Problem 1

1) The coefficients of the F.S. are

$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$ = integrate by parts

twice $= \frac{2}{\pi}\left(x^2 \frac{\sin nx}{n} - 2x\,\frac{-\cos nx}{n^2} + 2\,\frac{-\sin nx}{n^3}\right)\Big|_0^{\pi} = \frac{4\pi \cos n\pi}{\pi\, n^2}$

$= 4\,\frac{\cos n\pi}{n^2}$; $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$

$$x^2 \approx 4\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}\cos nx + \frac{\pi^2}{3} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

2) • The function x^2 is continuous on $[-\pi,\pi]$
$f(t_+) = f(t_-)$ are equal at each point, and certainly exist

• Derivatives $f'(t_\pm)$ exist at every point

By 4.8.2 the Fourier series converges to $\frac{1}{2}(f(t_+)+f(t_-)) = f(t)$

3) Evaluate at π:

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2} =$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \implies \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

Problem 2 (see page 122)

Set $\varphi(x) = 1 - \frac{x}{\pi}$ and consider

$v(x,t) = u(x,t) - \varphi(x)$

v solves:

$\{\ v_t = k\,v_{xx}$

Therefore, v satisfies a homog. problem, (2)
and

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\kappa n^2 t} \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} v(x,0)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \frac{x}{\pi}\sin nx\, dx =$$

$$= \frac{2}{\pi^2}\int_0^{\pi} -\frac{1}{n} x\, d\cos nx = -\frac{2}{\pi^2 n}\left[x\cos nx\Big|_0^{\pi} - \int_0^{\pi}\cos nx\, dx\right]$$

$$= -\frac{2}{\pi^2 n}\left[\pi(-1)^n - \sin nx\Big|_0^{\pi}\right] = \frac{2}{\pi}\frac{(-1)^{n+1}}{n}$$

and

$$\boxed{u(x,t) = \frac{2}{\pi}\sum_1^{\infty} e^{-\kappa n^2 t}\frac{(-1)^{n+1}\sin nx}{n} + \left(1-\frac{x}{\pi}\right)}$$

Problem 3

1) $|f(t)| \le \sum_{n=1}^{\infty}\frac{1}{(n-1)!\,n^2} \le \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

To show uniform convergence carefully:

$\forall \varepsilon > 0,\ \exists N$, s.t. <u>for all $t \in \mathbb{R}$</u>

$$\left|f(t) - \sum_{n=1}^{N}\frac{1}{(n-1)!(t^2+n^2)}\right| = \left|\sum_{n=N+1}^{\infty}\frac{1}{(n-1)!(t^2+n^2)}\right|$$

$$\le \sum_{n=N+1}^{\infty}\frac{1}{n^2} < \varepsilon \text{ by convergence of } \sum_{n=1}^{\infty}\frac{1}{n^2}$$

2) First, we recall that $F[e^{-|t|}](\omega) =$

$= \frac{2}{1+\omega^2}$ (ex 6.2.2)

Now, $\hat{f}(\omega) = \int_{-\infty}^{\infty}\sum\frac{1}{(n-1)!(t^2+n^2)}e^{-it\omega}\,dt$

$$\underset{\substack{\text{Dominated}\\ \text{convergence}\\ \text{and 1)}}}{=} \sum_{1}^{\infty}\frac{1}{(n-1)!}\int\frac{e^{-it\omega}}{(n^2+t^2)}\,dt =$$

$$\sum^{\infty}\frac{1}{(n-1)!}(-1)\int^{-\infty}\frac{e^{i\omega(-t)}}{\ldots}\,d(-t) =$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{-\infty}^{\infty} \frac{e^{iwu}}{n^2 + u^2}\, du$$

