Problem: 1.1.5(a)

$$\begin{cases} x u_x + y u_y + u_z = u \\ u(x,y,0) = h(x,y). \end{cases}$$

Method of characteristics

u is a function of 3 variables: (x,y,z). We will call the coordinate in $\mathbb{R}^4$, corresponding to u as w.

Characteristic eq-ns:

$$\frac{dx}{dt} = x \; ; \; \frac{dy}{dt} = y \; ; \; \frac{dz}{dt} = 1 \; ; \; \frac{dw}{dt} = w$$

$$x = x_0 e^t \; ; \; y = y_0 e^t \; ; \; z = t + z_0 \; ; \; w = w_0 e^t$$

IC: $x = x_0 e^t \; ; \; y = y_0 e^t \; ; \; z = t \; ; \; w = h(x_0, y_0) e^t$

We now need to obtain w as a function of x, y and z only, eliminating "constants" x_0, y_0 and parameter t:

$$w = h(x_0, y_0) e^z \; ; \; x_0 = \frac{x}{e^z} \; ; \; y_0 = \frac{y}{e^z}$$

$$w = h(xe^{-z}, ye^{-z}) e^z$$

$$\boxed{u = h(xe^{-z}, ye^{-z}) e^z}$$

Problems 1.1.6(b) and 1.1.8 done in class (problem session)

Problem: Find solution of the eq-n $u_x + 2u_y = 0$, such that its graph passes through the curve:

$$\Gamma: \; x = s + s^2, \; y = 2s^2, \; z = s^2$$

Characteristic eq-ns:

$$\frac{dx}{dt} = 1; \quad \frac{dy}{dt} = 2; \quad \frac{dz}{dt} = 0$$

$$x = t + x_0; \quad y = 2t + y_0; \quad z = z_0$$

At $t=0$ point (x_0, y_0, z_0) lies on Γ:

$$x = t + s + s^2; \quad y = 2t + 2s^2, \quad z = s^2$$

Express s from first and second eq-ns:

$$\begin{cases} 2x = 2t + 2s + 2s^2 \\ y = 2t + 2s^2 \end{cases} \Rightarrow 2s = 2x - y \Rightarrow s = x - \frac{y}{2}$$

I.e., $z = s^2 = \left(x - \frac{y}{2}\right)^2$

$$\boxed{u = \left(x - \frac{y}{2}\right)^2}$$

Problem: 1.7.(a)

$$(x+u)u_x + (y+u)u_y = 0$$

Characteristics:

$$\frac{dx}{dt} = x + z; \quad \frac{dy}{dt} = y + z; \quad \frac{dz}{dt} = 0$$

$$\frac{dx}{dt} = x + z_0; \quad \frac{dy}{dt} = y + z_0; \quad z = z_0$$

$$\frac{d(x+z_0)}{dt} = x + z_0 \Rightarrow x + z_0 = c_1 e^t$$

Similarly, $y + z_0 = c_2 e^t$

At $t=0$: $x_0 + z_0 = c_1 \Rightarrow c_1 = x_0 + z_0$

$y_0 + z_0 = c_2$, i.e.

$$\begin{cases} x = (x_0+z_0)e^t - z_0 \\ y = (y_0+z_0)e^t - z_0 \\ z = z_0 \end{cases}$$

are eq-ns of characteristics starting at point (x_0, y_0, z_0)

Find two functions constant on characteristics. One can be taken to be $\varphi(x,y,z) = z$, the other

$$\psi(x,y,z) = \frac{x+z}{y+z} = \frac{x+z_0}{y+z_0} = \frac{(x_0+z_0)e^t}{(y_0+z_0)e^t} = \frac{x_0+z_0}{y_0+z_0} = \text{const}$$

