Characterizing digital straightness by means of difference operators

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***Abstract*—We characterize straightness in the digital plane using difference operators.**

I. Introduction

Digital straightness will be characterized here using methods from Cartesian geometry as well as from word combinatorics, Diophantine inequalities, and the calculus of difference operators. While the first three methods are not new, the use of difference operators seems to be so. Briefly, we can state the purpose of the article as finding discrete analogues of the differential equation $F'' = 0$, which characterizes straight lines in the Euclidean plane.

We shall see that we can characterize refined digital lines (equivalently: balanced binary words) with the help of difference operators. The chord property of Rosenfeld (1974) will also be studied and it is shown that it can be characterized using difference operators.

The study of these discrete analogues of the differential equation $F'' = 0$ in the function space $\mathbf{Z}^{\mathbf{Z}}$ is equivalent to the study of straight lines in the digital plane $\mathbf{Z}^2$, and therefore also to the theory of balanced words from an alphabet of two letters. This theory is highly developed; see, e.g., Morse & Hedlund (1940), Hung & Kasvand (1984), Rosenfeld & Klette (2001), Lothaire (2002), Pytheas Fogg (2002), Vuillon (2003), Klette & Rosenfeld (2004), Samieinia (2007), Uscka-Wehlou (2009), Berthé (2009; with 94 references), Samieinia (forthc.), Bédaride et al. (forthc.). Nevertheless, the analogy with $F'' = 0$ may lead to a new, more numerical aspect of the theory, and certain results, like Theorem 5.3 on the extension of rectilinear segments, receive easy proofs. Viewed as a problem in combinatorics, this theorem says that a balanced finite binary word can be extended to a periodic balanced infinite word, moreover to infinitely many words with different periods, and with control over the periods obtained—and also to infinitely many balanced nonperiodic infinite words.

II. Difference operators

Definition 2.1: Given any $a \in \mathbf{R}$ we define a difference operator $D_a\colon \mathbf{R}^{\mathbf{R}} \to \mathbf{R}^{\mathbf{R}}$ by

$$\begin{aligned}(D_aF)(x) &= F(x+a) - F(x),\\ &\qquad x \in \mathbf{R},\ a \in \mathbf{R},\ F \in \mathbf{R}^{\mathbf{R}}.\end{aligned} \tag{1}$$

If $a \in \mathbf{N}$, D_a operates also from $\mathbf{R}^{\mathbf{Z}}$ to $\mathbf{R}^{\mathbf{Z}}$ and from $\mathbf{Z}^{\mathbf{Z}}$ to $\mathbf{Z}^{\mathbf{Z}}$; we shall use the same symbol for its restrictions to $\mathbf{R}^{\mathbf{Z}}$ and $\mathbf{Z}^{\mathbf{Z}}$.

We combine two of these operators to obtain the *Jensen operator* $J_{a,b}$,

$$\begin{aligned}(J_{a,b}F)(x) &= \tfrac{a}{a+b}D_bF(x+a) - \tfrac{b}{a+b}D_aF(x)\\ &= \tfrac{b}{a+b}F(x) - F(x+a) + \tfrac{a}{a+b}F(x+a+b),\\ &\qquad x \in \mathbf{R},\ a,b > 0.\end{aligned} \tag{2}$$

A function $F \in \mathbf{R}^{\mathbf{R}}$ is convex if and only if $J_{a,b}F \geq 0$ for all positive real numbers a, b.

Another second-order difference operator is D_bD_a, given by

$$(D_bD_aF)(x) = F(x+a+b) - F(x+b) - F(x+a) + F(x). \tag{3}$$

It is well known that a continuous function $F\colon \mathbf{R} \to \mathbf{R}$ is convex if and only if $D_aD_aF \geq 0$ for all real $a > 0$; equivalently $D_bD_aF \geq 0$ for all $a, b > 0$. We note that

$$D_bD_a = J_{a,b} + J_{b,a}. \tag{4}$$

For functions in $\mathbf{R}^{\mathbf{Z}}$, the condition $D_1D_1f = 0$ gives easy and satisfying results. For functions in $\mathbf{Z}^{\mathbf{Z}}$, on the other hand, this condition yields a very narrow class of functions. But if we relax it to $|D_1D_1f| \leq 1$, we get a class of functions which is much too wide to be of interest. It turns out, perhaps surprisingly, that a simple compromise, intermediate between the two conditions, viz.

$$|D_bD_af| \leq 1, \qquad a, b \in \dot{\mathbf{N}} = \mathbf{N} \setminus \{0\},$$

yields a class with good properties. These inequalities are equivalent to $|J_{a,b}f| < 1$ for all $a, b \in \dot{\mathbf{N}}$.