Now, $\int_{-\infty}^{\infty} \frac{e^{iwu}}{n^2+u^2}\, du$ is the inverse F.T. of the function $\frac{1}{n^2+u^2}$ upto $\frac{1}{2\pi}$. If $n=1$, then we would have $\int_{-\infty}^{\infty} \frac{e^{iwu}}{1+u^2}\, du = 2\pi \frac{1}{2} e^{-|w|}$. For general n, we use the scaling property:

$$\frac{2}{n^2+w^2} = \frac{1}{n}\left(\frac{1}{n}\,\frac{2}{1+(w/n)^2}\right) = \frac{1}{n}\left(\frac{1}{n}\,\hat{g}\left(\frac{w}{n}\right)\right)$$

with $g(t) = e^{-|t|}$, thus $\frac{2}{n^2+w^2} = \frac{1}{n}\,\widehat{g(nt)}$ and $\frac{2}{n^2+w^2}$ is a F.T. of $\frac{1}{n} e^{-|nt|}$, hence every integral

$$\int_{-\infty}^{\infty} \frac{e^{iwu}}{n^2+u^2}\, du = \frac{1}{2}\int_{-\infty}^{\infty} e^{iwu}\,\frac{2}{n^2+u^2}\, du \quad \text{– is}$$

the inverse transform of $\frac{2}{n^2+u^2}$ upto $\frac{1}{\pi}$, and $\int_{-\infty}^{\infty} \frac{e^{iwu}}{n^2+u^2}\, du = \pi \frac{1}{n} e^{-|nw|}$, thus

$$\hat{f}(w) = \pi \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\,\frac{1}{n}\, e^{-|nw|} = \pi \sum_{n=1}^{\infty} \frac{1}{n!}\, e^{-n|w|}$$

3) First, notice that

$$\int_{-\infty}^{\infty} |f(t)|\,dt = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\,\frac{1}{(t^2+n^2)}\, dt \le$$

$$\le \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{-\infty}^{\infty} \frac{1}{t^2+1} = \sum_{n=1}^{\infty} \frac{\text{const}}{(n-1)!} \le$$

$\leq \sum_{n=3}^{\infty} \frac{const}{(n-2)^2} + const = \sum_{k=1}^{\infty} \frac{const}{k^2} + const < \infty$ (4)

by convergence of $\sum \frac{1}{k^2}$, hence $\int_{-\infty}^{\infty} |f(t)|$ is bounded and $f \in L^1(\mathbb{R})$ (also in $L^2(\mathbb{R})$)

Next, consider ~~[illegible]~~

$g(t) = \chi_{[-a,a]}$ for some $a > 0$

$\hat{g}(w) = 2\frac{\sin aw}{w}$ by ex. 6.2.1, and

$\hat{g}(w)$ is in $L^2(\mathbb{R})$ since the integral

$$4\int_{-\infty}^{\infty} \frac{\sin^2 aw}{w^2} dw = 4a \int_{-\infty}^{\infty} \frac{\sin^2 aw}{w^2 a^2} daw =$$

$$= 4a \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} du < \infty \text{ for every fixed } a.$$

Plancherel's formula 2):

$$\int_{\mathbb{R}} f(t)\overline{g(t)}\, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(w)\, \overline{\hat{g}(w)}\, dw$$

Hence

$$\int_{-a}^{a} f(t)\, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \pi \sum_{n=1}^{\infty} \frac{1}{n!} e^{-n|w|} \cdot 2 \frac{\sin aw}{w} dw$$

$$\underset{D.C.}{=} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} e^{-n|w|} \frac{\sin aw}{w} dw =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} e^{-|nw|} \frac{\sin \frac{a}{n} wn}{wn} dwn = \sum \frac{1}{n!} \int_{\mathbb{R}} e^{-|u|} \frac{\sin(\frac{a}{n})u}{u} du$$

$$= \sum \frac{1}{n!}\, 2 \int_{0}^{\infty} e^{-u} \frac{\sin(\frac{a}{n})u}{u} dt =$$

$$= \sum \frac{1}{n!}\, 2\, L\left[\frac{\sin(\frac{a}{n})u}{u}\right](+1) = 2\sum \frac{1}{n!} \left(\frac{a}{n}\right) \times$$