Now $F(\varphi, \psi) = 0$ for an arbitrary C^1 function F, assume $F_\varphi \neq 0$, then

$$\varphi = f(\psi) \Rightarrow \boxed{z = f\left(\frac{x+z}{y+z}\right)}$$

an implicit solution

Problem: Proof that IVP

$$x u_x + t u_t = u^3, \quad u(x,0) = x$$

has no solutions

Characteristic eq-ns: (t here is a variable, we call parameter along the characteristics, τ)

$$\frac{dx}{d\tau} = x; \quad \frac{dt}{d\tau} = t$$

$$\frac{dz}{d\tau} = z^3$$

$$x = x_0 e^{\tau}; \; t = t_0 e^{\tau}; \quad \frac{d\left(-\frac{1}{2} z^{-2}\right)}{d\tau} = 1$$

$$x = x_0 e^{\tau}; \; t = t_0 e^{\tau}; \quad -\frac{1}{2} z^{-2} = \tau + const$$

$$z^2 = -\frac{1}{2} \frac{1}{\tau + const}$$

What is the value of const so that at $\tau = 0$ the characteristic passes through point (x_0, t_0, z_0)?

$$z_0^2 = -\frac{1}{2} \frac{1}{0 + const} \Rightarrow const = -\frac{1}{2} \frac{1}{z_0^2}$$

$$z^2 = -\frac{1}{2} \frac{1}{\tau - \frac{1}{2 z_0^2}} = \frac{z_0^2}{1 - 2\tau z_0^2}$$

$$\begin{cases} x = x_0 e^{\tau} \\ t = t_0 e^{\tau} \\ z^2 = \frac{z_0^2}{1 - 2\tau z_0^2} \end{cases}$$

the only possibility for $t = 0$ for a finite value of parameter τ is $t_0 = 0$, that is, characteristics are in the z-x plane, i.e. $u = u(x)$ only!! But then, the solution of the ODE $x u_x = u^3$ is

$$-\frac{1}{2} u^{-2} = \ln|x| + C \Rightarrow u^2 = \frac{1}{-2C - 2\ln|x|}$$

$$u = \pm \sqrt{\frac{1}{-2C - 2\ln|x|}}$$ which does not match the IC.

Problem: Solve the IVP

$$u^2u_x + u_t = u, \quad u(x,0) = x \quad (*)$$

$\boxed{\text{Remark}}$: on the problem set this eq-n had a typo:

$$u^2u_x + t\,u_t = u.$$

If t is kept, then the problem has no solution similarly to the previos one.

We work on $(*)$:

$$\frac{dx}{d\tau} = z^2, \quad \frac{dt}{d\tau} = 1, \quad \frac{dz}{d\tau} = z$$

$x = \frac{1}{2}z_0^2e^{2\tau} + c, \quad t = \tau + t_0, \quad z = z_0e^{\tau}$

$x_0 = \frac{1}{2}z_0^2 + const \Rightarrow const = x_0 - \frac{1}{2}z_0^2$

$$\begin{cases} x = \frac{1}{2}z_0^2(e^{2\tau} - 1) + x_0 \\ t = \tau + t_0 \\ z = z_0e^{\tau} \end{cases}$$
– a characteristic passing through (x_0, y_0, z_0) at $\tau = 0$

IVP: $u(s,0) = s \Rightarrow t_0 = 0$

$z_0 = s$

$s = x(0) = \frac{1}{2}z_0^2(1-1) + x_0$

$x_0 = s$, i.e.

$$\begin{cases} z = se^{\tau} \\ t = \tau \\ x = \frac{1}{2}s^2(e^{2\tau} - 1) + s \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} z = se^{t} \\ x = \frac{1}{2}z^2 - \frac{1}{2}s^2 + s \end{cases} \Rightarrow \begin{cases} s = z\cdot e^{-t} \\ x = \frac{1}{2}z^2 - \frac{1}{2}z^2\cdot e^{-2t} + ze^{-t} \end{cases}$$

$$\boxed{\frac{1}{2}z^2(1 - e^{-2t}) + ze^{-t} - x = 0}$$ is an implicit solution

Problem 1.3.2

Clairaut's eq-n

$u = xu_x + yu_y + \frac{u_x^2 + u_y^2}{2}$, $u(x,0) = \frac{1}{2}(1-x^2)$

1) Method of characteristics (strips)

We write the eq-n as

$F(x,y,z,p,q) = 0$

$F(x,y,z,p,q) = xp + yq + \frac{p^2+q^2}{2} - z.$

Characteristic eq-ns:

$\frac{dx}{dt} = F_p = x + p$; $\frac{dz}{dt} = pF_p + qF_q =$

$\frac{dy}{dt} = F_q = y + q$ $\quad = p\cdot(x+p) + q\cdot(y+q)$

$\frac{dp}{dt} = -F_x - F_z \cdot p = -p + 1\cdot p = 0$

$\frac{dq}{dt} = -F_y - F_z \cdot q = -q + q = 0$

Solution:

$$\begin{cases} p = p_0 \\ q = q_0 \\ x + p_0 = C_1 e^t \\ y + q_0 = C_2 e^t \\ \frac{dz}{dt} = p_0(x+p_0) + q_0(y+q_0) = p_0 \cdot C_1 e^t + q_0 \cdot C_2 e^t \end{cases}$$

$\Rightarrow$

$$\Rightarrow \begin{cases} p = p_0,\ q = q_0 \\ x = (x_0 + p_0)e^t - p_0 \\ y = (y_0 + q_0)e^t - q_0 \\ \frac{dz}{dt} = p_0(x_0+p_0)e^t + q_0(y_0+q_0)\cdot e^t \end{cases}$$

$$\Rightarrow \begin{cases} p = p_0,\ q = q_0 \\ x = (x_0 + p_0)e^t - p_0 \\ y = (y_0 + q_0)e^t - q_0 \end{cases}$$

$$z = [p_0(x_0+p_0) + q_0(y_0+q_0)][e^t - 1] + z_0$$

Initial conditions on the curve $y=0$ (x-axis):

$x_0 = s$; $y_0 = 0$; $z_0 = \frac{1}{2}(1-s^2)$

We now solve for $p_0 = p_0(s)$ and $q_0 = q_0(s)$

<u>The strip condition ($\mathcal{E}_2$):</u>

$$z_0'(s) = p_0(s)\,x_0'(s) + q_0(s)\cdot y_0'(s)$$

$$-s = p_0(s)\cdot 1 \Rightarrow p_0(s) = -s$$

<u>The eq-n itself ($\mathcal{E}_1$):</u>

$$x_0(s)\cdot p_0(s) + y_0(s)\,q_0(s) + \frac{p_0^2(s)+q_0^2(s)}{2} - z_0(s) = 0$$

$$s\cdot(-s) + 0\cdot q_0(s) + \frac{s^2+q_0^2}{2} - \frac{1}{2}(1-s^2) = 0$$

$$\cancel{-s^2} + \cancel{\frac{s^2}{2}} + \frac{q_0^2}{2} - \frac{1}{2} + \cancel{\frac{1}{2}s^2} = 0$$

$$q_0 = \pm 1, \text{ i.e.}$$

$$\begin{cases} p = -s \\ q = \pm 1 \\ x = (s-s)e^t + s = s \\ y = (0 \pm 1)e^t \mp 1 \\ z = [-s(s-s) \pm 1(0 \pm 1)][e^t - 1] + \frac{1}{2}(1-s^2) \end{cases}$$

$$= e^t - 1 + \frac{1}{2}(1-s^2)$$

$$\Rightarrow \begin{cases} x = s \\ y = \pm e^t \mp 1 \\ z = e^t - 1 + \frac{1}{2}(1-s^2) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = s \\ z = \pm y + \frac{1}{2}(1-s^2) \end{cases} \Rightarrow$$

$$\Rightarrow z = \pm y + \frac{1}{2}(1-x^2).$$

Check: $x u_x + y u_y + \frac{u_x^2 + u_y^2}{2} = u$

$$x \cdot (-x) + y(\pm 1) + \frac{x^2+1}{2} = \pm y + \frac{1}{2}(1-x^2)$$

$$-x^2 \pm y + \frac{x^2+1}{2} = \pm y + \frac{1}{2}(1-x^2)$$

$$\pm y + \frac{1}{2}(1-x^2) = \pm y + \frac{1}{2}(1-x^2) \quad \text{V}$$

and

$$u(x,0) = \frac{1}{2}(1-x^2)$$

$$\boxed{u(x,y) = \pm y + \frac{1}{2}(1-x^2)}$$

Notice, solution is not unique. This can be seen also from Cauchy-Kovalevsky: on the "hypersurface" $\{y=0\}$ the eq-n can not be written in a normal form, or, rather, it has two "normal forms":

$$u_y = \frac{-2y \pm \sqrt{4y^2 - 8(xu_x + \frac{u_x^2}{2} - u)}}{2}.$$