III. Defining convexity

It is most convenient to define convex functions with the help of convex sets. This also has the advantage that we can treat functions with infinite values without difficulty.

A. Basic definitions

A subset A of $\mathbf{R}^n$ is said to be *convex* if

$$\{a, b\} \subset A \text{ implies } [a, b] \subset A, \tag{5}$$

where

$$[a, b] = \{(1-t)a + tb;\ t \in \mathbf{R},\ 0 \leq t \leq 1\}$$

is the *segment* with a and b as endpoints. A segment $[a, b]$ with endpoints a, b in A will be called a *chord*, and we define the *chord set of* A as

$$\operatorname{chord}(A) = \bigcup_{a,b \in A} [a, b] \subset \mathbf{R}^n.$$

Thus a set is convex if and only if

$$\mathrm{chord}(A) \subset A. \tag{6}$$

The smallest convex set containing a set A is called its *convex hull* and will be denoted by $\mathrm{cvx}(A)$; it is well defined since any intersection of convex sets is convex.

The operation cvx is increasing, idempotent, and extensive, in other words, a cleistomorphism (closure operator) in the complete lattice of all subsets of $\mathbf{R}^n$. The operation chord, on the other hand, is increasing and extensive, but not idempotent in dimension $n \geq 2$.

Since the chord property of Euclid (6) is unreasonable in a digital setting, it has been weakened by Azriel Rosenfeld in a sense which turned out to be successful: We shall say that a set $A \subset \mathbf{R}^2$ has the *chord property in the sense of Rosenfeld* (1974) if

$$\mathrm{chord}(A) \subset A + U, \tag{7}$$

where U is the open unit ball in $\mathbf{R}^2$ for the l^∞ norm $\|x\|_\infty = \max(|x_1|, |x_2|)$,

$$U = \{x \in \mathbf{R}^2;\ \|x\|_\infty < 1\}.$$

B. Discrete convexity

We shall now generalize the notion of convexity as follows.

Definition 3.1: Given a subset W of $\mathbf{R}^n$ we shall say that a subset A of W is *W-convex* (or just *convex* if W is understood) if there exists a convex subset C of $\mathbf{R}^n$ such that $A = C \cap W$.

When $W = \mathbf{R}^n$ we get usual convexity. Of interest in this paper are the cases $W = \mathbf{Z}^n$ and $W = \mathbf{Z}^{n-1} \times \mathbf{R}$.

Since we always have $A \subset \mathrm{cvx}(A) \cap W$, W-convexity of A is equivalent to the inclusion

$$\mathrm{cvx}(A) \cap W \subset A. \tag{8}$$

Kim & Rosenfeld (1982) established a perfect digital analogue in $\mathbf{Z}^2$ of the Euclidean definition of convexity (5): they proved that a subset of $\mathbf{Z}^2$ is $\mathbf{Z}^2$-convex if and only if any two of its points can be connected by a digital straight line segment in the sense of Rosenfeld (1974).

Proposition 3.2: For subsets of $\mathbf{Z}^2$, the chord property (7) in the sense of Rosenfeld implies $\mathbf{Z}^2$-convexity. The converse implication does not hold.

Definition 3.3: Given a subset X of $\mathbf{R}^n$, a subset Y of $\mathbf{R}$, and a subset W of $X \times Y$, we shall say that a function $f\colon X \to Y \cup \{-\infty, +\infty\}$ is *W-convex* (or just *convex* if W is understood) if its *finite epigraph*

$$\mathrm{epi}^{\mathrm{F}}(f) = \{(x, y) \in X \times Y;\ f(x) \leq y\}$$

is a W-convex set in the sense of Definition 3.1.

Thus f is W-convex if and only if $\mathrm{cvx}(\mathrm{epi}^{\mathrm{F}}(f)) \cap W \subset \mathrm{epi}(f)$.

