

FINAL EXAMINATION

1MA052 Ordinary Differential Equations II

Code/Name: _____

Problem 1. (Method of Frobenius) (11 points)

Use the method of Frobenius to find the two solutions of

$$2x^2y'' + 3xy' - (x^2 + 1)y = 0$$

(it is sufficient to find the recursion relations for the coefficients and write out several first terms of the solution).

Problem 2. (Sturm-Liouville problems) (12 points)

Consider the following boundary value problem:

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y(1) = y'(1).$$

- 1) Show that $\lambda = 0$ is an eigenvalue with the associated eigenfunction $y_0(x) = x$.
- 2) Show that the remaining eigenfunction are $y_n(x) = \sin \beta_n x$, where β_n is the n -th positive root of the equation

$$\tan z = z.$$

- 3) Draw a sketch showing these roots. Deduce from the sketch, that

$$\beta_n \approx \frac{(2n+1)\pi}{2}$$

when n is large.

Problem 3. (Stability) (8 points)

Find all equilibrium solutions of

$$x'(t) = 1 - x(t)y(t), \quad y'(t) = x(t) - y(t)^3,$$

and determine whether they are stable or unstable.

Problem 4. (Poincaré-Bendixson, Limit cycles) (9 points)

Find all limit sets for the system

$$x'(t) = -y(t) - \frac{x(t)(x(t)^2 + y(t)^2 - 2)}{\sqrt{x(t)^2 + y(t)^2}}, \quad y'(t) = x(t) - \frac{y(t)(x(t)^2 + y(t)^2 - 2)}{\sqrt{x(t)^2 + y(t)^2}}.$$

(*Hint:* you will have to compute $\frac{d(x^2+y^2)}{dt}$).