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Tangents of plurisubharmonic functions

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Resumo: *Tanĝantoj de plursubharmonaj funkcioj*

Oni studas lokajn ecojn de plursubharmonaj funkcioj per la nocio de tanĝanto kiu priskribas la konduton de la funkcio en ĉirkaŭaĵo de donita punkto. Estas montrite ke ekzistas plursubharmonaj funkcioj kun pluraj tanĝantoj, male al konjekto de Reese Harvey.

Abstract: *Tangents of plurisubharmonic functions*

Local properties of plurisubharmonic functions are studied by means of the notion of tangent which describes the behavior of the function near a given point. We show that there are plurisubharmonic functions with several tangents, disproving a conjecture of Reese Harvey.

1. Introduction

Let f be a plurisubharmonic function in an open set ω in $\mathbf{C}^n$; we shall write this as $f \in PSH(\omega)$. If we want to study the behavior of f near a point $x \in \omega$, it is natural to consider $f(x+rz)$ for a small positive number r. However, if f takes the value $-\infty$ at x, then $f(x+rz)$ just tends to minus infinity as $r \to 0$. So we subtract a constant to prevent this from happening: let us define, for any point x in ω such that f is not identically minus infinity near x, and any positive number r,

$$f_{x,r}(z) = f(x+rz) - \sup_{x+rB} f, \qquad |z| < \frac{1}{r} d_\omega(x), \tag{1.1}$$

and let us write $f_r = f_{0,r}$ for brevity. Here B is the open unit ball in $\mathbf{C}^n$ for the Euclidean metric, and $d_\omega(x)$ is the distance from x to the complement of ω. This means that we are looking at f with a microscope magnifying $1/r$ times, but have adjusted the level by an additive constant so that $\sup_B f_{x,r} = 0$. Of the properties of the operation $f \mapsto f_r$ we may note the following: $(af+b)_r = af_r$ if a and b are constants with $a \geq 0$, b real; $(f+g)_r \geq f_r + g_r$; and $(f \vee g)_r \leq f_r \vee g_r$, where $\vee$ denotes the supremum.

Given any compact set K in $\mathbf{C}^n$, the domain of $f_{x,r}$ contains K if r is small. Moreover, as we shall see, the family $(f_{x,r})_{0<r<\delta}$ is bounded in $L^1(K)$ if δ is a sufficiently small positive number. To every sequence $(f_{x,r_j})_j$ there is a subsequence converging in $L^1(K)$ and even a subsequence converging in $L^1_{loc}(\mathbf{C}^n)$. The limit must be a plurisubharmonic function. Let us agree to say that $g \in PSH(\mathbf{C}^n)$ is ***tangent to*** f ***at*** $x \in \omega$ if there is a sequence $r_j \to 0$ such that $f_{x,r_j} \to g$ in $L^1_{loc}(\mathbf{C}^n)$ (or, which turns out to be the same thing, for the weak topology in $\mathcal{D}'(\mathbf{C}^n)$; see Hörmander 1983, Theorem 4.1.9.b)). We shall denote by $T_x(f)$ the set of all tangents, ***the tangent space of*** f ***at*** x.

Example 1.1. If f is the constant $-\infty$, then $T_x(f) = \emptyset$. For $f_{x,r}$ is never defined.

Example 1.2. If f is finite at x, then $T_x(f) = \{0\}$. To see this, note first that for $|z| \le R$,

$$f_{x,r}(z) \le \sup_{x+rRB} f - \sup_{x+rB} f \to 0, \qquad r \to 0,$$

so that $\limsup_{r\to 0} f_{x,r} \le 0$ everywhere, hence $g \le 0$ for all elements $g \in T_x(f)$. On the other hand, $f_{x,r}(0) = f(x) - \sup_{x+rB} f \to 0$ in view of the upper semicontinuity of f, so that g must be identically zero.