When X is all of $\mathbf{R}^n$, Y is all of $\mathbf{R}$, and $W = \mathbf{R}^n \times \mathbf{R}$, thus for $(\mathbf{R}^n \times \mathbf{R})$-convexity, we get usual convexity for functions $F \in (\mathbf{R} \cup \{-\infty, +\infty\})^{\mathbf{R}^n}$.

IV. Characterizations of Straightness

A. Rosenfeld: the chord property

In order to characterize straightness of finite subsets of $\mathbf{Z}^2$, Azriel Rosenfeld (1974) introduced the chord property already mentioned in (7).

We may define the P-digitization of a subset M of $\mathbf{R}^n$ as the set

$$\mathrm{dig}_P(M) = (M + P) \cap \mathbf{Z}^n, \qquad M \subset \mathbf{R}^n.$$

Here P is a pixel or voxel located at the origin—it may in fact be any subset of $\mathbf{R}^n$. The role of P is to fatten M before intersecting it with the grid $\mathbf{Z}^n$.

Rosenfeld took P as the cross

$$R = \left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \{0\}\right) \cup \left(\{0\} \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right[\right) \subset \mathbf{R}^2.$$

Then the straight line L in $\mathbf{R}^2$ defined by an equation $x_2 = F(x_1) = \alpha x_1 + \beta$ with $|\alpha| < 1$ gives rise to a function $f\colon \mathbf{Z} \to \mathbf{Z}$. Indeed, for such a line, given $z_1 \in \mathbf{Z}$, there is one and only one z_2 such that (z_1, z_2) belongs to $L + R$. Actually, $f(z_1) = \lceil \alpha z_1 + \beta - \frac{1}{2} \rceil$, so that this digitization of the real line with equation $x_2 = F(x_1)$ has the equation $z_2 = \lceil \alpha z_1 + \beta - \frac{1}{2} \rceil$.

Rosenfeld proved that a finite digital arc, in particular the graph of a function $f\colon [c, d]_{\mathbf{Z}} \to \mathbf{Z}$ with $|D_1 f| \leq 1$, has the chord property if and only if $A = \mathrm{dig}_R(L)$ for some rectilinear segment $L = [p, q]$ in $\mathbf{R}^2$.

Theorem 4.1: Let $f\colon \mathbf{Z} \to \mathbf{Z}$ be a function with integer values. Then its graph has the chord property if and only if $|D_1 f(x)| \leq 1$ and $|J_{a,b} f(x)| < 1$ for all $(x, a, b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$. The corresponding result holds also for a function defined on an interval $[c, d]_{\mathbf{Z}}$ or $[c, +\infty[_{\mathbf{Z}}$ or $]-\infty, d]_{\mathbf{Z}}$ of $\mathbf{Z}$.

We shall also establish a partial converse to Proposition 3.2:

Theorem 4.2: The graph of an integer-valued function defined on an interval of $\mathbf{Z}$ and satisfying $|D_1 f| \leq 1$ is $\mathbf{Z}^2$-convex if and only if it possesses the chord property.

B. Characterizations by means of balanced words

Theorem 4.3: A function $f \in \mathbf{Z}^{\mathbf{Z}}$ with $0 \leq D_1 f \leq 1$ satisfies

$$|D_b D_a f(x)| \leq 1, \qquad x \in \mathbf{Z}, \quad a, b \in \dot{\mathbf{N}}, \tag{9}$$

if and only if the binary word $D_1 f$ is balanced.

For the proof we recall some notions from word theory. By a *word* we understand here a doubly infinite sequence $(w_j)_{j \in \mathbf{Z}}$ of letters w_j; it is *binary* if there are only two letters; we shall then take them as 0 and 1.

