Takehome

1MA208 Ordinary Differential Equations II

Due March 12, 2018

Name: ____

45% to 62% of the maximum point total - 3
62% to 80% of the maximum point total - 4
$\geq$ 80% of the maximum point total - 5

Problem 1. (Picard-Lindelöf Theorem). 10 points

a) How many solutions does the initial value problem

$$y' = 3(y-1)^{\frac{2}{3}}, \quad y(0) = 1$$

have? Is the right hand side in the equation Lipschitz? Make a conclusion about the applicability of the Picard-Lindelöf Theorem.

b) How many solutions does the initial value problem

$$y' = \frac{2y}{x-1}, \quad y(0) = 1$$

have? Is the right hand side in the equation Lipschitz? Make a conclusion about the applicability of the Picard-Lindelöf Theorem.

Problem 2. (Dependence on the initial conditions). 10 points

a) Let $F(x,t)$ be a continuous non-autonomous vector field on $\mathbb{R}^n \times \mathbb{R}$ that satisfies

$$\|F(x,t) - F(y,t)\| \leq L(t)\|x - y\|.$$

Show that the solution $\phi_t(x_0)$ of

$$x' = F, \ x(0) = x_0$$

satisfies

$$\|\phi_t(x_0) - \phi_t(y_0)\| \leq \|x_0 - y_0\| e^{\left|\int_0^t L(s)ds\right|}.$$

b) Suppose that $F(x,t)$ is a continuous non-autonomous vector field on $\mathbb{R} \times \mathbb{R}$ which is continuously differentiable in x. Show that we have

$$\frac{\partial \phi_t(x)}{\partial x} = \exp\left(\int_0^t F_1(\phi_s(x), s)ds\right),$$

where $F_1(x,t) := \frac{\partial F(x,t)}{\partial x}$,

Remark: This expression shows how quickly the solution for a smooth vector field in the 1D case (n=1) changes as the initial condition is changed.

Problem 3. (Linearization. Bifurcations). 10 points

Consider the system

$$x'(t) = x(t)^2 + y(t),$$
$$y'(t) = x(t) - y(t) + a,$$

where a is a real parameter.

a) Find all equilibrium points and derive the linearized equation at each.

b) Describe the behaviour of the linearized system at each equilibrium point.

c) Describe any bifurcation that occur.

Problem 4 (Stable/unstable theorem). 9 points

Consider the non-linear system

$$x_1' = \, - x_1, \tag{0.1}$$
$$x_2' = x_2 + x_1^2 \tag{0.2}$$

Find stable and unstable manifolds explicitly.

Problem 5. (Lyapunov function). 10 points
Consider the system

$$\begin{aligned} x'(t) &= -x(t) + y(t) + x(t)y(t), \\ y'(t) &= x(t) - y(t) - x(t)^2 - y(t)^3, \end{aligned} \tag{0.3}$$

a) Find the equilibrium points.

b) Construct a Lyapunov function and use it to analyze stability of the equilibria.

Problem 6 (Limit sets). 10 points

Prove that ω and α limit sets are invariant and closed. Additionally, show that if the flow ϕ_t preserves some compact set $D \subset \mathbb{R}^n$, then $\omega(X)$ and $\alpha(X)$ are non-empty for every $X \in D$.

Problem 7. (Poincare-Bendixson Theorem). 10 points
a) Consider the system

$$x' = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1 + x), \tag{0.4}$$

$$y' = y(1 - 4x^2 - y^2) - 2x(1 + x). \tag{0.5}$$

Perform the change of coordinates $\tilde{x} = 2x$. Write the system in the new coordinates $(\tilde{x}, y)$.

Pass to the polar coordinates. Find all periodic orbits. Which ones are α-limit cycles, which ones are ω-limit cycles?

b) Consider

$$r' = r(1 - r^2) + \mu r \cos(\theta)$$

Show that a closed orbit exists for small $\mu > 0$.

Construct a trapping region $r_{min} \le r \le r_{max}$ with

$$r_{min} < \sqrt{1 - \mu}, \quad \sqrt{1 + \mu} < r_{max}.$$

What type of limiting behaviour (equilibria, periodic orbits) can we have inside of this trapping region?

Problem 8. (Trapping regions, bifurcations), 10 points

Consider a modified Lotka-Volterra system

$$\begin{aligned} x' &= x\left(1 - \frac{x}{K}\right) - xy, \\ y' &= \rho(xy - y), \end{aligned}$$

where $K > 1$ and $\rho > 0$.

a) Find the equilibrium point different from $(0, 0)$. Linearize the system at that point and find its stability type (sink, saddle, etc).

b) Find the nullclines.

c) Draw an approximate phase portrait.

d) Construct a trapping region: a triangular region with one vertical side and one horizontal would be sufficient. The sides have to be sufficiently large - demonstrate that.

e) Use Poincare-Bendixson Theorem to analyze which limit sets may exist in the trapping region.

Problem 9. (Bifurcations), 10 points

Consider the system

$$\begin{aligned} x'(t) &= y, \\ y'(t) &= -\sin x - \beta y + \mu, \end{aligned} \tag{0.6}$$

a) Sketch the bifurcation diagram for this system: a graph of the equilibrium value of x vs μ.

b) Show that the equilibrium that satisfies $x \in (0, \pi/2)$ is a stable node, and that the equilibrium for $x \in (\pi/2, \pi)$ is a saddle.

c) Let us suppose that β is large: i.e., there is a lot of damping. Argue that for $\mu < 1$ the two halves of the unstable manifold from the saddle point fall into the stable equilibrium, one being very short and the other making a nearly complete revolution. Together these make up a "homoclinic cycle". Sketch the phase portrait.

(When μ crosses 1, the equilibria disappear and the homoclinic cycle becomes a limit cycle).