FINAL EXAMINATION

1MA208 Ordinary Differential Equations II

Code/Name: ____

Problem 1. (Continuity of solutions)
Suppose that $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are continuous and each are Lipschitz with respect to the second argument.

Suppose that $x(t)$ is the global solution to $x' = f(t, x), \quad x(t_0) = a$, and $y(t)$ is the global solution to $y' = g(t, y), \quad y(t_0) = b$.

1) If $f(t, p) < g(t, p)$ for every $(t, p) \in \mathbb{R}^2$ and $a < b$, show that $x(t) < y(t)$ for every $t \geq t_0$.

2) If $f(t, p) \leq g(t, p)$ for every $(t, p) \in \mathbb{R}^2$ and $a \leq b$, show that $x(t) \leq y(t)$ for every $t \geq t_0$.

Problem 2. (Hartman-Grobman and conjugacies)
Let a and b be distinct constants and consider the equations $x' = ax$ and $x' = bx$ for $x \in \mathbb{R}$. Under what conditions on a and b does their exist a topological conjugacy h taking solutions of one equation to solution of the other?

Let $f(x) \equiv ax$, $g(x) \equiv bx$. The equation for the topological conjugacy

$$\phi_t^f(h(x)) = h(\phi_t^g(x)), \quad \phi_t^f(x) = xe^{at}, \quad \phi_t^g(x) = xe^{bt}$$

tells us that

$$h(x)e^{at} = h(xe^{bt}).$$

Try a power function $h(x) = C|x|^r, r > 0$ (we want this to be defined for both positive and negative x, that is why x comes with the absolute values sign):

$$C|x|^r e^{at} = C|x|^r e^{rbt}) \implies r = a/b.$$

However, for a fixed C, $h(x)$ is not a topological conjugacy (it is not injective). But we do not need to fix C: choose $C = 1$ for $x > 0$, $C = 0$ for $x = 0$ and $C = -1$ for $x < 0$, i.e., take

$$h(x) = \begin{cases} x|x|^{\frac{a}{b}-1}, x \neq 0 \\ x = 0. \end{cases}$$

This is a conjugating homeomorphism if $a/b > 0$.

Problem 3. (Limit sets, Stability)
Consider the system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= \sin^2\left(\frac{\pi}{x(t)^2 + y(t)^2}\right) y(t) - x(t). \end{aligned}$$

1) *Show that the origin is a fixed point. Is it stable or unstable?*

We have

$$y = 0 \implies x' = 0,$$

at the origin, while

$$\lim_{r\to 0} \sin^2\left(\frac{\pi}{r^2}\right) y = 0,$$

hence

$$\lim_{r\to 0} \sin^2\left(\frac{\pi}{r^2}\right) y - x = 0,$$

and $y' = 0$ at the origin.

2) *Show that the circles $x(t)^2 + y(t)^2 = \frac{1}{n}$, for integer $n \geq 1$, are periodic orbits.*

$$(r^2)' = 2\sin^2\left(\frac{\pi}{r^2}\right) y^2 \geq 0,$$

and at $r^2 = 1/n$,

$$(r^2)' = 2\sin^2(n\pi)\, y^2 = 0.$$

At the same time

$$\begin{aligned} x'(t) = y(t) &\implies r'(t)\cos(\theta(t)) - r(t)\sin(\theta(t))\theta'(t) = r(t)\sin(t), \\ y'(t) = -x(t) &\implies r'(t)\sin(\theta(t)) + r(t)\cos(\theta(t))\theta'(t) = -r(t)\cos(t), \end{aligned}$$

multiplying the first equation by sin, the second by cos and subtracting:

$$r(t)\theta'(t) = -r(t) \implies \theta'(t) = -1, \text{if } r \neq 0,$$

and the system has no equilibria other than the origin. Hence, every level set $r^2 = 1/n$ is a closed orbit (of period 2π).

3) *Draw the phase portrait.*

Trajectories spiral clockwise from $\{r^2 = 1/(n+1)\}$ to $\{r^2 = 1/n\}$, $n \in \mathbb{N}$.

4) *Determine all α and ω-limit sets.*

The origin is the ω/α-limit set of itself only, since any trajectory with a non-zero initial condition will be separated from the origin by an invariant curve $\{r^2 = 1/n\}$ for some $n \in \mathbb{N}$.
Circles $\{r^2 = 1/n\}$ are ω-limit sets of points in $\{1/(n+1) < r^2 \leq 1/n\}$, $n \in \mathbb{N}$, and α-limit sets of points $\{1/n \leq r^2 < 1/(n-1)\}$, $n \in \mathbb{N}$ (here, by convention, $1/0 = \infty$: this happens for $n = 1$).