A factor $w' = (w_j)_{j=p}^{q}$ of a word w is said to have *length* $q - p + 1$:

$$\mathrm{length}(w') = q - p + 1.$$

If w is binary, the number of ones in a factor $w' = (w_j)_{j=p}^{q}$ is called its *height*:

$$\mathrm{height}(w') = \sum_{j=p}^{q} w_j.$$

A function $f \in \mathbf{Z}^{\mathbf{Z}}$ is said to have *chain code* $c = c(f) = (c_j)_{j\in\mathbf{Z}}$, where

$$c_j = f(j+1) - f(j) = D_1 f(j), \qquad j \in \mathbf{Z}.$$

A binary word w is said to be *balanced* if for any two factors w' and w'' of w we have

$$\begin{aligned}&\text{length}(w') = \text{length}(w'') \text{ implies}\\ &|\text{height}(w') - \text{height}(w'')| \le 1.\end{aligned} \tag{10}$$

Let now $w' = (w_j)_{j=p'}^{q'}$, $w'' = (w_j)_{j=p''}^{q''}$ be two factors of the same binary word w. That they have the same length means that $q' - p' + 1 = q'' - p'' + 1$. Their heights are

$$\text{height}(w') = \sum_{j=p'}^{q'} w_j, \quad \text{height}(w'') = \sum_{j=p''}^{q''} w_j.$$

Now, writing $w_j = D_1 f(j)$, we obtain

$$\text{height}(w') = \sum_{j=p'}^{q'} D_1 f(j) = D_a f(p'),$$

where $a = q' - p' + 1$.
Proof of Theorem 4.3. Given f, let $w' = (w_j)_{j=p'}^{q'}$ and $w'' = (w_j)_{j=p''}^{q''}$ be two factors of the same length of the binary word $w = D_1 f$. For reasons of symmetry we may assume that $p' \le p''$. Define $x = p'$, $a = q' - p' + 1 = q'' - p'' + 1$, the common length of the intervals, and $b = p'' - p' = q'' - q'$, the distance between their left endpoints. Then $x + a = q' + 1$, $x + b = p''$, and $x + a + b = q'' + 1$, so that

$$\begin{aligned}&\text{height}(w'') - \text{height}(w') = D_a f(p'') - D_a f(p')\\ &= D_b D_a f(p') = D_a D_b f(p').\end{aligned}$$

We see that condition (9) translates directly to condition (10).

Thus the equality $D_1 D_1 f = 0$ for functions in $\mathbf{R}^{\mathbf{Z}}$ is replaced by the inequality (9) for functions in $\mathbf{Z}^{\mathbf{Z}}$, which we can understand as a kind of approximate equality.

C. Hyperplanes in the sense of Reveillès

Jean-Pierre Reveillès (1991:45) introduced digital lines in the digital plane as solutions to a double Diophantine inequality: he considered sets of the form

$$\{(x,y) \in \mathbf{Z}^2;\ \gamma \le \alpha x + \beta y < \gamma\,'\}, \tag{11}$$

where α and β are real numbers, not both of them zero, and γ and $\gamma\,'$ are real numbers. We shall refer to such a set as a *digital straight line in the sense of Reveillès*. The definition is easy to generalize to hyperplanes in $\mathbf{Z}^n$.

D. Refined digital hyperplanes

In Kiselman (2004:456) we generalized the notion of digital hyperplanes to the following. Let us denote by $\pi_k\colon \mathbf{Z}^n \to \mathbf{Z}^{n-1}$ the projection which forgets the coordinate x_k, $k = 1, \dots, n$. A set D is a *refined digital hyperplane* if D is $\mathbf{Z}^n$-convex, if the strict slab $\{x \in \mathbf{Z}^n;\ \beta < \alpha \cdot x < \gamma\}$ is contained in D, which in turn is contained in the non-strict slab

$$\{x \in \mathbf{Z}^n;\ \beta \le \alpha \cdot x \le \gamma\}$$

for some choice of $\alpha \in \mathbf{R}^n \setminus \{0\}$ and $\beta, \gamma \in \mathbf{R}$, and if in addition, for some k, the sets

$$\pi_k(D \cap T^0) \text{ and } \pi_k(D \cap T^1)$$

are disjoint and together fill all of $\pi_k(T^0 \cup T^1)$, where $T^0 = \{x \in \mathbf{Z}^n;\ \alpha x = \beta\}$ and $T^1 = \{x \in \mathbf{Z}^n;\ \alpha x = \gamma\}$.

