FINAL EXAMINATION

1MA052 Ordinary Differential Equations II

Code/Name: ____

Problem 1. (Method of Frobenius)

Use the method of Frobenius to find the two solutions (for $x > 0$) of

$$2xy''(x) - y'(x) - y(x) = 0$$

(it is sufficient to find the recursion relations for the coefficients and write out several first terms of the solution).

Problem 2. (Sturm-Liouville problems)

Consider the following boundary value problem:

$$y'' + 2y' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0.$$

1) Show that all eigenvalues are positive, and the $n - th$ positive eigenvalue is $\lambda_n = \alpha_n^2 + 1$, where α_n is the $n - th$ root of the $\tan z = z$.
2) Find eigenfunctions associated with λ_n.

Problem 3. (Limit sets, Stability)

Consider the system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= \sin^2\left(\frac{\pi}{x(t)^2 + y(t)^2}\right) y(t) - x(t). \end{aligned}$$

1) Show that the origin is a fixed point. Is it stable or unstable?
2) Show that the circles $x(t)^2 + y(t)^2 = \frac{1}{n}$, for integer $n \geq 1$, are periodic orbits.

3) Draw the phase portrait.
4) Determine all α and ω-limit sets.

Problem 4. (Poincaré-Bendixson, Limit cycles)
Consider the system

$$x'(t) = -y(t) + x(t)(1 - x(t)^2 - y(t)^2),$$
$$y'(t) = x(t) + y(t)(1 - x(t)^2 - y(t)^2).$$

1) Use the fact that $r^2 = x^2 + y^2$ to find the derivative $r'(t)$.
2) Prove that all trajectories eventually enter the region $r < C$ for some constant C.
3) Use the Poincaré-Bendixson theorem to prove that the system has a limit cycle.