1MA216, Partial Differential Equations

Code: ____________________

Problem 1.

1) Describe, in your own words, the method of parallelograms for the boundary/initial value problem for the wave equation.
2) Solve the IVP
$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz}, \\ u(x, y, z, 0) = x^2 + y^2 \end{cases}$$
First using the *Kirchhoff's formula* in 3D, and second using the *method of descent* to 2D from 3D.

Problem 2.

1) Give the definition of a weak solution for differential operator L.
2) State the Cauchy-Kovalevskaya theorem.
3) What is the principle symbol of an operator?
4) Define what is a hyperbolic PDE using the notion of a characteristic surface.

Problem 3.
Find the potential generated by an infinitely long charged wire with linear charge density ρ.

Problem 4.

1) What is the difference between Lax-Milgram and Riesz representation theorems?
2) For a bounded domain $\Omega \subset \mathbb{R}^n$, let
$$\lambda_1 = \inf_{u \in C_0^1(\Omega)} \frac{\left[\int_\Omega |\nabla u|^2\right]^{\frac{1}{2}}}{\left[\int_\Omega |u|^2\right]^{\frac{1}{2}}}.$$

- Prove that $\lambda_1 > 0$.
- Prove that for every $f \in L^2(\Omega)$ and constant $c < \lambda_1$, the Dirichlet problem

$$\begin{cases} \Delta u + cu = f & \text{in} \quad \Omega \\ \quad u = 0 & \in \quad \partial\Omega \end{cases}$$

has a weak solution in $H_0^{1,2}(\Omega)$.

Problem 5.
Consider the functional

$$F(u) = \int_\Omega L(\nabla u, x)dx,$$

where the *Lagrangian* $L(p,x) \in C^{1,1}(\mathbb{R}^n, \Omega)$, $\Omega \in \mathbb{R}^n$ - bounded. Assume a *uniform Legendre convexity condition*:

$$\sum_{i,j}^{n} L_{p_i p_j}(p,x)\xi_i\xi_j \geq \epsilon|\xi|^2$$

for all $p \in \mathbb{R}^n, x \in \Omega, \xi \in \mathbb{R}^2$. Here, $\epsilon > 0$.

Show that F is strictly convex in

$$A = \{u \in H^{1,2}(\Omega) : u - u|_{\partial\Omega} \in H_0^{1,2}(\Omega)\}.$$