In two dimensions this definition can be expressed in a simple way. We take $n = 2$, $(\alpha_1, \alpha_2) = (-\alpha, 1)$ and define strips in $\mathbf{R}^2$ as follows.

$$\begin{aligned}S(\alpha,\beta,\gamma) &= \{(x,y) \in \mathbf{R}^2;\ \beta \le y - \alpha x \le \gamma\},\\ S^*(\alpha,\beta,\gamma) &= \{(x,y) \in \mathbf{R}^2;\ \beta \le y - \alpha x < \gamma\},\\ S_*(\alpha,\beta,\gamma) &= \{(x,y) \in \mathbf{R}^2;\ \beta < y - \alpha x \le \gamma\},\\ S_*^*(\alpha,\beta,\gamma) &= \{(x,y) \in \mathbf{R}^2;\ \beta < y - \alpha x < \gamma\}.\end{aligned} \tag{12}$$

Then a straight line in $\mathbf{Z}^2$ in the sense of Reveillès is, possibly after a permutation of the coordinates, equal to the intersection $S^*(\alpha,\beta,\gamma) \cap \mathbf{Z}^2$, for some α, β, γ, $|\alpha| \le 1$.

A refined digital line with $|\alpha| \le 1$ and $\gamma = \beta + 1$ is either a digital line in the sense of Reveillès or, possibly after a reflection, of the form

$$\begin{aligned}D(\alpha,\beta,p) = \{(x,y) &\in \mathbf{Z}^2 \cap S^*(\alpha,\beta,\beta+1);\ x < p\}\\ &\cup \{(x,y) \in \mathbf{Z}^2 \cap S_*(\alpha,\beta,\beta+1);\ x \ge p\}\end{aligned}$$

for some $\alpha, \beta \in \mathbf{R}$ and some $p \in \mathbf{Z}$. This is because the only pairs of $\mathbf{Z}$-convex complementary subsets of the digital line are $(\mathbf{Z}, \emptyset)$ and $(]-\infty, p[_{\mathbf{Z}}\,,\ [p, +\infty[_{\mathbf{Z}})$, $p \in \mathbf{Z}$.

Theorem 4.4: Every digital line in the sense of Reveillès is a refined digital line.

Conversely, given $|\alpha| \le 1$ and β real, we consider four cases for the set

$$D = S(\alpha,\beta,\beta+1) \cap \mathbf{Z}^2,$$

defining $D^j = \{(x,y) \in D;\ y - \alpha x = \beta + j\}$, $j = 0, 1$:

(A). *The slope α is rational and $\beta \in \mathbf{Z} + \alpha\mathbf{Z}$. Then D^0 and D^1 contain infinitely many points and D is not a refined digital line. For any integer p, the set $D(\alpha,\beta,p)$, obtained by removing from D certain points in $D^0 \cup D^1$, is a refined digital line. The sets $D \setminus D^0$ and $D \setminus D^1$ are digital lines in the sense of Reveillès.*

(B). *The slope α is rational and $\beta \notin \mathbf{Z} + \alpha\mathbf{Z}$ (for instance when β is irrational). Then D^0 and D^1 are empty, so that $D = S_*^*(\alpha,\beta,\beta+1) \cap \mathbf{Z}^2$ and D is a digital straight line in the sense of Reveillès.*

(C). *The slope α is irrational and D^0 is empty. Then $D = S(\alpha,\beta,\beta+1) \cap \mathbf{Z}^2 = S_*^*(\alpha,\beta,\beta+1) \cap \mathbf{Z}^2$ is a digital straight line in the sense of Reveillès.*

(D). *The slope α is irrational and D^0 is a singleton set. Then D^1 is also a singleton set, and D is not a refined digital line. But $D \setminus D^0$ and $D \setminus D^1$ are digital straight lines in the sense of Reveillès.*

V. Extending rectilinear segments

Let us consider functions defined on an interval: let c and d be two integers and consider functions $f\colon [c,d]_{\mathbf{Z}} \to \mathbf{Z}$. We can then form $D_b D_a f(x)$ only for $c \le x \le d-a-b$, $a,b \in \dot{\mathbf{N}}$. A natural question is whether the conditions $|D_b D_a f(x)| \le 1$ for these finitely many a,b,x are sufficient to ensure that f represents a straight line segment; in other words, whether we can find an extension g to all of $\mathbf{Z}$ of the function f which satisfies the conditions everywhere. The answer is in the affirmative, but the extension is never unique.