= this Laplace transform has been computed in class $= 2\sum \frac{1}{n!}\left(\frac{\pi}{2} - \arctan\frac{n}{a}\right)$

We have:

$$\int_{-a}^{a} f(t)\,dt = 2\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\pi}{2} - \arctan\left(\frac{n}{a}\right)\right)$$

$$\|f\|_1 = \lim_{a\to\infty} \int_{-a}^{a} f(t)\,dt = 2\sum_{n=1}^{\infty} \frac{\pi}{n!} = \boxed{2\pi\sum_{n=1}^{\infty}\frac{1}{n!}}$$

Problem 4:

1) $$\widehat{H(t)e^{-a|t|}}(w) = \int_{-\infty}^{\infty} H(t)e^{-a|t|}e^{-iwt}\,dt$$

$$= \int_0^{\infty} e^{-at}e^{-iwt}\,dt = \frac{1}{-a-iw}\, e^{(-a-iw)t}\Big|_0^{\infty}$$

$$= \frac{1}{a+iw}$$

$$\widehat{e^{-a|t|}}(w) = \int_{-\infty}^{0} e^{at}e^{-iwt} + \int_0^{\infty} e^{-at}e^{-iwt}\,dt$$

$$= \frac{1}{a-iw}\, e^{(a-iw)t}\Big|_{-\infty}^{0} + \frac{1}{a+iw} = \frac{1}{a-iw} + \frac{1}{a+iw} =$$

$$= \frac{2a}{a^2+w^2}$$

2) Fourier transform the eq-n

$$-w^2\hat{x} + 2\gamma iw\hat{x} + q^2\hat{x} = \hat{f}$$

$$\hat{x} = \hat{f}\cdot\frac{1}{q^2+2\gamma iw - w^2}$$

Define:

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{q^2+2\gamma i\omega-\omega^2}\, d\omega \text{, then}$$

this is nothing but the inverse formula for $\hat{G}(\omega) \equiv \frac{1}{q^2+2\gamma i\omega-\omega^2}$. But we need to check that the inverse formula $G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{G}(\omega)\, d\omega$ applies here. Compare with theorem 6.6.2. Function $\hat{G}(\omega)$ is in $L^1(\mathbb{R})$ since $\frac{1}{-\omega^2+a}$ is integrable ($\hat{G}(\omega)$ is $\approx \frac{1}{\omega^2}$ for large ω). The fact that $G(t) \in L^1(\mathbb{R})$ will be checked after $G(t)$ is computed.

We have: $\hat{x} = \hat{f}\cdot\hat{G} \Rightarrow x(t) = f * G(t)$

3) $$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{q^2+2\gamma i\omega-\omega^2}\, d\omega =$$

$$= \frac{1}{2\pi}\left\{\int \frac{e^{i\omega t}}{q^2-\gamma^2+\gamma^2+2\gamma i\omega-\omega^2}\, d\omega\right\} = \frac{1}{2\pi}\int\frac{e^{i\omega t}}{\rho^2+(i\omega+\gamma)^2}\, d\omega$$

$$= \frac{1}{2\pi}\int\frac{e^{i\omega t}}{\rho^2-(\omega-i\gamma)^2} = \frac{1}{2\pi}\left\{\int\frac{e^{i\omega t}}{\rho-(\omega-i\gamma)}+\int\frac{e^{i\omega t}}{\rho+(\omega-i\gamma)}\right\}\cdot\frac{1}{2\rho}$$

$$= \frac{1}{4\pi\rho}\left\{\int\frac{e^{i\omega t}}{\rho+i\gamma-\omega}+\int\frac{e^{i\omega t}}{\rho-i\gamma+\omega}\right\} =$$

$$= \frac{1}{4\pi\rho}\left\{\frac{1}{i}\int\frac{e^{i\omega t}}{(-i\rho+\gamma)+i\omega}-\frac{1}{i}\int\frac{e^{i\omega t}}{(i\rho+\gamma)+i\omega}\right\} = \text{compare}$$

with part 1) $$= \frac{1}{4\pi\rho i}\left\{\int\overline{H(t)\, e^{-(-i\rho+\gamma)|t|}}\, e^{i\omega t} - \int\overline{H(t)\, e^{-(i\rho+\gamma)|t|}}\, e^{i\omega t}\right\} =$$