As these two examples show, the tangent space can be of possible interest only if f takes the value $-\infty$ at the point x but is not identically $-\infty$ near x. The model example is this:

Example 1.3. If $f = \log|h|$ for a holomorphic function h which has a zero of order $m \ge 1$ at 0, then we can write $h = P + H$ where P is a homogeneous polynomial of degree m and $H(z) = O(|z|^{m+1})$, $z \to 0$. With the notation $a = \sup_B |P| > 0$ we can estimate f as follows:

$$\sup_{rB} f \le \log|ar^m + Cr^{m+1}| = \log a + m\log r + \log(1 + Cr/a),$$

and similarly from below

$$\sup_{rB} f \ge \log|ar^m - Cr^{m+1}| = \log a + m\log r + \log(1 - Cr/a),$$

so that, as r tends to zero,

$$f_r(z) = f(rz) - \sup_{rB} f = \log|P(rz) + H(rz)| - \sup_{rB} f =$$

$$= \log|P(z) + r^{-m}H(rz)| + m\log r - \sup_{rB} f \to \log|P(z)/a|$$

pointwise. The convergence holds in fact also in L^1_{loc}, i. e., $f_r \to \log|P/a|$ in $L^1_{loc}(\mathbf{C}^n)$. To see this, note that the convergence is uniform on compact sets which avoid the zeros of P, and that integrals over small sets can be estimated uniformly in the parameter r. So here $T_0(f) = \{\log|P/a|\}$, reflecting, as a tangent should, the main term in the expansion of h.

Example 1.4. If $f \in PSH(\mathbf{C}^n)$ satisfies a homogeneity property

$$f(tz) = C\log t + f(z), \qquad t > 0, \quad z \in \mathbf{C}^n,$$

and if $\sup_B f = 0$, then $f_r = f$ for all $r > 0$ so that f is an eigenfunction for all operations $f \mapsto f_r$. In this case the tangent space at the origin is $T_0(f) = \{f\}$. The homogeneity property mentioned is equivalent to the formally stronger

$$f(tz) = C\log|t| + f(z), \qquad t \in \mathbf{C} \setminus \{0\}, \quad z \in \mathbf{C}^n;$$

see the proof of Proposition 3.1. Conversely it can be shown that these functions are the only ones that are eigenfunctions simultaneously for all $r > 0$.

The notion of a tangent has been defined in a more general setting. If T is a current we can define its push-forward $(1/r)_*T$ under the mapping $x \mapsto x/r$. The current $\lim_{r\to 0}(1/r)_*T$, if this limit exists in a suitable topology, is called ***the tangent cone to** T **at*** 0; see Harvey 1977:332. To a plurisubharmonic function f we associate the current $i\partial\bar\partial f$ of type $(1,1)$ and bidimension $(n-1, n-1)$; it is closed and (strongly and weakly) positive. Then $(1/r)_*(i\partial\bar\partial f) = i\partial\bar\partial f_r$. Therefore the tangent cone to the positive closed current $i\partial\bar\partial f$ exists at 0 if and only if the tangent space $T_0(f)$ of f is a singleton.

Harvey formulates as a conjecture that the tangent cone to a strongly positive closed current always exists (1977:332, Conjecture 1.32). This was so in Examples 1.2, 1.3 and 1.4. We shall see, however, that a plurisubharmonic function may have infinitely many tangents if $n \geq 2$.

The tangent space $T_x(f)$ is analogous to the limit set of a plurisubharmonic function of finite order studied by Sigurđsson 1986. Actually it is in many ways simpler, since the behavior of a plurisubharmonic function is more restricted near a point than at infinity. To give but one example, the logarithmic homogeneity is automatic here (see Proposition 3.1), whereas homogeneity of degree ρ has to be imposed as an extra and very restrictive condition in the case of limit sets of plurisubharmonic functions of finite order.

