# FINAL EXAMINATION

**1MA208 Ordinary Differential Equations II**

Code/Name: ____________________________________________

**Problem 1. (Continuity of solutions)**
Suppose that $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are continous and each are Lipschitz with respect to the seconf argument.

Suppose that $x(t)$ is the global solution to $x' = f(t, x), \quad x(t_0) = a$, and $y(t)$ is the global solution to $y' = g(t, y), \quad y(t_0) = b$.

1) If $f(t, p) < g(t, p)$ for every $(t, p) \in \mathbb{R}^2$ and $a < b$, show that $x(t) < y(t)$ for every $t \geq t_0$.

2) If $f(t, p) \leq g(t, p)$ for every $(t, p) \in \mathbb{R}^2$ and $a \leq b$, show that $x(t) \leq y(t)$ for every $t \geq t_0$.

**Problem 2. (Hartman-Grobman and conjugacies)**
Let $a$ and $b$ be distinct constants and consider the equations $x' = ax$ and $x' = bx$ for $x \in \mathbb{R}$. Under what conditions on $a$ and $b$ does their exist a topological conjugacy $h$ taking solutions of one equation to solution of the other?

**Problem 3. (Limit sets, Stability)**
Consider the system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= \sin^2\left(\frac{\pi}{x(t)^2 + y(t)^2}\right) y(t) - x(t). \end{aligned}$$

1) Show that the origin is a fixed point. Is it stable or unstable?
2) Show that the circles $x(t)^2 + y(t)^2 = \frac{1}{n}$, for integer $n \geq 1$, are periodic orbits.
3) Draw the phase portrait.
4) Determine all $\alpha$ and $\omega$-limit sets.

**Problem 4. (Poincaré-Bendixson, Limit cycles)**
Consider the system

$$\begin{aligned} x'(t) &= -y(t) + x(t)(1 - x(t)^2 - y(t)^2), \\ y'(t) &= x(t) + y(t)(1 - x(t)^2 - y(t)^2). \end{aligned}$$

1) Use the fact that $r^2 = x^2 + y^2$ to find the derivative $r'(t)$.
2) Prove that all trajectories eventually enter the region $r < C$ for some constant $C$.
3) Use the Poincaré-Bendixson theorem to prove that the system has a limit cycle.

**Problem 4. (Lyapunov function)**
Consider the system

$$\begin{aligned} x' &= x(a + bx + cy), \\ y' &= y(d + ex + fy). \end{aligned}$$

Suppose that this "two species Lotka-Volterra" system has a unique equilibrium point $(x^*, y^*)$ in the first quadrant $\mathbb{R}^2_{>0}$. Thus $bf - ce \neq 0$.

Show that

$$L(x, y) = \alpha \left( x - x^* \left( 1 - \ln \frac{x}{x^*} \right) \right) + \beta \left( y - y^* \left( 1 - \ln \frac{y}{y^*} \right) \right),$$

is a Lyapunov function for the system <u>with an appropriate choice of $\alpha > 0$ and $\beta > 0$</u>. Find the conditions on $a, b, c, d, e, f$ so that the equilibrium would be asymptotically stable.