1MA211, Fourieranalys

Kod: ____

Problem 1.

1) Betrakta den 2π-periodiska funktionen definerad på $(-\pi, \pi)$ som

$$f(t) = e^t + e^{-t}.$$

Bestäm fourierserien.

2) Bestäm om fourierserien konvergerar (använd en av konvergenssatserna).
3) Bestäm fourierserien för $f'(t)$ (använd deriveringssatsen: $\hat{f}'(n) = \ldots$).
4) Kan man använda deriveringssatsen ($\hat{f}'(n) = \ldots$) att bestämma fourierserien för $f''(t)$? Förklara.

Solution:

1) Use partial integration twice to get

$$\begin{aligned}\int_0^\pi e^t \cos(nt)dt &= e^t \cos(nt)|_0^\pi + n\int_0^\pi e^t \sin(nt)dt \\ &= ((-1)^n e^\pi - 1) + n\int_0^\pi e^t \sin(nt)dt \\ &= ((-1)^n e^\pi - 1) + n\left(e^t \sin(nt)|_0^\pi - n\int_0^\pi e^t \cos(nt)dt\right) \\ &= ((-1)^n e^\pi - 1) - n^2\int_0^\pi e^t \cos(nt)dt,\end{aligned}$$

and similarly

$$\begin{aligned}\int_0^\pi e^{-t} \cos(nt)dt &= -e^{-t} \cos(nt)|_0^\pi - n\int_0^\pi e^{-t} \sin(nt)dt \\ &= \left(1 - (-1)^n e^{-\pi}\right) - n\int_0^\pi e^{-t} \sin(nt)dt \\ &= \left(1 - (-1)^n e^{-\pi}\right) - n\left(-e^{-t} \sin(nt)|_0^\pi + n\int_0^\pi e^{-t} \cos(nt)dt\right) \\ &= \left(1 - (-1)^n e^{-\pi}\right) - n^2\int_0^\pi e^{-t} \cos(nt)dt.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^\pi \left(e^t + e^{-t}\right)\cos(nt)dt &= \left((-1)^n e^\pi - 1\right) + \left(1 - (-1)^n e^{-\pi}\right) - n^2 \int_0^\pi \left(e^{-t} + e^{-t}\cos(nt)dt\right) \\ &= (-1)^n \left(e^\pi - e^{-\pi}\right) - n^2 \int_0^\pi \left(e^{-t} + e^{-t}\cos(nt)dt\right),\end{aligned}$$

and one can express the integral $\int_0^\pi \left(e^t + e^{-t}\right)\cos(nt)dt$ out of this identity:

$$\frac{2}{\pi}\int_0^\pi \left(e^t + e^{-t}\right)\cos(nt)dt = 2(-1)^n \frac{e^\pi - e^{-\pi}}{\pi(1+n^2)}.$$

Thus Fourier series is given by

$$f(t) = \frac{e^\pi - e^{-\pi}}{\pi}\left(1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2}\cos(nt)\right).$$

2) We use Theorem 4.8.3.: *if an $L^1(\mathbb{T})$ function is Hölder continuous, then its Fourier series converges to the value of the function at every point.*

The derivative of the function $e^{\pm t}$ is $\pm e^{\pm t}$, and the absolute value of this derivative is bounded by e^π on $[-\pi, \pi]$. We have

$$\begin{aligned}|e^x - e^{-x} - (e^y - e^{-y})| &\le |e^x - e^y| + |e^{-x} - e^{-y}| \\ &\le \max_{z\in[-\pi,\pi]}\left|\frac{de^z}{dz}\right||x-y| + \max_{z\in[-\pi,\pi]}\left|\frac{de^{-z}}{dz}\right||x-y| \\ &\le 2e^\pi|x-y|.\end{aligned}$$

Thus this function is Lipschitz, and the theorem applies.