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2. Basic estimates

If f is a plurisubharmonic function in an open subset ω of $\mathbf{C}^n$ and x is an arbitrary point of ω, we shall use two functions, u and U, to describe its behavior near x. We define for $x \in \omega$, $t < \log d_\omega(x)$,

$$u(x,t) = u_f(x,t) = \fint_{z\in S} f(x + e^t z)$$

and

$$U(x,t) = U_f(x,t) = \sup_{z\in S} f(x + e^t z).$$

Here S is the Euclidean unit sphere, and the barred integral sign indicates the mean value: in general

$$\fint_A f = \int_A f \Big/ \int_A 1 \quad \text{provided} \quad 0 < \int_A 1 < +\infty.$$

So $u(x,t)$ is the mean value of f over the sphere $x+e^t S$, and $U(x,t)$ is the supremum of f over the same sphere; therefore obviously $u \le U$. To get an estimate in the opposite direction we shall use Harnack's inequality which has the form

$$\frac{1+|x|/r}{(1-|x|/r)^{m-1}} h(0) \le h(x) \le \frac{1-|x|/r}{(1+|x|/r)^{m-1}} h(0)$$

for harmonic functions which satisfy $h \le 0$ in the ball of radius r in $\mathbf{R}^m$. If f is subharmonic in a neighborhood of the ball $e^s \bar{B}$ in $\mathbf{C}^n$ we can consider its harmonic majorant h there, which satsfies $f(x) \le h(x)$ and

$$h(0) = \fint_{z \in S} h(e^s z) = \fint_{z \in S} f(e^s z) = u(0,s).$$

Therefore

$$U(0,t) = \sup_{e^t S} f \le \sup_{e^t S} h \le \frac{1-e^{t-s}}{(1+e^{t-s})^{2n-1}} u(0,s), \quad t < s,$$

provided only $f \le 0$ in $e^s B$. If we apply this inequality to the function $f - U(0,s)$ which is ≤ 0 in $e^s B$, we get, writing $U(t)$ instead of $U(0,t)$ for simplicity:

$$U(t) - U(s) \le \frac{1-e^{t-s}}{(1+e^{t-s})^{2n-1}} \big(u(s) - U(s)\big),$$

i. e.,

$$U(t) \le (1-\lambda_{s-t}) U(s) + \lambda_{s-t} u(s), \quad t < s, \tag{2.1}$$

where λ_s is defined for $s > 0$ as

$$\lambda_s = \frac{1-e^{-s}}{(1+e^{-s})^{2n-1}}.$$

As a consequence of the maximum principle, $u(x,t)$ and $U(x,t)$ are increasing in t; by Hadamard's three-circle theorem, they are convex functions of t. Therefore their slopes at $-\infty$ exist:

$$\nu_f(x) = \lim_{t \to -\infty} \frac{u(x,t)}{t} \quad \text{and} \quad N_f(x) = \lim_{t \to -\infty} \frac{U(x,t)}{t} \tag{2.2}$$

both exist. This follows from the fact that the slopes

$$\frac{u(x,t) - u(x,t_0)}{t - t_0} \quad \text{and} \quad \frac{U(x,t) - U(x,t_0)}{t - t_0}$$

are increasing in t. The first limit $\nu_f(x)$ is ***the Lelong number of*** f ***at*** x. The Lelong number is usually defined as a density of a measure (in the present case the

$(2n-2)$-dimensional density of the measure $(2\pi)^{-1}\triangle f$ at x). See e. g. Kiselman 1979 for the equivalence of the two definitions. The two limits in (2.2) are equal. In fact, since $u \le U$ we immediately get $\nu_f(x) \ge N_f(x)$. In the other direction it follows from (2.1), taking $s = t+1$, that

$$U(t) \le (1-\lambda_1)U(t+1) + \lambda_1 u(t+1),$$

whence

$$\frac{U(t)}{t} \ge (1-\lambda_1)\frac{U(t+1)}{t} + \lambda_1\frac{u(t+1)}{t}, \quad t < 0.$$

Letting t tend to $-\infty$ we see that $N_f(x) \ge \nu_f(x)$.

