FINAL EXAMINATION

1MA208 Ordinary Differential Equations II

Code/Name: ____

Problem 1. (Sturm-Liouville problems)

Consider the SLP:

$$-(xu')' = \lambda x^{-1}u, \quad 1 < x < e, \quad u(1) = 0, \quad u'(e) = 0.$$

a) Find all eigenvalues and eigenfunctions.

b) Expand the constant function f(x) 1 in terms of the eigenfunctions.

c) Discuss the convergence of the series obtained in b).

d) Use b) and c) to determine the value of

$$1 + 1/3 - 1/5 - 1/7 + 1/9 + 1/11 - 1/13 - 1/15 + 1/17 + \dots$$

Problem 2. (Sturm-Liouville problems)
Find all the eigenvalues and eigenfunctions of the problem

$$-u'' = \lambda u \quad 0 < x < \pi, \quad u(0) - au'(0) = 0, \quad u(\pi) + bu'(\pi) = 0,$$

where $a, b > 0$.

Problem 3. (Lorenz system)
Prove that there is a periodic solution γ of the geometric model for the Lorenz system that meets the rectangle R at precisely two distinct points.

Problem 4. (Lorenz system)
Consider the system

$$x' = 10(y - x), \tag{0.1}$$
$$y' = 28x - y + xz, \tag{0.2}$$
$$z' = xy - (8/3)z. \tag{0.3}$$

Note the difference with the Lorenz system (extra term in the second equation). Show that most orbits escape to ∞.

Problem 5. (Limit cycles, Hopf bifurcation)
Consider

$$x' = ax - y + x(x^2 + y^2)(2 - x^2 - y^2) \tag{0.4}$$
$$y' = x + ay + y(x^2 + y^2)(2 - x^2 - y^2). \tag{0.5}$$

a) Find all periodic orbits for $-1 < a < 0$ and $a > 0$. Determine their stability.

b) Show that the system undergoes a subcritical Hopf bifurcation at $a = 0$.