$= \frac{1}{2pi}\{H(t)e^{-\gamma t}e^{ipt} - H(t)e^{-\gamma t}e^{-ip}\} =$

$= \boxed{\frac{H(t)}{p}e^{-\gamma t}\sin pt}$

Problems 5

$y''' - 3y'' + 3y' - y = t^2 e^t$

$L[y'] = s\,L[y] - y(0)$

$L[y''] = s\,L[y'] - y'(0) = s^2 L[y] - sy(0) - y'(0)$

$L[y'''] = s\,L[y''] - y''(0) = s^3 L[y] - s^2 y(0) - sy'(0) - y''(0)$

Then, Laplace transform the eq-n

$s^3 L[y] - s^2 y_0 - s y_0' - y_0'' - 3s^2 L[y] + 3 s y_0 + 3y_0' +$

$+ 3sL[y] - 3y_0 - L[y] = \int_0^\infty e^{-st} t^2 e^t dt$

$(s-1)^3 \tilde{y} + (-s^3 + 3s - 3) y_0 + (3-s) y_0' - y_0'' = \int_0^\infty e^{-st} t^2 e^t dt$

Next, $\int_0^\infty e^{-st} t^2 e^t dt = \int_0^\infty \frac{1}{1-s} t^2 de^{-st+t} = \frac{1}{1-s} t^2 \cdot$

$\cdot e^{-st+t}\Big|_0^\infty - \frac{2}{1-s}\int_0^\infty t e^{-st+t} dt = \frac{2}{s-1}\int_0^\infty \frac{1}{1-s} t\, de^{-st+t} dt$

$= \frac{2}{s-1}\left(0 - \frac{1}{1-s}\int_0^\infty e^{-st+t} dt\right) = \frac{2}{(s-1)^2}\frac{1}{1-s} e^{-st+t}\Big|_0^\infty$

$= \frac{2}{(s-1)^3}$, then

$\tilde{y} = \frac{2}{(1-s)^6} + y_0 \frac{s^2 - 3s + 3}{(s-1)^3} + y_0' \frac{s-3}{(s-1)^3} + y''\frac{1}{(s-1)^3}$

$$= \frac{2}{(s-1)^6} + y_0 \cancel{s} \frac{(s^2-2s+1)+2-s}{(s-1)^3} + y_0' \frac{(s-1)-2}{(s-1)^3} + y_0'' \frac{1}{(s-1)^3} =$$

$$= \frac{2}{(s-1)^6} + y_0\left(\frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}\right) + y_0'\left(\frac{1}{s-1} - \frac{2}{(s-1)^3}\right) +$$

$$+ y'' \frac{1}{(s-1)^3} = \frac{2}{5!} \mathcal{L}[t^5 e^t](s) + y_0 \mathcal{L}\left[e^t - te^t + \tfrac{1}{2}t^2 e^t\right]$$

$$+ y_0' \mathcal{L}[e^t - t^2 e^t] + y_0'' \mathcal{L}\left[\tfrac{1}{2} t^2 e^t\right]$$

$$\boxed{\begin{aligned} y(t) = \frac{2}{5} t^5 e^t + y_0\left(e^t - te^t + \frac{t^2}{2} e^t\right) \\ y_0'(e^t - t^2 e^t) \\ y_0''\left(\tfrac{1}{2} t^2 e^t\right) \end{aligned}}$$