Lemma 2.1. *Let $f \in PSH(\omega)$ be plurisubharmonic in an open set in $\mathbf{C}^n$, let $x \in \omega$ be such that f is not identically minus infinity near x, and define $f_{x,r}$ by (1.1). Let R be a positive number. Then the family $(f_{x,r})_{0<r<\delta}$ is bounded in $L^1(RB)$ for some $\delta > 0$.*

Proof. We may assume without loss of generality that $x = 0$ and that f is defined and plurisubharmonic in a neighborhood of the closed unit ball in $\mathbf{C}^n$. The function U, being convex and increasing, satisfies

$$\nu_f(0)(s-t) \le U(s) - U(t) \le C(s-t), \quad t \le s \le 0, \tag{2.3}$$

for some constant C; when $t \le s \ll 0$ we even get

$$U(s) - U(t) \le (\nu_f(0)+\varepsilon)(s-t) \tag{2.4}$$

for any preassigned $\varepsilon > 0$.

We can now estimate f_r from above in the ball RB as follows, using (2.3):

$$\sup_{RB} f_r = U_f(\log R + \log r) - U_f(\log r) \le C\log^+ R$$

for $0 < r \le \min(1/R, 1)$. Hence $(f_r)_{0<r<\delta}$ is bounded from above in RB by a constant if only $\delta \le \min(1/R, 1)$.

There need of course not exist a pointwise bound from below, but we can estimate the mean value of f_r over a sphere. We note that (2.1) can be written

$$u(s) - U(s) \ge \frac{1}{\lambda_{s-t}}\big(U(t) - U(s)\big), \quad t < s,$$

so that, using (2.3) with $s - t = 1$, we get

$$u(s) - U(s) \ge -\frac{C(s-t)}{\lambda_{s-t}} = -\frac{C}{\lambda_1}, \quad s \le 0.$$

This shows that the mean value of f_r over e^tS is

$$u_{f_r}(t) = u_f(t+\log r) - U_f(\log r) \ge U_f(t+\log r) - U_f(\log r) - \frac{C}{\lambda_1} \ge -Ct^- - \frac{C}{\lambda_1}$$

for $0 < r \le \min(e^{-t}, 1)$; the last inequality holds in view of (2.3). Hence $(f_r)_{0<r<\delta}$ is bounded from below in $L^1(RS)$ when $\delta \le \min(1/R, 1)$. The integral over RB is

$$\int_{RB} f_r\,d\lambda = A_{2n-1}\int_0^R u_{f_r}(\log\rho)\rho^{2n-1}\,d\rho = A_{2n-1}\int_{-\infty}^{\log R} u_{f_r}(t)e^{2nt}\,dt \geq$$
$$\geq A_{2n-1}\int_{-\infty}^{\log R}(-Ct^- - C/\lambda_1)e^{2nt}\,dt = -C_R;$$

here $d\lambda$ is Lebesgue measure in $\mathbf{C}^n$ and A_{2n-1} is the area of the unit sphere S in $\mathbf{C}^n$.

The norm in $L^1(RB)$, finally, is bounded since $|f_r| \leq 2\sup f_r - f_r$, so that

$$\int_{RB}|f_r|\,d\lambda \leq 2\sup_{RB} f_r\int_{RB}1\,d\lambda - \int_{RB} f_r\,d\lambda \leq 2C\log^+ R\int_{RB}1\,d\lambda + C_R.$$

Proposition 2.2. *Let $f \in PSH(\omega)$ be plurisubharmonic in an open set in $\mathbf{C}^n$, assume that f is not identically minus infinity near a given point $x \in \omega$, and define $f_{x,r}$ by (1.1). Let R be a positive number. Then the family $(f_{x,r})_{0<r<\delta}$ is relatively compact in $L^1(RB)$ for some positive number δ. The tangent space $T_x(f)$ is connected and compact in $L^1_{loc}(\mathbf{C}^n)$.*

Proof. We know from Lemma 2.1 that $\|f_{x,r}\|_{L^1(RB)}$ is bounded. We can apply Theorem 4.1.9.a) of Hörmander 1983 to deduce that every sequence (f_{x,r_j}) has a subsequence which converges in $L^1(RB)$. Repeating this for every natural number R and taking a diagonal subsequence we get convergence in $L^1_{loc}(\mathbf{C}^n)$. The set of all limits obtained in this way must of course be closed, and the connectedness follows as for limit sets in Sigurđsson 1986:241.

