FINAL EXAMINATION

1MA208 Ordinary Differential Equations II

Problem 1. (Sturm-Liouville problems)

Consider the SLP:

$$-(xu')' = \lambda x^{-1}u, \quad 1 < x < e, \quad u(1) = 0, \quad u'(e) = 0.$$

a) Find all eigenvalues and eigenfunctions.

i) First, consider the case $\lambda > 0$, write $\lambda = \omega^2$, $\omega > 0$:

$$x^2u'' + xu' + \omega^2 u = 0.$$

This is the Euler equation (also sometimes called the equidimensional equation). It is known to possess solutions of the form $u = x^r$. Substituting this into the equation we obtain

$$x^2 r(r-1)x^{r-2} + xrx^{r-1} + \omega^2 x^r = 0$$

Upon simplifying this becomes

$$(r^2 + \omega^2)x^r = 0$$

so that we must have $r = \pm i\omega$, and the corresponding solutions are

$$u = x^{\pm i\omega} = e^{\pm i\omega \ln x} = \cos(\omega \ln x) \pm i \sin(\omega \ln x).$$

Consequently the general solution is given by

$$u = A\cos(\omega \ln x) + B\sin(\omega \ln x).$$

Next we need to satisfy the boundary conditions. At $x = 1$ we have $0 = u(1) = A$ and at $x = e$ we have

$$0 = u'(e) = \omega B e^{-1}\cos(\omega \ln e).$$

This can only be true if either $\omega = 0$ or if ω is an odd multiple of $\pi/2$. But if $\omega = 0$ then $u = 0$ which can not be an eigenfunction. We therefore arrive at the following collection of eigenpairs:

$$(\lambda_k, u_k) = (\omega_k^2, \sin(\omega_k \ln x)),$$

where

$$\omega_k = \frac{(2k-1)\pi}{2}, \quad k = 1, 2, \ldots$$

ii) The cases $\lambda = 0$ and $\lambda = -\sqrt{\nu^2} < 0$ to show that $A = B = 0$. Hence there are no non-positive eigenvalues.

b) Expand the constant function $f(x) = 1$ in terms of the eigenfunctions.

$$f = \sum_{k=1}^{\infty} c_k u_k, \quad \text{where } c_k = \frac{< \rho f, u_k >}{< \rho u_k, u_k >},$$

ρ being the weight in the SL problem. In order to compute the generalized Fourier coefficient we obtain

$$< \rho u_k, u_k >= \int_1^e x^{-1} \sin^2(\omega_k \ln x) dx$$

If we make the substitution $z = \omega_k \ln x$ this integral becomes

$$\omega_k^{-1} \int_0^{\omega_k} \sin^2 z dz = (2\omega_k)^{-1} \int_0^{\omega_k} [1 - \cos(2z)] dz = \frac{1}{(2k-1)\pi} \left[z - \frac{1}{2} \sin(2z) \right]_0^{(2k-1)\pi/2} = \frac{1}{2}.$$

Therefore

$$c_k = \int_1^e x^{-1} \sin(\omega_k \ln x) dx = 2\omega^{-1} \int_0^{\omega_k} \sin z dz = \frac{4}{(2k-1)\pi} [-\cos z)]_0^{(2k-1)\pi/2} = \frac{4}{(2k-1)\pi}.$$

So we have the generalized Fourier series

$$1 = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin\left(\frac{(2k-1)\pi}{2} \ln x \right).$$

c) Discuss the convergence of the series obtained in b).

Since the function $f = 1$ does not satisfy the boundary conditions while each of u_k does, we do not have uniform convergence on the interval $[1, e]$ (any truncation of the series for 1 satisfies the boundary condition), however we do have pointwise convergence to 1 for all x on the interval $1 < x < e$. Also, we have uniform convergence on any closed subinterval $[a, b]$ with $1 < a < b < e$.

d) Use b) and c) to determine the value of

$$1 + 1/3 - 1/5 - 1/7 + 1/9 + 1/11 - 1/13 - 1/15 + 1/17 + \ldots$$

Since $1 < \sqrt{e} < e$, we see that the above series converges to 1 at $x = \sqrt{e}$. This tells us that

$$1 = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin\left(\frac{(2k-1)\pi}{4}\right),$$

or

$$\begin{aligned} \frac{\pi}{4} &= \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin\left(\frac{(2k-1)\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}} \left(1 + 1/3 - 1/5 - 1/7 + 1/9 + 1/11 - 1/13 - 1/15 + 1/17 + \ldots\right). \end{aligned}$$

Hence, the series converges to $\pi\sqrt{2}/4$.

