TAKE HOME EXAM (DUE DEC. 25th)

1MA151, Applied Dynamical Systems

Name: ____

Problem 1. (Period doubling in quadratic families), 10 points

1) Use Maple/Matlab/C/C++/etc to compute and draw the bifurcation diagram for a perturbation of the quadratic family (ex, $f_r(x) = rx(1-x)(1+ax^4)$ for some small a, or a similar function). Make sure that the function chosen is such that it maps a unit interval to itself.
2) Compute the Feigenbaum parameter from the diagram (the value to which the parameters in the bifurcation cascade accumulate). Don't just look it up on the diagram itself, but rather find it numerically as the limit of values r_n for which $(f_r^{2^n})'(x) = -1$ at a point x in the 2^n-th attracting periodic orbit.
3) Using a similar parameter search ($(f_r^{2^n})'(x) = -1$) compute the accumulation rate of parameters
$$\lim_{n\to\infty} \frac{r_{n+1}-r_n}{r_{n-1}-r_n}.$$
Compare with the value for the quadratic map itself (see Strogatz).
4) Find the parameters for which the period three window (an attracting orbit of period 3) occurs.

Problem 2. (Cantor set), 7 points

Consider a generalized Cantor set built by removing γl_m, $0 < \gamma < 1$, from the middle of each segment of length l_m at the m-th iteration.

Find its Hausdorff (and not just box counting) dimension.

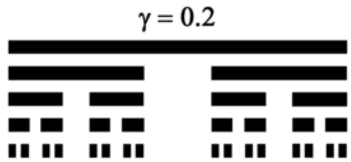

Figure 1. A generalized Cantor set with $\gamma = 0.2$

Problem 3. (Box-counting dimension), 6 points

Use the “Box counting dimension of coastline, applet” on the course page to compute the box counting dimension of a randomly generated coastline. Attach the pictures of the grid covers.

Problem 6 (Limit cycles), 7 points

Consider the system

$$x' = \sin x \,(-0.1 \cos x - \cos y)\,,$$
$$y' = \sin y \,(\cos x - 0.1 \cos y)\,.$$

What is the type of the equilibrium point $(\pi/2, \pi/2)$ here? Show that all solutions with an initial condition close to this equilibrium point have the ω-limit sets equal to the square bounded by $x = 0$, $x = \pi$ and $y = 0$, $y = \pi$.

Problem 5 (Belousov-Zhabotinsky reactions: an oscillating reaction), 10 points

One particular oscillating chemical reaction is given by a chlorine dioxide-iodine-malonic acid interaction. The exact differential equations modeling this reaction are complicated. However, there is a planar nonlinear system that closely approximates the concentrations of two of the reactants. The system is

$$x' = a - x - \frac{4xy}{1+x^2},$$
$$y' = bx\left(1 - \frac{y}{1+x^2}\right),$$

where x and y represent concentrations of I^- (iodine ions) and ClO_2^- (chlorine dioxide ions), respectively, and a and b are positive parameters.

1) Begin the exploration by investigating these reaction equations numerically. What qualitatively different types of phase portraits do you find?
2) Find all equilibrium points for this system.
3) Linearize the system at your equilibria and determine the type of each equilibrium.
4) In the ab-plane sketch the regions where you find asymptotically stable and unstable equilibria.
5) Identify the a, b values for which the system undergoes a bifurcation.
6) Using nullclines for the system together with the Poincaré-Bendixson theorem find the a, b values for which a stable limit cycle exists. Why do these values correspond to an oscillating chemical reaction?

Problem 6. (Horseshoe), 8 points

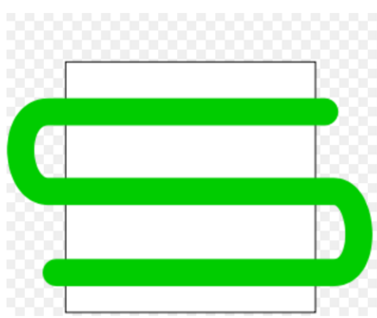

Figure 2. A three component horseshoe.

1) Assume that the map is reversible, that is $H^{-1} = R \circ H \circ R$, $R(x,y) = (y,x)$. Draw the action of the inverse map.

2) Repeat the construction of the sets Δ_ω and Δ^ω for the three component horseshoe (notice, you will have to use an alphabet of three symbols now 0, 1 and 2).
3) Draw some of the sets $\Delta_{\omega_1..\omega_n}$, $\Delta^{\omega_{-n}..\omega_0}$ and $\Delta_{\omega_1..\omega_n}^{\omega_{-n}..\omega_0}$ for your choice of n. Track the index, that is indicate in the picture which index that set has.
4) What kind of shift is the dynamical system $H|_\Lambda$ is conjugate to?

Problem 7. (Strange attractor in the Hénon map), 8 points

1) Consider the Hénon map

$$H(x, y) = (y + 1 - ax^2, bx),$$

with the "classical" parameters $a = 1.4$, $b = 0.3$. Let Q denote the quadrilateral with vertices $(-1.33, 0.42)$, $(1.32, 0.133)$, $(1.245, -0.14)$, $(-1.06, -0.5)$.
Plot Q and its image $H(Q)$ in Matlab/Maple/etc. Demonstrate that $T(Q)$ is contained in Q. This is a *trapping region*.
2) Explore where the points from Q converge under dynamics (you can simply iterate $H^n(Q)$). Draw this "strange attractor". Zoom on a piece of this attractor to show that it has a Cantor-like structure.

Problem 8. (Lorenz attractor), 10 points

1) For the Lorenz equations, show that the characteristic equation for the eigenvalues of the Jacobian matrix at $Q_\pm$ is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$

Find the third eigenvalue.
2) Assume that the Lorenz flow is linear in the cube $\{|x| \leq 5, |y| \leq 5, |z| \leq 5\}$:

$$\begin{aligned} x' &= 2x, \\ y' &= -3y, \\ z' &= -z. \end{aligned}$$

A trajectory starting from point (x, y) in the upper face $\{z = 5\}$ of the box, ends up at some (y, z) on the right face of the box $\{x = 5\}$.
Express these (y, z) as a function of (x, y), and find the Poincaré map $h(x, y) = (y, z)$ from the upper face to the right.

3) Using the analytic results obtained about the bifurcation in the Lorenz equations, give a partial sketch of the stability diagram. Specifically, assume $b = 1$, and then plot the pitch fork bifurcation n ($Q_{\pm}$ pass from $\mathbb{C}^3$ to $\mathbb{R}^3$) and the Hopf bifurcation curves in the (σ, r) plane.