3. Properties of tangent functions

Proposition 3.1. *Let $f \in PSH(\omega)$, ω an open subset of $\mathbf{C}^n$. Assume that f is not identically minus infinity near a point x of ω. Then every tangent function $g \in T_x(f)$ satisfies the homogeneity relation*

$$g(tz) = \nu_f(x)\log|t| + g(z), \quad t \in \mathbf{C},\ z \in \mathbf{C}^n,$$

where $\nu_f(x)$ is the Lelong number of f at x. In particular, g is a plurisubharmonic function of minimal growth.

Corollary 3.2. *In one dimension, $T_x(f)$ is a singleton with the only function $g(z) = \nu_f(x)\log|z|$ as element.*

Proof of Proposition 3.1. We take $x = 0$ and write $U(t)$ for $U(0,t)$. From (2.4) we get

$$U_{f_r}(t) = U_f(t + \log r) - U_f(\log r) \leq (\nu_f(0)+\varepsilon)t, \qquad t \geq 0,\quad r < r_\varepsilon e^{-t}.$$

Using a sequence (r_j) such that $f_{r_j} \to g$, we see that $U_g(t) \leq (\nu_f(0)+\varepsilon)t$ for every $t \geq 0$, and, since ε is arbitrary, $U_g(t) \leq \nu_f(0)t$ for $t \geq 0$. We may write this as

$$g(z) \leq \nu_f(0)\log|z| \tag{3.1}$$

for $|z| \geq 1$. Similarly,

$$U_{f_r}(t) = U_f(t + \log r) - U_f(\log r) \le \nu_f(0)t, \quad t \le 0,$$

for all sufficiently small r, and therefore $U_g(t) \le \nu_f(0)t$ for $t \le 0$, i. e., (3.1) holds also for $|z| \le 1$. Now fix $z \in \mathbf{C}^n$. Since g is plurisubharmonic, the function $h_z(t) = g(tz) - \nu_f(0)\log|t|$ is subharmonic in $\mathbf{C} \setminus \{0\}$. But h_z is bounded from above in view of (3.1), so it can be extended to a subharmonic function in all of $\mathbf{C}$. Moreover it is constant by Liouville's theorem:

$$h_z(t) = g(tz) - \nu_f(0)\log|t| = h_z(1) = g(z).$$

This proves the proposition.

Knowing g is equivalent to knowing the open set $\Omega = \{z \in \mathbf{C}^n; \;\; g(z) < 0\}$ and the number $\nu_f(0)$.

4. Plurisubharmonic functions with prescribed tangents

Theorem 4.1. *Let M be a subset of $PSH(\mathbf{C}^n)$. Then M is the tangent space $T_x(f)$ of some plurisubharmonic function f defined in a neighborhood of x if and only if the following four conditions hold:*

1. *Every element $g \in M$ is homogeneous:*

   $$g(tz) = C\log|t| + g(z), \quad t \in \mathbf{C},\; z \in \mathbf{C}^n, \tag{4.1}$$

   the constant C being equal to $\nu_f(x)$ by necessity;
2. *Every element $g \in M$ satisfies*

   $$\sup_B g = 0; \tag{4.2}$$

3. *M is closed in $L^1_{loc}(\mathbf{C}^n)$; and*
4. *M is connected for the topology induced by $L^1_{loc}(\mathbf{C}^n)$.*

Proof. First let f be a given plurisubharmonic function and denote its tangent space at a point x by $M = T_x(f)$. Property 1 was proved in Proposition 3.1, and property 2 is an easy consequence of the fact that every function $f_{x,r}$ satisfies (4.2). Properties 3 and 4 were noted above in Proposition 2.2.