3) $\hat{f'}(n) = in\hat{f}(n)$. Therefore, since $\hat{f}(n) = \frac{a_n - ib_n}{2}$, $n \ge 0$, and $\hat{f}(-n) = \frac{a_n + ib_n}{2}$, $n \ge 0$, where a_n and b_n are the coefficients of the cosine-sine Fourier series, we have

$$\hat{f'}(n) = in\hat{f}(n) = in\frac{a_{|n|}}{2},$$

and

$$f'(t) \approx \frac{e^\pi - e^{-\pi}}{\pi}\sum_{n=-\infty}^{\infty} \frac{(-1)^n in}{1+n^2}e^{int},$$

and since

$$\frac{-(-1)^n in}{1+n^2}e^{-int} + \frac{(-1)^n in}{1+n^2}e^{int} = \frac{-2(-1)^n n}{1+n^2}\sin(nt),$$

we have

$$f'(t) \approx -\frac{e^\pi - e^{-\pi}}{\pi}\sum_{n=1}^{\infty} \frac{2(-1)^n n}{1+n^2}\sin(nt).$$

This series converges to the value of the derivative at all poins, but $\pi + 2n\pi$, $n \in \mathbb{Z}$, where it converges to

$$\frac{f'((\pi+2n\pi)_+) + f'((\pi+2n\pi)_-)}{2},$$

where $f'((\pi+2n\pi)_\pm)$ are the left and right derivatives at points $\pi + 2n\pi$.

4) Theorem 3.5.2 that says that $\hat{f''}(n) = in\hat{f'}(n)$ *if* f' *is itself continuously differentiable* does not apply since f' is discontinuous at points $\pi + 2n\pi$, $n \in \mathbb{Z}$.

One can see that $\hat{f''}(n) = in\hat{f'}(n)$ does not hold, because, on one hand, $f''(t) = e^t - e^{-t} = f(t)$ at all $t \neq \pi + 2n\pi$, $n \in \mathbb{Z}$. On the other hand,

$$\sum_{-\infty}^{\infty} in\hat{f'}(n)e^{int} = -\sum_{-\infty}^{\infty} n^2 \frac{a_{|n|}}{2} e^{int} = -\frac{e^\pi - e^{-\pi}}{\pi} \sum_{n=1}^{\infty} \frac{2(-1)^n n^2}{1+n^2} \cos(nt). \tag{0.1}$$

This last series is clearly not equal to $e^t - e^{-t}$ on $(-\pi, \pi)$. For example, $f''(0) = e^0 - e^{-0} = 0$, while the series (0.1), equal to

$$-\frac{e^\pi - e^{-\pi}}{\pi} \sum_{n=1}^{\infty} \frac{2(-1)^n n^2}{1+n^2},$$

does not converge since the necessary test of convergence fails here:

$$\lim_{n\mapsto\infty} \frac{n^2}{1+n^2} = 1 \neq 0.$$

Problem 2.

$$\begin{cases} 1, & 0 \le |x| \le d, \\ 0, & d \le |x| \le \pi. \end{cases}$$

1) Bestäm fourierserien.
2) Använd Parsevals formler att beräkna $\sum_{n=1}^{\infty} \frac{\sin^2 nd}{n^2}$.
3) I vilket rum kan man använda Parsevals formler?

Solution:

1) The function is even, therefore, the Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos(nx),$$

with

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^d \cos(nx)dx = \frac{2}{\pi}\frac{\sin(nd)}{n},$$

and $b_n = 0$, thus

$$f(x) = \frac{d}{\pi} + \frac{2}{\pi}\sum_1^\infty \frac{\sin(nd)}{n}\cos(nx).$$

2) We have the following Parseval's formula in the trigonometric form:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_n |f_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^\infty(|a_n|^2 + |b_n|^2),$$

i.e.

$$\frac{1}{2\pi}\int_{-d}^{d} 1dx = \frac{d^2}{\pi^2} + \frac{1}{2}\sum_{n=1}^\infty \frac{4}{\pi^2}\frac{\sin^2(nd)}{n^2},$$

or

$$\frac{d}{\pi} = \frac{d^2}{\pi^2} + \frac{2}{\pi^2}\sum_{n=1}^\infty \frac{\sin^2(nd)}{n^2} \implies \sum_{n=1}^\infty \frac{\sin^2(nd)}{n^2} = \frac{d\pi}{2}\left(1 - \frac{d}{\pi}\right).$$

3) Whenever $f \in L^2(\mathbb{T})$.

Problem 3.
Bestäm lösningen till

$$u_t(x,y) = \kappa u_{xx}(x,t), \quad x \in \mathbb{R}, \quad t \geq 0,$$

med begynnelsevillkoret

$$u(x,0) = e^{-x}.$$

(Tips: reducera integralen till $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}du = 1$.)

