 2)r^4 + (8a^2 + 4a)r^2 + 4a^3 + 6a^2 + 4a + 1 + 16x^2y^2,$$

from which the result is obvious.

We see that for all a, the function is very close to being harmonic far away from the origin: $\Delta E_a = O(r^{-4})$ as $r \to +\infty$. When $a = 0$ the function is superharmonic in sectors of openings 45° around the x-axis and y-axis far away from the origin, and subharmonic in sectors of openings 45° around the diagonals far away from the origin.

The fundamental solution E satisfies the differential equation $\Delta E = \delta$. On taking the Fourier transform in the Schwartz space $\mathscr{S}'(\mathbf{R}^2)$ we get

$$-(\xi^2 + \eta^2)\widehat{E}(\xi, \eta) = 1,$$

so that $\widehat{E}$ is a tempered distribution defined in $\mathbf{R}^2$ as an extension of the homogeneous function $-(\xi^2 + \eta^2)^{-1}$ defined in $\mathbf{R}^2 \smallsetminus \{0\}$. One such extension is the pseudofunction $-\mathsf{pf}(\xi^2 + \eta^2)^{-1}$ defined as the finite part of an integral.

Similarly, a fundamental solution for the $\mathbf{Z}^2$-Laplacian shall satisfy $\Delta_{\mathbf{Z}^2} F = \delta$, which implies that its Fourier transform, defined as a distribution in $\mathbf{R}^2$, or rather in $(\mathbf{R} \bmod 2\pi)^2$, shall satisfy

$$(2\cos\xi + 2\cos\eta - 4)\widehat{F}(\xi, \eta) = 1.$$

The inverse of the factor in front of $\widehat{F}$ has the same kind of singularity at the origin as $-(\xi^2 + \eta^2)^{-1}$. Therefore $\widehat{F}$ can be defined by the finite part of an integral of $(2\cos\xi + 2\cos\eta - 4)^{-1}$. The fundamental solution F is its inverse Fourier transform. Can one find an explicit formula for F?

For the corresponding problem in $\mathbf{Z}^3$, Duffin (1953:239) determined the first terms in the asymptotic development as $\|x\|_2 \to +\infty$ to be

$$F(x) = \frac{1}{4\pi\|x\|_2} + \frac{1}{32\pi\|x\|_2^3}\left[-3 + \frac{5(x_1^4 + x_2^4 + x_3^4)}{\|x\|_2^4}\right] + O\big(\|x\|_2^{-5}\big), \quad x \in \mathbf{Z}^3.$$

Burkhardt (1997:1159) proved that there exist asymptotic developments to any order and gave formulas that allow us to calculate them; he gave explicit formulas for the terms of order -5 and -7.

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