Now let M be a given set of plurisubharmonic functions satisfying the four properties. If $M = \emptyset$ we take f as the constant $-\infty$. If M is non-empty and $C = 0$, then $M = \{0\}$ and we can take f as any finite constant or more generally as any plurisubharmonic function with $\nu_f(x) = 0$. For the rest of the proof we shall suppose that $M \ne \emptyset$ and that C is positive. We shall construct f very much like Sigurđsson 1986: Theorem 1.2.1, ii). There is only a small difference in the proof due to a glueing procedure of a different kind.

Let (g_j) be a sequence of functions in M which is dense for the topology in $L^1(B)$. Next let h_j be plurisubharmonic functions which are continuous on $\mathbf{C}^n \setminus \{0\}$ and satisfy (4.1) and (4.2) with the same constant C, and such that $\|h_j - g_j\|_{L^1(B)} \to 0$. We can obtain such functions h_j by convolving over the space of matrices as in Sigurđsson

1986:244, or over the unitary group as in Kiselman 1967:10 (such a convolution, as opposed to the usual smoothing by convolution on $\mathbf{C}^n$, preserves the homogeneity (4.1)), and then adjust by a constant to satisfy (4.2). Therefore also the functions h_j are eigenfunctions for the operation (1.1): they satisfy $(h_j)_r = h_j$. We shall define the function f as

$$f = \sup_j (c_j h_j + n_j), \tag{4.3}$$

where n_j are constants to be determined, and where c_j are numbers tending to 1 and satisfying $c_j > c_{j+1} > 1$. These strict inequalities are essential for our construction. To be precise we shall choose (c_j) tending to one so fast that

$$\|c_k h_k - h_k\|_{L^1(kB)} = (c_k - 1)\|h_k\|_{L^1(kB)} \to 0.$$

The idea is to choose the level constants n_j in such a way that in certain shells $s_k/k \le |z| \le ks_k$, the term of index k dominates the others: $f = c_k h_k + n_k$ there. To be specific, let Z_k be the set where $c_k h_k + n_k = f$. Then Z_k shall be so wide that it contains the spherical shell $s_k/k \le |z| \le ks_k$. If this is so, then we can perform the operation $f \mapsto f_{s_k}$ on (4.3) (see (1.1) for the definition) to obtain

$$f_{s_k} = (c_k h_k + n_k)_{s_k} = (c_k h_k)_{s_k} = c_k h_k \qquad \text{for} \qquad \frac{1}{k} \le |z| \le k. \tag{4.4}$$

To determine the constants n_j, first choose $n_1 = 0$ and $s_1 = 1$. Next assume that n_j and s_j have been chosen for $j = 1, 2, ..., k-1$. We then choose n_k so that

$$c_k h_k(z) + n_k \le c_{k-1} h_{k-1}(z) + n_{k-1} \qquad \text{when} \qquad |z| = \frac{s_{k-1}}{k-1}.$$

This is possible since h_k and h_{k-1} are bounded on that sphere. We note that the term of index k will not influence f where $|z| \ge s_{k-1}/(k-1)$. Next choose s_k so that

$$c_k h_k(z) + n_k \ge c_{k-1} h_{k-1}(z) + n_{k-1} \qquad \text{when} \qquad |z| = ks_k.$$

This is possible since $c_k h_k$ and $c_{k-1} h_{k-1}$ are homogeneous of different degrees: $0 < c_k C < c_{k-1} C$. As a consequence the term of index $k-1$ in (4.3) will not influence f where $|z| \le ks_k$.