Solution:

The solution of the heat equation on a real line is equal to the convolution of the initial data with the heat kernel:

$$\begin{aligned}
u(x,t) = \left(e^{-x} * H_{2kt}\right)(x) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} e^{-y} dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2+4kty}{4kt}} dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{\left[(y-x)^2+4kt(y-x)+(2kt)^2\right]-(2kt)^2+x}{4kt}} dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x+2kt)^2}{4kt}+kt-x} dy \\
&= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{-\infty}^{\infty} e^{-\frac{(y-x+2kt)^2}{4kt}} d\frac{y-x+2kt}{\sqrt{4kt}} \\
&= e^{kt-x} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du \\
&= e^{kt-x}.
\end{aligned}$$

Notice, that this solution is simply the "wave" e^{-x} propagating to the right with speed k.

Problem 4.
Erinra dig om sats 7.2.2 on faltningen av en funktion med värmeledningskärnan:

Antag $f \in L^1(\mathbb{R})$. Då är

$$f * H_\tau(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) e^{-\tau\omega^2/2} e^{i\omega t} d\omega.$$

Visa med hjälp av satsen att funktionerna $f * H_\tau(t)$ är oändligt deriverbara.

(Tips: Skriv derivatorna som "lim" av en differens, och använd Dominerad Konvergensen.)

Solution:

We proceed by induction.

Step 1: First, we show that the first derivative exists

$$\begin{aligned}
\frac{f * H_\tau(t+\epsilon) - f * H_\tau(t)}{\epsilon} &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) e^{-\tau\omega^2/2} \frac{e^{i\omega(t+\epsilon)} - e^{i\omega t}}{\epsilon} d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) e^{-\tau\omega^2/2} e^{i\omega t} \omega \frac{e^{i\omega\epsilon} - 1}{\omega\epsilon} d\omega.
\end{aligned}$$

The functions $g_\epsilon(\omega) = \frac{e^{i\omega\epsilon}-1}{\omega\epsilon}$ are uniformly bounded from above for all $\omega \in \mathbb{R}$ and all $\epsilon \in \mathbb{R}$:

$$\left|\frac{e^{i\omega\epsilon}-1}{\omega\epsilon}\right| \leq \left|\frac{\cos(\omega\epsilon)-1}{\omega\epsilon}\right| + \left|\frac{\sin(\omega\epsilon)}{\omega\epsilon}\right| < 2$$

(look at the graphs of the functions $|\cos x - 1|$ and $|\sin x|$ over $\mathbb{R}$ and compare them with the graph of $|x|$).

Notice, that the function

$$h(\omega) = \hat{f}(\omega)e^{-\tau\omega^2/2}e^{i\omega t}\omega$$

under the integral is in $L^1(\mathbb{R})$ as a function of ω. Indeed, f is in $L^1(\mathbb{R})$ by the condition of the problem, and, by Theorem 6.3.1, $|\hat{f}| \leq \|f\|_1$, and hence $\hat{f}$ is bounded on $\mathbb{R}$. Thus, the absolute value of the function $h(\omega)$ is bounded from above by *const* $\omega e^{-\tau\omega^2/2}$ - which is in Ł$^1(\mathbb{R})$ for every $\tau > 0$. Now we can use the Dominated Convergence theorem (4.1.1 in the book, with $h(\omega)$ playing the role of $f(t)$ in the Theorem, and $g_\epsilon(\omega)$ - that of $g_n(t)$):

$$\begin{aligned}\lim_{\epsilon\mapsto 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-\tau\omega^2/2}e^{i\omega t}\omega\frac{e^{i\omega\epsilon}-1}{\omega\epsilon}d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-\tau\omega^2/2}e^{i\omega t}\omega\lim_{\epsilon\mapsto 0}\frac{e^{i\omega\epsilon}-1}{\omega\epsilon}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-\tau\omega^2/2}e^{i\omega t}i\omega d\omega.\end{aligned}$$

The last integral converges since the integrand is equal to $ih(\omega)$, which is in $L^1(\mathbb{R})$, as we have just argued.

