FINAL EXAMINATION

1MA052 Ordinary Differential Equations II

Code/Name: ______________________________

Problem 1. (Method of Frobenius)

Use the method of Frobenius to find the solutions of

$$(x+2)x^2y'' - xy' + (1+x)y = 0$$

(it is sufficient to find the recursion relations for the coefficients and write out several first terms of the solution).

Problem 2. (Sturm-Liouville problems)

Find the eigenvalues λ and eigenfunctions for

$$y'' + 2y' + (1+\lambda)y = 0.$$

Problem 3. (Limit sets, Stability)
Consider the system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= \sin^2\left(\frac{\pi}{x(t)^2 + y(t)^2}\right) y(t) - x(t). \end{aligned}$$

1) Show that the origin is a fixed point. Is it stable or unstable?
2) Show that the circles $x(t)^2 + y(t)^2 = \frac{1}{n}$, for integer $n \geq 1$, are periodic orbits.
3) Draw the phase portrait.
4) Determine all α and ω-limit sets.

Problem 4. (Poincaré-Bendixson, Limit cycles)
Consider the system

$$x'(t) = -y(t) + x(t)(1 - x(t)^2 - y(t)^2),$$
$$y'(t) = x(t) + y(t)(1 - x(t)^2 - y(t)^2).$$

1) Use the fact that $r^2 = x^2 + y^2$ to find the derivative $r'(t)$.
2) Prove that all trajectories eventually enter the region $r < C$ for some constant C.
3) Use the Poincaré-Bendixson theorem to prove that the system has a limit cycle.