This procedure will define a function f satisfying (4.4): $f_{s_k} = c_k h_k$ in $kB \setminus k^{-1}B$. Therefore the norm of $f_{s_k} - c_k h_k$ in $L^1(kB)$ is equal to its norm in $L^1(k^{-1}B)$:

$$\|f_{s_k} - c_k h_k\|_{L^1(kB)} = \|f_{s_k} - c_k h_k\|_{L^1(k^{-1}B)}.$$

This last quantity evidently tends to zero. Moreover, by the choice of c_k, we know that $\|c_k h_k - h_k\|_{L^1(kB)} \to 0$, so $\|f_{s_k} - h_k\|_{L^1(kB)} \to 0$, which implies that $f_{s_k} - h_k \to 0$ in $L^1_{loc}(\mathbf{C}^n)$. To every given $g \in M$ there is a subsequence of (h_k) which tends to g, so the corresponding subsequence of (f_{s_k}) tends to the same limit. Therefore g is tangent to f, and we have proved that $T_0(f)$ contains M.

Example 4.2. To get a specific example of a function whose tangent space is not a singleton we can now take g_j equal to $\log|z|$ for j odd, and equal to $\log|z_1| \vee \cdots \vee \log|z_n|$

for j even—these functions are different if $n \geq 2$. In the construction above we can take $h_j = g_j$, and it will yield a function whose tangent space contains both g_0 and g_1. But it will also contain every function

$$G_a(z) = \log|z_1| \vee \cdots \vee \log|z_n| \vee (a + \log|z|),$$

where a is a constant satisfying $-\log\sqrt{n} < a < 0$. We note that $G_a = g_0$ if $a \leq -\log\sqrt{n}$ and that $G_a = g_1$ if $a = 0$, so G_a describes, as a moves from $-\log\sqrt{n}$ to 0, a curve in $PSH(\mathbf{C}^n)$ connecting g_0 to g_1. The tangent space in this case is precisely this curve.

Example 4.3. If we take $g_j(z) = \log|z_1|$ for j odd and $g_j(z) = \log|z_2|$ for j even we get a function whose tangent space contains all functions

$$G_{a,b}(z) = (\log|z_1| + a) \vee (\log|z_2| + b),$$

where $-\infty \leq a, b \leq 0$ and $a \vee b = 0$. Here $G_{0,b}$ describes a curve from g_1 to $g_1 \vee g_2$ as b moves from $-\infty$ to 0, and then $G_{a,0}$ goes from $g_1 \vee g_2$ to g_2 as a moves from 0 to $-\infty$. This time it is not possible to take $h_j = g_j$, at least if we want suprema of the type (4.3) which are locally finite in $\mathbf{C}^n \setminus \{0\}$.

To complete the proof we must be a bit more careful to get $T_0(f)$ as a subset of M. The technique to achieve this will be as in Sigurđsson 1986; specifically we shall use his Lemma 1.2.5 which we quote for reference:

Lemma 4.4. *Let X be a compact connected metric space with metric d. Then there exists a sequence (x_k) in X such that its elements form a dense subset and $d(x_{k+1}, x_k)$ tends (decreasingly) to zero.*

In the given set M we shall use the metric induced by $L^1_{loc}(\mathbf{C}^n)$, which we can take as

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \min(1, \|f-g\|_{L^1(jB)}).$$

Then $\|f-g\|_{L^1(jB)} < 1/j$ when $d(f,g) < 2^{-j}/j$. Using the lemma we can arrange that $d(h_{k+1}, h_k)$ tends to zero, so that

$$\|h_{k+1} - h_k\|_{L^1(jB)} < 1/j$$

if $k \geq k_j$ for suitable indices k_j. We may assume $k_j \geq j$. Defining $R_k = j$ for $k_j \leq k < k_{j+1}$, we may rewrite this as

$$\|h_{k+1} - h_k\|_{L^1(R_k B)} < 1/R_k \quad \text{for} \quad k \geq k_1. \tag{4.5}$$

We note that $R_k \leq k$. Moreover $R_k \to +\infty$ as $k \to +\infty$, so that convergence with respect to all the seminorms $\|\cdot\|_{L^1(R_k B)}$ is equivalent to convergence in the metric of $L^1_{loc}(\mathbf{C}^n)$.