Step 2: Assume that the k-th derivative exists and is equal to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-\tau\omega^2/2}e^{i\omega t}(i\omega)^k d\omega.$$

Step 3: To complete induction, we proof existence of $k+1$-st derivative. Consider the difference of the k-th derivatives evaluated at points $t+\epsilon$ and t, divided by ϵ:

$$\frac{1}{\epsilon}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-\tau\omega^2/2}\left(e^{i\omega(t+\epsilon)} - e^{i\omega t}\right)(i\omega)^k d\omega.$$

Repeat arguments from the Step 1) with $g_\epsilon(\omega)$ as before, and

$$h(\omega) = \hat{f}(\omega)e^{-\tau\omega^2/2}e^{i\omega t}(i\omega)^k\omega.$$

We have: $|h(\omega)| <$ *const* $e^{-\tau\omega^2/2}\omega^{k+1}$, this bound being a function in $L^1(\mathbb{R})$ (since the exponential always decreases much faster than any power of ω). Again, by the

Dominated Convergence Theorem,

$$\lim_{\epsilon\mapsto 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-\tau\omega^2/2}\frac{e^{i\omega(t+\epsilon)}-e^{i\omega t}}{\epsilon}(i\omega)^k d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-\tau\omega^2/2}(i\omega)^k\omega e^{i\omega t}\lim_{\epsilon\mapsto 0}\left(\frac{e^{i\omega\epsilon}-1}{\omega\epsilon}\right)d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-\tau\omega^2/2}(i\omega)^{k+1}e^{i\omega t}d\omega,$$

the last integral being convergent.

Problem 5.
Bestäm en lösning till

$$\begin{cases} y''(t)-3y'(t)+2y(t)=4e^{2t} \\ \quad y(0)=-3, \quad y'(0)=5 \end{cases}$$

genom Laplacetransformen.

Solution:

Take the Laplace transform of the equation:

$$L[y'']-3L[y']+2L[y]=4L[e^{2t}].$$

Now,

$$\begin{aligned} L[y''](s) &= s^2\tilde{y}(s)-sy(0)-y'(0), \\ L[y'](s) &= s\tilde{y}(s)-y(0), \\ L[e^{2t}](s) &= \frac{1}{s-2}, \end{aligned}$$

then

$$s^2\tilde{y}(s)+3s-5-3s\tilde{y}(s)-9+2\tilde{y}(s)=\frac{4}{s-2},$$

and

$$\begin{aligned} \tilde{y}(s) &= \frac{4}{(s-2)^2(s-1)}+\frac{14-3s}{(s-2)(s-1)} = \frac{-3s^2+20s-24}{(s-2)^2(s-1)} \\ &= \frac{A}{s-2}+\frac{B}{(s-2)^2}+\frac{C}{s-1} = \frac{A(s-2)(s-1)+B(s-1)+C(s-2)^2}{(s-2)^2(s-1)}. \end{aligned}$$

Equating the powers of s:

$$A+C=-3,\ -3A+B-4C=20,\ 2A-B+4C=-24 \implies A=4,\ C=-7,\ B=4,$$

and

$$\begin{aligned}
\tilde{y}(s) &= \frac{4}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{s-1} \implies \\
y(t) &= 4e^{2t} + 4te^{2t} - 7e^t
\end{aligned}$$

Problem 6.
Bestäm en lösning till

$$\begin{aligned}
u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad 0 < x < 1, t > 0, \\
u(0,t) &= u(1,t) = 0, \quad t > 0, \\
u(x,0) &= \sin(5\pi x) + 2\sin(7\pi x), \quad 0 < x < 1, \\
u_t(x,0) &= 0, \quad 0 < x < 1.
\end{aligned}$$

Solution:

We look for the solution in the form

$$u(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x)\left(\alpha_k \cos(ck\pi t) + \beta_k \sin(ck\pi t)\right).$$

The second initial condition gives:

$$\begin{aligned}
u_t(x,0) &= \sum_{k=1}^{\infty} \sin(k\pi x)\left(-ck\pi\alpha_k \sin(ck\pi 0) + ck\pi\beta_k \cos(ck\pi 0)\right) \\
&= c\pi \sum_{k=1}^{\infty} \sin(k\pi x) k\beta_k = 0
\end{aligned}$$

which means that all $\beta_k = 0$.

The first initial condition gives:

$$\begin{aligned}
u(x,0) &= \sum_{k=1}^{\infty} \sin(k\pi x)\left(\alpha_k \cos(ck\pi 0) + \beta_k \sin(ck\pi 0)\right) \\
&= \sum_{k=1}^{\infty} \sin(k\pi x)\alpha_k = \\
&= \sin(5\pi x) + 2\sin(7\pi x),
\end{aligned}$$

which implies that all $\alpha_5 = 1$, $\alpha_7 = 2$ and all other $\alpha_k = 0$.

Therefore,

$$u(x,t) = \sin(5\pi x)\cos(5c\pi t) + 2\sin(7\pi x)\cos(7c\pi t).$$