Problem 2. (Sturm-Liouville problems)
Find all the eigenvalues and eigenfunctions of the problem

$$-u'' = \lambda u \quad 0 < x < \pi, \quad u(0) - au'(0) = 0, \quad u(\pi) + bu'(\pi) = 0,$$

where $a, b > 0$.

i) First one show that in the cases $\lambda = 0$ and $\lambda < 0$ the solution is trivial and thus is not an eigenfunction.

ii) Case $\lambda = \omega^2$ with $\omega > 0$, and the equation becomes $-u'' = \omega^2 u$, which has the general solution

$$u = A\cos(\omega x) + B\sin(\omega x), \omega > 0.$$

$$u'(x) = -\omega A\sin(\omega x) + B\omega\cos(\omega x)$$

so that the boundary conditions read $A - a\omega B = 0$, $A\cos(\omega\pi) + B\sin(\omega\pi) + b[-\omega A\sin(\omega\pi) + \omega B\cos(\omega\pi)] = 0$, which may be written as

$$\begin{bmatrix} 1 & -a\omega \\ [\cos(\omega\pi) - b\omega\sin(\omega\pi)] & [\sin(\omega\pi) + b\omega\cos(\omega\pi)] \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In order for this equation to have a nontrivial solution (i.e. a solution besides the solution $A = B = 0$) the determinant of the coefficient matrix must be zero:

$$\sin(\omega\pi) + b\omega\cos(\omega\pi) + a\omega\cos(\omega\pi) - ab\omega 2\sin(\omega\pi) = 0. \tag{0.1}$$

Let us assume that $\cos(\omega\pi) \neq 0$ and let us divide the above equation by $\cos(\omega\pi)$. After rearranging terms we then obtain

$$\tan(\omega\pi) = \frac{(a+b)\omega}{ab\omega^2 - 1}. \tag{0.2}$$

Let us define $\omega_* = (ab)^{-1/2}$. The solutions $\omega > 0$ of equation (0.2) correspond to the intersections of the curves $y = \tan(\pi\omega)$ and $y = (a+b)\omega/(ab\omega^2 - 1)$ in the half plane $\omega > 0$. We label the first coordinates of the intersections with subscripts: $\omega_1 < \omega_2 < \ldots$. These values may be obtained by some numerical method such as Newton's Method. As far as we are concerned we will consider the problem of finding the eigenvalues completed at this point. We have $\lambda_k = \omega_k^2$

The corresponding eigenfunctions have the form $u_k(x) = A\cos(\omega_k x) + B\sin(\omega_k x)$, however we saw from our boundary conditions that $A = a\omega_k B$ so that we may write $u_k(x) = B[a\omega_k\cos(\omega_k x) + \sin(\omega_k x)]$. Finally, since we only need one eigenfunction per eigenvalue, we may set $B = 1$, so that

$$u_k(x) = a\omega_k\cos(\omega_k x) + \sin(\omega_k x).$$

The previous arguments are based on the assumption that $\cos(\omega\pi) \neq 0$. Is it possible to have solutions ω of equation for the zero determinant (0.1) for which $\cos(\omega\pi) = 0$. Clearly this would imply $\omega\pi = (2m-1)\pi/2$ for some positive integer m. Equation (0.1) then implies that $\sin(\omega\pi) = ab\omega^2\sin(\omega\pi)$, where $\sin(\omega\pi) = \pm 1$, and hence may be canceled to yield $ab\omega^2 = 1$, so that also $\omega = \omega_*$. To summarize, in the exceptional case that $(ab)^{-1/2} = m - 1/2$ for some positive integer m there will be another eigenpair

$$(\lambda_*, u_*) = \left((m-1/2)^2, a\omega_*\cos(\omega_* x) + \sin(\omega_* x)\right).$$

Remark: whenever you divide by $\cos$ **or** $\sin$ **of something to get an equation for** $\tan$ **or** $\cot$ **make sure to justify the division!**