We have to prove that if (r_j) is a sequence tending to zero and such that (f_{r_j}) converges, then the limit of the latter sequence is in M. The construction gives us full control over (f_{r_j}) if $r_j = s_j$, the special radii selected above. We need to study the behavior of f_r for intermediate values of r. Then more than one term in (4.3) must be taken into account. However, since the s_j tend to zero faster than any geometric progression, it will not be necessary to consider more than two terms.

Let us study the function f in a spherical shell $s_{k+1}/(k+1) \le |z| \le ks_k$ where only two terms may be of influence:

$$f = (c_{k+1}h_{k+1} + n_{k+1}) \vee (c_k h_k + n_k).$$

Performing the operation $f \mapsto f_r$ for an r such that $s_{k+1} \le r \le s_k$ we get

$$f_r = (c_{k+1}h_{k+1} + a) \vee (c_k h_k + b) \tag{4.6}$$

in the shell

$$\frac{s_{k+1}}{(k+1)r} \le |z| \le \frac{ks_k}{r},$$

where a and b are certain constants which must satisfy $a \vee b = 0$ since $\sup_B f_r = 0$ by definition.

The shell where (4.6) holds contains $kB \setminus (k+1)^{-1}B$, for the outer and inner radii of the former are, respectively,

$$\frac{ks_k}{r} \ge k, \quad \text{and} \quad \frac{s_{k+1}}{(k+1)r} \le \frac{1}{k+1}.$$

Therefore the norm in $L^1(kB)$ of the difference between the two sides in (4.6) is equal to the norm in $L^1((k+1)^{-1}B)$ which tends to zero.

It remains to estimate the distance from the right-hand side of (4.6) to M. To do so we shall use the inequality

$$\|F \vee (G+H) - F\|_{L^1(A)} \le \|G - F\|_{L^1(A)} + \|H \vee 0\|_{L^1(A)}$$

which holds for arbitrary functions $F, G, H \in L^1(A)$. We apply it to the supremum in (4.6), putting

$$A = A(k) = R_k B \setminus (k+1)^{-1}B \subset kB,$$

where R_k is the radius occurring in (4.5),

$$F = c_k h_k, \qquad G = c_{k+1}h_{k+1}, \quad H = a \le 0 \quad \text{if} \quad b = 0,$$

and

$$F = c_{k+1}h_{k+1}, \quad G = c_k h_k, \qquad H = b \le 0 \quad \text{if} \quad a = 0.$$

We then get $f_r = F \vee (G+H)$ and $H \vee 0 = 0$ in both cases, so

$$\|f_r - c_p h_p\|_{L^1(A(k))} \le \|c_{k+1}h_{k+1} - c_k h_k\|_{L^1(A(k))} \tag{4.7}$$

for $p = k$ or $p = k+1$. Given any sequence (r_j) tending to zero we can choose k_j such that $s_{k_j+1} \le r_j \le s_{k_j}$. Then (4.7) shows that

$$\|f_{r_j} - c_p h_p\|_{L^1(A(k_j))} \le \|c_{k_j+1}h_{k_j+1} - c_{k_j}h_{k_j}\|_{L^1(A(k_j))}$$

for $p = k_j$ or $p = k_j + 1$. We recall that, by our choice of c_k,

$$\|c_k h_k - h_k\|_{L^1(R_k B)} \le \|c_k h_k - h_k\|_{L^1(kB)} \to 0$$

and that, by (4.5), the quantity $\|h_{k+1} - h_k\|_{L^1(R_k B)}$ tends to zero. This proves that f_{r_j} is as close as we wish to either h_{k_j} or h_{k_j+1} in the metric of $L^1_{loc}(\mathbf{C}^n)$, and therefore tends to an element of M: we have proved that $T_0(f)$ is contained in M.

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