Theorem 5.1: If $f\colon [c,d]_{\mathbf{Z}} \to \mathbf{Z}$ satisfies $|D_b D_a f(x)| \le 1$ for all x,a,b for which the expression is defined, then its graph is contained in an open strip $S^*_*(\alpha,\beta,\gamma)$ with rational α and of height $\gamma-\beta<1$. If a function $f\colon \mathbf{Z} \to \mathbf{Z}$ defined on the whole integer axis satisfies $|D_b D_a f| \le 1$, its graph is contained in a closed strip $S(\alpha,\beta,\beta+1)$ of height 1.

Theorem 5.2: If the graph of a function $f\colon \mathbf{Z} \to \mathbf{Z}$ or $f\colon [c,d]_{\mathbf{Z}} \to \mathbf{Z}$ is contained in a half-open strip $S^*(\alpha,\beta,\beta+1)$ or $S_*(\alpha,\beta,\beta+1)$, then $|(D_b D_a f)(x)| \le 1$ for all x and $a,b \in \dot{\mathbf{N}}$ for which the expression is defined.

Theorem 5.3: Let $f\colon [c,d]_{\mathbf{Z}} \to \mathbf{Z}$ be given such that $|D_b D_a f(x)| \le 1$ for all a, b, x for which the expression is defined, i.e., for $c \le x \le d-a-b$, $a,b \in \dot{\mathbf{N}}$. Then f can be extended to a function $g\colon \mathbf{Z} \to \mathbf{Z}$ such that $|D_b D_a g(x)| \le 1$ for all $x \in \mathbf{Z}$ and all $a,b \in \dot{\mathbf{N}}$.

If we look at this as a combinatorial problem for chain codes, i.e., for binary words, the theorem says, in case $0 \le D_1 f \le 1$, that a balanced finite binary word can be extended to a periodic balanced infinite word, moreover to infinitely many words with different periods—and also to infinitely many balanced nonperiodic infinite words.

VI. Digital straightness

Combining what we have learned about digital straightness so far we obtain the following result.

Theorem 6.1: Let $f \in \mathbf{Z}^{\mathbf{Z}}$, assume that $0 \le D_1 f \le 1$, and consider the following properties.

(A). *The graph of f has the chord property;*

(B). *Both f and $-f$ are convex;*

(C). *The graph of f is a $\mathbf{Z}^2$-convex set;*

(D). *The inequality $|(D_b D_a f)(x)| \le 1$ holds for all $(x,a,b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$;*

(E). *The inequality $|(J_{a,b} f)(x)| < 1$ holds for all $(x,a,b) \in \mathbf{Z} \times \dot{\mathbf{N}} \times \dot{\mathbf{N}}$;*

(F). *The binary word $D_1 f\colon \mathbf{Z} \to \mathbf{Z}$ is balanced.*

(G). *The function f defines a refined digital hyperplane in $\mathbf{Z}^2$ in the sense of Kiselman (2004);*

(H). *The function f defines a digital straight line in the sense of Reveillès (1991).*

All conditions (A), (B), (C), (D), (E), (F) *and* (G) *are equivalent, and they are implied by* (H).

Remark 6.2: Some of the equivalences in this theorem have a long history. Morse & Hedlund (1940) proved that Sturmian words (aperiodic words of minimal complexity) are balanced, and conversely. That balance of a binary word is equivalent to the property of being a mechanical word is proved in the case of irrational slope in Lothaire (2002: Theorem 2.1.13).

We also note the following result on locality of the various properties. Let us say that a property of functions $f \in (\mathbf{Z} \cup \{-\infty,+\infty\})^A$, where A is an arbitrary subinterval of $\mathbf{Z}$, is *local* if it is true that it has the property if and only if all its restrictions $f|_{[c,d]_{\mathbf{Z}}}$ to finite intervals $[c,d]_{\mathbf{Z}}$ have the property.

Proposition 6.3: The properties (A), (B), (C), (D), (E), (F) *and* (G) *of Theorem 6.1, understood respectively for functions defined on $\mathbf{Z}$ and on subintervals of $\mathbf{Z}$, are local properties. The property* (H) *is not local.*

VII. Conclusion

We have found a set of difference operators that can be used to give a convenient characterization of digital straightness. This makes the theory of straightness more like differential calculus and the study of the differential equation $F''=0$.

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