Problem 4. (Poincaré-Bendixson, Limit cycles)
Consider the system

$$\begin{aligned} x'(t) &= -y(t) + x(t)(1 - x(t)^2 - y(t)^2), \\ y'(t) &= x(t) + y(t)(1 - x(t)^2 - y(t)^2). \end{aligned}$$

2) *Prove that all trajectories eventually enter the region $r < C$ for some constant C.*

$$rr' = xx' + yy' = r^2(1 - r^2),$$

For all $r > 1$, $r' = r(1 - r^2) < 0$. Hence, given any $C > 1$, any trajectory with $r(0) > C$ enters $\{r < C\}$.

3) *Use the Poincaré-Bendixson theorem to prove that the system has a limit cycle.*

The origin is unstable: for small r_0, $r(t) = r_0 e^t + O(r_0^2)$, hence there is an open disk D_ϵ of radius ϵ, such that $\phi_\tau\left(\mathbb{R}^2 \setminus D_\epsilon\right) \subset \mathbb{R}^2 \setminus D_\epsilon$.
Let $A = \{(x, y) \in \mathbb{R} : \epsilon \leq r \leq C\}$. This is an invariant compact set, thus it contains an equilibrium or a closed orbit by Corollary 2 of PB. We now verify that the only equilibrium of the system is at zero: suppose r is non-zero (we can divide by it), then

$$xy' - yx' = r^2 \implies r^2\left(\cos(\theta(t))^2\theta'(t) + \sin(\theta(t))^2\theta'(t)\right) = r^2 \implies \theta'(t) = 1.$$

Thus the angular projection of the vector field is never 0 if $r \neq 0$.
By Corollary 2 of PB, there is a closed orbit in A.
One of these orbits is at $r = 1$ (this is the only one, but we will not prove that), and since $r' < 0$ for $r > 1$ and $r' > 0$ for $r < 1$, this orbit is a double-sided ω-limit cycle.

Problem 4. (Lyapunov function)
Consider the system

$$\begin{aligned} x' &= x(a + bx + cy), \\ y' &= y(d + ex + fy). \end{aligned}$$

Suppose that this "two species Lotka-Volterra" system has a unique equilibrium point (x^*, y^*) *in the first quadrant* $\mathbb{R}^2_{>0}$. *Thus* $bf - ce \neq 0$.
Show that

$$L(x, y) = \alpha\left(x - x^*\left(1 - \ln\frac{x}{x^*}\right)\right) + \beta\left(y - y^*\left(1 - \ln\frac{y}{y^*}\right)\right),$$

is a Lyapunov function for the system <u>*with an appropriate choice of* $\alpha > 0$ *and* $\beta > 0$</u>. *Find the conditions on* a, b, c, d, e, f *so that the equilibrium would be asymptotically stable.*

$L(x^*, y^*) = 0$ and $L(x, y) > 0$ for all $(x, y) \neq (x^*, y^*)$ (proved by using the fact that $\ln(t) < t - 1$ for all positive $t \neq 1$).
Moreover, in the first quadrant

$$\begin{aligned} \hat{L} &= L_x(x, y)x' + L_y(x, y)y' = \\ &= \alpha\left(1 - \frac{x^*}{x}\right)x(a + bx + cy) + \beta\left(1 - \frac{y^*}{y}\right)y(d + ex + fy) \\ &= \alpha\left[x - x^*\right](a + bx + cy) + \beta\left[y - y^*\right](d + ex + fy). \end{aligned}$$

Subtract in the first parenthesis $a + bx^* + cy^* = 0$ and $d + ex^* + fy^* = 0$, and rename $x - x^* = \xi$, $y - y^* = \eta$. Then

$$\hat{L} = \alpha\xi(b\xi + c\eta) + \beta(e\xi + f\eta) = \alpha b\xi^2 + \beta f\eta + (\alpha c + \beta e)\xi\eta = [\xi \quad \eta] \begin{bmatrix} b & c \\ e & f \end{bmatrix}^T \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Denote $A = \begin{bmatrix} b & c \\ e & f \end{bmatrix}$, $D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$. The transpose of a scalar is the scalar itself: $\hat{L} = [\xi \quad \eta] A^T D \begin{bmatrix} \xi \\ \eta \end{bmatrix} = [\xi \quad \eta] D^T A \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, therefore, $A^T D = DA$ and

$$\hat{L} = \frac{1}{2}[\xi \quad \eta] \left(A^T D + D^T A\right) \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Hence if we can choose the parameters b, c, d, e such that the symmetric matrix $A^T D + DA$ is negative definite, we will have $\hat{L} \leq 0$ with equality if and only if $(\xi, \eta) = (0, 0)$, i.e. $x = x^*, y = y^*$. Therefore we require that

$$M = A^T D + DA = \begin{bmatrix} 2b\alpha & c\beta + e\alpha \\ c\beta + e\alpha & 2f\beta \end{bmatrix}.$$

has negative eigenvalues. This is the case if trace$(M) < 0$ and $\det(M) > 0$. That is

$$b\alpha + f\beta < 0, \quad 4fb\alpha\beta - (c\beta + e\alpha)^2 > 0.$$